A SCHOOL ALGEBRA

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A SCHOOL ALGEBRA

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PREFACE.

THE present work is not a mere revision of any of the text-books on Algebra with which my name is connected The last edition of Hall and Knight's *Elementary Algebra* published in Dr Knight's life-time was the sixth, issued in 1890 Since that time there have been fifteen reprints besides three new editions Several of the chapters have been re-written, and I have added more than 100 pages of new matter But it is not easy to make changes satisfactorily in a text-book which is being widely used, it is inconvenient both to teachers and pupils if two or more editions are in use in the same class Some teachers, fully alive to this inconvenience, have implored me to make no more changes Others, with equal insistence have urged that the *Elementary Algebra* should now undergo a thorough revision, involving a new presentment of the subject-matter covered by the existing contents, and the addition of much that is new, both in text and examples The former suggestion would only satisfy a few, while the latter would entail insuperable difficulties in a book that is stereotyped

Guided by such considerations I have undertaken the present work To a large extent it will be found to differ in plan and detail from my other works, though I have not sacrificed utility to novelty of treatment At the same time I have attempted to preserve those characteristic features which contributed so largely to the success of my earliest effort in mathematical writing

In planning the succession of the various parts of the subject, the needs of beginners have constantly been kept in view A few articles and sets of examples have been marked with an asterisk to indicate that they may conveniently be postponed for a second reading As the chapters are usually divided into short sections, each complete in itself, further deviations from the order of the text can be easily made, if necessary. The full Table of Contents will enable teachers to map out for themselves the course best suited to their own classes

The following special features may be mentioned

(i) The early use of symbols is made to arise naturally out of Generalized Arithmetic

(ii) New technical terms and definitions are introduced as they become necessary, and are not crowded together in an initial chapter

(iii) Only the easier cases of Multiplication and Division are at first dealt with See Chaps IV and V

(iv) To relieve the wearisome monotony of that part of the subject, Resolution into Factors has been divided into two separate sections Chap XIV furnishes an easy first course of Factorization, suitable for beginners At the end of this chapter, suggestions are offered for practice in the simpler applications of factors See page 147

Factorization is resumed in Chap XVII, which concludes with examples in the Converse Use of Factors, Easy Identities, and the solution of Quadratic Equations by means of Factors

(v) Chap XV contains harder cases of Multiplication and Division In this chapter some prominence is given to Detached Coefficients, Functional Notation, and the Remainder Theorem

(vi) All the sections on Equations and Problems are unusually full in detail Irrational Roots are usually given to two places of decimals, and a Table of Square Roots is given on page 263

(vii) Graphical work has been kept within reasonable limits The elementary principles of graphs are discussed fully in Chap XI, and some specially useful types in Chaps XXIV, XXVIII, and XXXVIII Elsewhere graphs are interwoven with the text, not so much for their own sake as for the purposes of illustration

(viii) A chapter for revision has been given to illustrate harder applications of elementary processes It also includes some miscellaneous theorems and examples, useful at this stage In particular, it contains further applications of the Remainder Theorem, and some of the uses of Undetermined Coefficients, incidentally preparing the way for Partial Fractions in Chap XLII

(ix) In the chapter on Permutations and Combinations, an attempt has been made to provide against the mental confusion so common in this part of Algebra By careful classification of different cases, illustrated by suitable examples, the student is led on by easy stages to a set of miscellaneous examples which he should be able to attack with some degree of confidence Similar remarks apply to the chapter on Variation

(x) Ample provision has been made for exercise in the practical use of Logarithms and the Binomial Theorem as applied to approximations Tables of Logarithms and Antilogarithms are given on pages 364–367

(xi) Ten sets of Miscellaneous Examples have been provided These are arranged in the form of Revision Papers, of moderate length As papers of this kind sometimes defeat their object by being too hard, I have tried to make each set reasonably simple for the place it occupies

(xii) Pagination has received very careful attention *Throughout the whole book—whether in theorems or examples—the reader's attention is never distracted by turning over a page* This feature, if not unique, must at least be very rare in a mathematical text-book

My aim has been to provide all that is essential in a School Course of Elementary Algebra, sufficient for all students who are not specializing in mathematics. Those who are destined to become mathematicians in any real sense will pursue the study of Algebra in more advanced works, and in due course will fill some gaps which form part of a deliberate plan in the present text-book I refer, in particular, to the latter half of Chap XLI, pages 482–494, and Chap XLIII, where I have emphasized the *practical use* of the Binomial, Exponential, and Logarithmic Series, though omitting proofs of theorems which cannot be established satisfactorily without a considerable digression on Convergency and Divergency of Series

The difficulties connected with the Exponential Theorem and Logarithmic Series were discussed a few years ago at a meeting of the British Association, and in some subsequent papers in the *Mathematical Gazette* Among the methods of proof there suggested there was not one which was in the least suitable for an elementary book Further, from the views expressed by some eminent mathematicians who took part in the discussion, I quote the following remarks

"The less beginners are troubled with questions of convergency of series the better"

"To base the exponential theorem on the limit of $\left(1+\frac{x}{n}\right)^n$, when n is infinite, is logically quite wrong The logarithmic expansion is more difficult and may well be postponed for a while"

After a personal experience of nearly thirty years I have been brought to concur with these views, though I once thought differently.

The convergency of series is a part of Algebra which very few mathematicians really understand until they reach a much later stage in their reading; and to ask the immature schoolboy mind to grapple with all their inherent difficulties before being allowed to make any use of the binomial theorem for any exponent, or of the expansions for e^x and $\log_e(1+x)$, in some of their easy and useful applications, seems to me unwise and unpractical. Be this as it may, I venture to think that the examples and illustrations which I have given on pages 489–494, and in Chap XLIII, will furnish useful matter for numbers of pupils whose algebraical work will never go far beyond the limits of this book, and who in no circumstances would ever find a profitable study in a completely logical treatment of infinite series

It would have been easy to give incomplete, though plausible, proofs of the theorems in question, but it would have been at variance with the spirit of the times There is a growing feeling that it is better to give results without proof rather than to offer proofs, in which all the difficulties are glossed over, and which afterwards have to be abandoned as unsound

H. S HALL

Feb. 1912.

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PART I.

CHAPTER I

Generalized Arithmetic Symbols Substitution

1 Algebra in its simplest applications is a generalized form of Arithmetic Thus in the first place, algebraical usage includes all the definitions and processes of Arithmetic Such definitions and processes are afterwards extended in such a way as to make them of wider and more general use, so that they may be applied to numbers and quantities which have no place in ordinary Arithmetic.

2 In Arithmetic all the numbers we use are expressed by means of the digits 0, 1, 2, 3, 9, each of which has a single definite value In Algebra, besides the ordinary arithmetical numbers we use *symbols* which usually have not a single definite value In some cases the symbols may stand for any numerical values we choose to give them, in others the value or values of any symbol may be restricted by the conditions of the question we are considering

The symbols generally used are the letters of our own alphabet. The Greek letters α, β, γ, are also occasionally used

3 **Signs of operation and their use** The signs $+$, $-$, $\times$, $\div$, $=$ have the same meanings as in Arithmetic A few other signs will be introduced as occasion requires

(i) Thus $7+5=12$ means that by adding the numbers 7 and 5 we obtain 12 as the *sum*

In Algebra $a+b=c$ is a statement which asserts that the *sum* of two numbers denoted by the symbols a and b is equal to a number denoted by c

Thus if c stands for 15, a and b may stand for any pair of numbers whose sum is 15, such as 12 and 3, 1 and 14, 9 and 6, and so on

(ii) If Smith has x apples and Jones has y, they together have $x+y$ apples, an algebraical result which has a *general* value, and is true for any values we choose to give to x and y

Thus if $x=2$, $y=3$, then $x+y=2+3=5$,
if $x=7$, $y=5$, then $x+y=7+5=12$

From this we see that we can get *single definite* values from the *general* algebraical value, and when we say "let $x=2$" we do not

mean that x must always have the value 2, but that 2 is the value to be given to x in the particular example we are considering

Moreover, we see that we may work with symbols without giving them any particular value; indeed it is with such operations that Algebra is chiefly concerned

(iii) Again, $7-5=2$ states that the *difference* between 7 and 5 is 2

And $a-b=2$ states that a and b stand for two numbers whose *difference* is 2

Thus besides the values 7 and 5 for a and b respectively, we might have 8 and 6, 11 and 9, 30 and 28, and so on

(iv) When we say $5\times 7=7\times 5$ we merely state that 5 multiplied by 7 gives the same result as 7 multiplied by 5 But if we allow a and b to stand for any numbers whatever, then $a\times b=b\times a$ states the same principle quite generally, that is not for 5 and 7 only, but for any and every pair of numbers

(v) Division is often expressed by writing the divisor under or after the dividend with a line between them

Thus $30\div 5$ may be written $\frac{30}{5}$, or $30/5$,

$a\div b$ $\qquad \frac{a}{b}$, or a/b

The arithmetical statement $\frac{30}{5}=6$ admits of one and only one interpretation, whereas in the algebraical statement $\frac{a}{b}=6$, besides the values 30 and 5 for a and b respectively, we might have $a=6$, $b=1$; $a=48$, $b=8$; $a=60$, $b=10$; and so on

4 In the ordinary notation of Arithmetic, signs of operation are often omitted without risk of confusion Thus the number *seventy-three* is written 73, and we understand these figures to mean 7 *tens* together with 3 *units* If we use algebraical notation we ought to write the number in the form $7\times 10+3$ If now we replace 7 and 3 by the letters a and b, we may use $a\times 10+b$ to represent a *tens* together with b *units* In other words $a\times 10+b$ represents any number whose units' digit is b, and whose tens' digit is a That is by giving to a and b different numerical values from 0 to 9, we may make $a\times 10+b$ stand for any number less than 100

5 Again, we use £5 6*s* 8*d* as a short way of stating 5 *pounds together with* 6 *shillings together with* 8 *pence* If we wish to express algebraically the number of pence in the above sum we should write it in the form $5\times 240+6\times 12+8$

Similarly if we have x pounds, y shillings, and z pence, the equivalent in pence will be $x\times 240+y\times 12+z$

No numerical result can be given to this number of pence unless some definite numerical values are assigned to the symbols x, y, and z.

6 The foregoing illustrations will serve to explain

(i) the way in which letters may be used just as if they were the ordinary numbers of Arithmetic,

(ii) some points of difference in algebraical as compared with arithmetical usage,

(iii) the way in which a symbol, or collection of symbols, may have different numerical equivalents according to the numerical values the symbols are supposed to represent

7 The following easy examples will further illustrate these points

EXAMPLE 1 *Find a number greater than* x *by* a

If the answer is not obvious take a similar numerical question "Find a number greater than 70 by 6" The process of addition which gives the answer supplies the necessary hint, and just as the number which is greater than 70 by 6 is $70+6$, so the number which is greater than x by a is $x+a$

EXAMPLE 2 *By how much does* x *exceed* 12?

Take a numerical instance "By how much does 20 exceed 12?"

The answer is obviously 8, and is obtained by subtracting 12 from 20

Hence the *excess* of x over 12 is $x-12$

Similarly the *defect* of x from 12 is $12-x$

EXAMPLE 3 If x is one part of 45, the other part is $45-x$

If x is one part of y, the other part is $y-x$

EXAMPLE 4 If $x+6=11$, x must stand for 5

If $y-7=8$, y must be equal to 15

EXAMPLE 5 *A man is* x *years old, how old was he* y *years ago? How old will he be* z *years hence?*

By thinking the question out first with numbers instead of letters, it is easy to see that y years ago his age was $x-y$ years, and in z years' time his age will be $x+z$ years

EXAMPLES I. a

(Many of the following Examples may be taken orally)

1 The quantities a, b, c are to be added together Express this algebraically What is the answer if $a=5$, $b=7$, $c=11$?

2 The quantity r is to be taken from the quantity s How do you express this? What is the answer if $r=27$ and $s=41$?

3 A boy starts playing with x marbles and wins y Express the number he then has If $x=25$ and $y=9$, what number has he?

4 The same boy plays with his increased number and loses z Express the number he then has If $z=17$, how many has he left?

5. A farmer takes f sheep to market and sells g of them How many has he left? What is the remainder if $f=64$ and $g=48$?

6. Another farmer takes L sheep to market and returns with l of them How many has he sold? If $L=75$ and $l=32$, what is the number he has sold?

7 The quantity b is to be subtracted from a, and c added to the difference How do you express this? Give the result if $a=15$, $b=9$, $c=1$ If $a=13$, $b=7$, $c=0$, what is the answer?

8 If n denotes a certain number, how would you express (i) the number next *above* n, (ii) the next above that, (iii) the number that is greater than n by 5?

9 How would you express the number next *below* x, (ii) the next below that, (iii) the number less than x by 6?

10 By how much does x exceed 3? By how much is y less than 10?

11 What must be added to x to make 5? What must be taken from y to make 7?

12. What is the number greater than m by n?

13. What is the number less than n by m?

14. If 10 is greater than a by 4, what is a?

15 If 16 is less than b by 5, what is b?

16 The sum of two numbers is x, and one of them is 8, what is the other?

17 The difference between two numbers is y, and the greater of them is 11, what is the less?

18 I possess £50 How many pounds shall I have left (i) if I spend £16, (ii) if I spend £c, (iii) if I first spend £16 and then spend £c?

19. How many shillings have I left out of £1 if I spend (i) 4 shillings, (ii) x shillings, (iii) y shillings?

20 What is the cost of 16 books (i) at 3s each, (ii) at p shillings each? What is the cost of x books at y shillings each?

21. How would you express (i) the number of shillings in £b, (ii) the number of pence in c shillings, (iii) the number of cwt in p tons?

22 How many pounds are there (i) in 80 shillings, (ii) in d shillings? How many shillings are there (i) in 72 pence, (ii) in c pence?

23 A boy is 11 years old How old will he be (i) in 5 years, (ii) in 8 years, (iii) in m years? How old was he (i) 5 years ago, (ii) 8 years ago, (iii) m years ago?

24 A man is m years old, how old will he be in n years' time? How old was he p years ago?

25 A man is p years old, and his son is q years younger, how old is the son?

26 How old will a boy be in 12 years if he was x years old 3 years ago?

27. In 5 years' time a boy will be l years old, how old was he 5 years ago?

28. What number must be subtracted from $x+13$ in order to obtain x? If $x+13=20$, what is the value of x?

29 What number must be added to $y-6$ in order to obtain y? If $y-6=13$, what is the value of y?

If x stands for an unknown number, state its value when

30. $x+5=11$	31. $6+x=17$	32 $x-4=10$
33. $25-x=15$	34. $15=x+3$	35 $21=x-5$
36. $x\times4=20$	37. $x\times6=6$	38 $5\times x=30$

Give the value of y when

39. $y-2=8-2$	40 $27-y=17$	41. $y+2=3+5$
42. $y-3=8$	43. $7=y-2$	44. $\frac{y}{5}=7$

45. How would you express a number less than 100 if the units' digit was k, and the tens' digit was p?

46 How would you express the number whose three digits in order from left to right are p, q, and r?

8 When two or more numbers are multiplied together the result is called the **product** One important difference between the notation of Arithmetic and Algebra should be here remarked In Arithmetic the product of 2 and 3 is written 2×3, whereas in Algebra the product of a and b may be written in any of the forms $a\times b$, $a\cdot b$, or ab The form ab is the most usual Thus, if $a=2$, $b=3$, the product $ab=a\times b=2\times3=6$, but in Arithmetic 23 means *twenty-three*, or $2\times10+3$

It should be here carefully noted that, in Algebra, the multiplication signs ($\times$ or $\cdot$) must be expressed between *figures*, otherwise they retain their place values, as in Arithmetic Thus

254 means *two hundred and fifty-four*, or $2\times100+5\times10+4$.

But $2\cdot5\cdot4$ means $2\times5\times4$, or 40

When symbols are multiplied by a number, the number is usually placed before the symbols, with no sign of multiplication between.

Thus $3ab$ means 3 times the product ab, or $3\times a\times b$

$25xy$ „ 25 „ „ xy, or $25\times x\times y$

9 The distinction in meaning between *sum* and *product* of two algebraical symbols must be carefully noted For instance,

the *sum* of the two quantities a and b is written $a+b$,

and the *product* „ „ a and b is written ab

Thus, if $a=7$, $b=9$,

the *sum* of a and b is $7+9$, that is, 16:

the *product* of a and b is 7×9, that is, 63

10 Each of the quantities multiplied together to form a product is called a **factor** of the product

Thus 5, a, b, are the factors of the product $5ab$

11 When one of the factors of a product is a numerical quantity, it is called the **coefficient** of the remaining factors

Thus, in the product $5ab$, 5 is the coefficient

Sometimes it is convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors

Thus, in the product $6abc$, $6a$ is the coefficient of bc

A coefficient which involves *letters* is called a **literal coefficient.**

NOTE When the coefficient is unity it is usually omitted Thus we do not write $1a$, but simply a

EXAMPLES I. b

(*Examples* 1–13 *may be taken orally*)

1. What do you understand by 63 and by 6 · 3?

2 What is meant by $45xy$ and $4 \cdot 5xy$? If $x=4$, $y=5$, give the arithmetical value of each

3 Which is the greater 245 or 2 · 4 · 5, and by how much?

4. Write down the product of t and u in three ways

5 If 5 boys each have p shillings, express algebraically how many they have in all.

If $p=25$ what is the number?

6. Write down in two ways the quotient when t is divided by u

7. Write down in two ways the quotient when u is divided by t

8 If x cakes are to be shared equally among 6 boys, express algebraically how many each will have

If $x=42$ what is the number?

9 If 54 books are divided equally among c boys, express each boy's share algebraically What is the arithmetical value if $c=6$?

10 Write down the sum and product of the three quantities a, b, c If $a=5$, $b=7$, $c=6$, what is the value of each?

11 Suppose a day's work consists of 5 lessons, and a boy's mark for each of them is 0, what is his score for the day? What is the value of (i) 0×4, (ii) 0×9, (iii) 11×0, (iv) $0\times$ a million?

12 If a man earns 25 shillings a week, and a boy 8 shillings, at this rate what are the weekly earnings of (i) 4 men and 6 boys, (ii) of p men and q boys?

13 A bookshelf has m shelves, each holding p books, and n shelves, each holding q books Express algebraically the total number of books What is the numerical equivalent when $m=3$, $p=15$, $n=4$ $q=20$?

12 EXAMPLE 1 *If* $a=3$, $b=5$, $c=8$, *find the value of* (i) abc; (ii) $9b$, (iii) $7bc$

(i) $abc=3\times5\times8=120$, (ii) $9b=9\times5=45$, (iii) $7bc=7\times5\times8=280$

EXAMPLE 2 *If* $x=5$, $y=3$, $z=6$, *find the value of* $\dfrac{16xy}{25z}$

Here $$\frac{16xy}{25z}=\frac{16\times5\times3}{25\times6}=\frac{8}{5}=1\tfrac{3}{5}$$

EXAMPLES I. b (*Continued*)

If $a=3$, $b=2$, $c=1$, $x=4$, $y=10$, $z=5$, find the value of

14. $3a$	15. $5b$	16. $7c$	17. ax	18. yz.
19. $\dfrac{x}{b}$	20. $\dfrac{b}{c}$	21. $5ab$	22. $2xy$	23. $3bc$.
24. $20yz$	25. $25cx$	26. $\dfrac{6b}{x}$	27. $\dfrac{3y}{z}$	28. $\dfrac{7a}{c}$

If $m=6$, $n=4$, $s=3$, $x=1$, $y=2$, find the value of

29. mnx	30. nsx	31. $2sxy$	32. $5mxy$
33. $\dfrac{7}{12}ns$	34. $\dfrac{4s}{3n}$	35. $\dfrac{6y}{ns}$	36. $\dfrac{9ny}{16ms}$
37. $\dfrac{nsy}{24x}$	38. $\dfrac{nsxy}{2m}$	39. $\dfrac{19ns}{20mxy}$	40. $\dfrac{7}{12}mnsxy$

13 The product obtained by multiplying together several factors all equal to the same number is called a **power** of that number.

Thus 4×4 is called the second power of 4,

$6\times6\times6$ third power of 6,

$a\times a\times a\times a$ fourth power of a,

and so on

For the sake of brevity the following notation is used

$$4\times4=4^2,\ 6\times6\times6=6^3,\ a\times a\times a\times a=a^4,$$

and the small figure which indicates the number of equal factors is called the **index** or **exponent** of the power

Thus in 2^3, 4^5, x^6 the *indices* are 3, 5, and 6 respectively

14 The *second* and *third* powers of a number are known as its **square** and **cube** respectively

Thus the square of 8, or $8^2=8\times8 \quad =64$,

the cube of 7, or $7^3=7\times7\times7=343$

a^2 is usually read "a squared", a^3 is read "a cubed", a^4 is read "a to the fourth", and so on

The *first power* of a number is the number itself Hence we do not write a^1, but simply a

Thus a, $1a$, a^1, $1a^1$, all have the same meaning

15 The beginner must be careful to distinguish between *coefficient* and *index*

EXAMPLE 1 *What is the difference in meaning between* 3a *and* a^3?

By $3a$ we mean the product of the quantities 3 and a

By a^3 we mean the third power of a, that is, the product of the quantities a, a, a

Thus, if $a=4$, $\qquad 3a=3\times a=3\times 4=12,$

$$a^3=a\times a\times a=4\times 4\times 4=64$$

EXAMPLE 2 *If* b=5, *distinguish between* $4b^2$ *and* $2b^4$

Here $\qquad 4b^2=4\times b\times b=4\times 5\times 5=100,$

whereas $\qquad 2b^4=2\times b\times b\times b\times b=2\times 5\times 5\times 5\times 5=1250$

EXAMPLE 3 *If* a=4, x=1, *find the value of* $5x^a$

Here $\qquad 5x^a=5x^4=5\times x\times x\times x\times x=5\times 1\times 1\times 1\times 1=5$

NOTE Every power of 1 is 1.

16 In Arithmetic the factors of a product may be written in any order. Thus, for example,

$$3\times 4=4\times 3,$$

and

$$3\times 4\times 5=4\times 3\times 5=4\times 5\times 3$$

As all the symbols we are using denote arithmetical numbers, we shall assume for the present that the same principle holds good for an algebraical product Thus the products *abc, acb, bac, bca, cab, cba* have the same value It is usual, however, to write the factors of such a product in alphabetical order

17 Fractional coefficients which are greater than unity are usually kept in the form of improper fractions

EXAMPLE *If* a=6, x=7, z=5, *find the value of* $\frac{13}{10}axz$.

Here $\qquad \frac{13}{10}axz=\frac{13}{10}\times 6\times 7\times 5=273$

EXAMPLES I. c.

(*Examples* 1–15 *may be taken orally*)

1. What is the difference between "twice 3" and "3 squared"?

2 Write down the expression for "thrice d," also that for the "cube of d" Give the arithmetical values if $d=2$

3 Distinguish between "four times x" and "x to the fourth" Write down the respective values when $x=3$

4 The quantity c is to be multiplied by the quantity x How is this expressed? Write down the product if $c=7$ and $x=3$

5 If x factors, each equal to c, are to be multiplied together, express this algebraically What is the value if $x=3$, and the factor $c=7$?

6 If I walk y miles per hour for y hours, express the length of my walk algebraically If $y=4$, what is the answer?

7. What is the area of a square room each side of which is m feet? Give the numerical value if $m=16$?

If $a=7$, $b=5$, $c=1$, $x=3$, $y=2$, find the value of

8. $2a$	9 a^2	10 $3b$	11 b^3	12. $4c$
13. c^4	14. $7c^2$	15 $5y^3$	16. x^5	17. y^6
18 $2x^4$	19 $4a^2$	20 $3b^2$	21. $2b^3$.	22 $5c^5$

If $a=8$, $b=2$, $c=5$, $x=1$, $y=3$, find the value of

23 $\frac{1}{x^3}$	24. $\frac{b^2}{a}$	25 $\frac{x^4}{c^2}$.	26 $\frac{b^6}{a^2}$	27. $\frac{y^2}{3x^2}$
28. $\frac{2y^3}{3a^2}$	29. $\frac{25b^4}{16c^3}$	30. $\frac{4y^3}{9b^3}$	31 $\frac{cy}{30b^2}$	32 $\frac{acx}{25b^3}$.

If $a=3$, $b=5$, $c=4$, $x=1$, find the value of

33. 2^a	34 3^x	35 a^x	36 x^a	37. $\frac{x}{ac}$
38 $\frac{x}{a^c}$	39 $\frac{b^x}{acx}$	40. $\frac{27x^b}{a^3}$	41 $\frac{4bcx}{15a^x}$	42 $\frac{x^a}{b^a}$

18 When powers of several different quantities are multiplied together, a notation similar to that of Art 13 is adopted Thus $aabbbbcddd$ is written $a^2b^4cd^3$ And, conversely, $7a^3cd^2$ has the same meaning as $7\times a\times a\times a\times c\times d\times d$

EXAMPLE 1 *If* $c=3$, $d=5$, *find the value of* $16c^4d^3$

Here $16c^4d^3=16\times 3^4\times 5^3=(16\times 5^3)\times 3^4=2000\times 81=162000$

NOTE The beginner should observe that by a suitable combination of the factors some labour has been avoided

EXAMPLE 2 *If* $p=4$, $q=9$, $r=6$, $n=5$, *find the value of* $\frac{32qr^3}{81p^n}$

Here $\frac{32qr^3}{81p^n}=\frac{32\times 9\times 6^3}{81\times 4^5}=\frac{32\times 9\times 6\times 6\times 6}{81\times 4\times 4\times 4\times 4\times 4}=\frac{3}{4}$

19 If one factor of a product is equal to 0, the product must be equal to 0, *whatever values the other factors may have*

A factor 0 is usually called a **zero factor.**

For instance, if $x=0$, then ab^3xy^2 contains a zero factor Therefore $ab^3xy^2=0$ when $x=0$, whatever be the values of a, b, y

Again, if $c=0$, then $c^3=0$, therefore $ab^2c^3=0$, whatever values a and b may have

NOTE Every power of 0 is 0

EXAMPLES. I. d.

If $a=5$, $b=3$, $m=10$, $n=1$, $x=0$, $z=6$, find the value of

1. a^2b	2. ab^3	3. am^3.	4. a^3m	5. m^3n^2
6. m^2n^3	7. ax^2	8. a^2x	9. ab^3x	10. bxz^3
11. m^2n^3z	12. bm^3n^4	13. m^2nz^3	14. a^4x^3z	15. $m^3n^2z^2$

If $d=1$, $e=2$, $f=0$, $g=4$, $s=6$, find the value of

16. $5d^4e^3$	17. $3d^3e^2$	18. $7e^2f$	19. $7ef^2$	20. $2d^2gs$.
21. $3d^2e^2g^2$	22. $6d^4e^3s$	23. $4f^2gs^4$	24. $\frac{1}{2}d^2eg$	25. $\frac{d^2e}{de^2}$
26. $\frac{ef^3}{gs^2}$	27. $\frac{1}{d^4e^2}$	28. $\frac{13g^2s^2}{12e^3}$	29. $\frac{d^2ef^3}{17s}$	30. $\frac{gs^3}{18deg^2}$

20 Any collection of numbers and symbols connected by the signs $+$, $-$, $\times$, $\div$ is called an **algebraical expression.** Parts of an expression separated by the signs $+$ or $-$ are called **terms.** The signs $\times$ and $\div$ do not separate terms

Thus $7a+3b\times c-4d+xy-a\div b$ is an expression of *five* terms Here $3b\times c$ is a single term, so is $a\div b$

NOTE When no sign precedes a term the sign $+$ is understood

21 Expressions are either simple or compound A *simple expression* consists of *one* term, as $5a$ A *compound expression* consists of *two or more* terms An expression of *two* terms, as $3a-2b$, is called a **binomial** expression, one of *three* terms, as $2a-3b+c$, a **trinomial**, one of *more than three* terms a **multinomial** Simple expressions are also spoken of as **monomials**

22 In the case of expressions which contain more than one term, each term can be dealt with singly by the rules already given, and by combining the terms the numerical value of the whole expression is obtained When brackets ( ) are used, they will have the same meaning as in Arithmetic, indicating that the terms enclosed within them are to be considered as one quantity

EXAMPLE 1 *When* $c=5$, *find the value of* $c^4-4c+2c^3-3c^2$

Here
$$c^4=5^4=5\times5\times5\times5=625,$$
$$4c=4\times5=20,$$
$$2c^3=2\times5^3=2\times5\times5\times5=250,$$
$$3c^2=3\times5^2=3\times5\times5=75$$

Hence the value of the expression
$$=625-20+250-75=780$$

EXAMPLE 2 *If* $a=7$, $b=3$, $c=2$, *find the value of*
$$a(b+c)^2-c(a-b)^3$$

The expression $=7(3+2)^2-2(7-3)^3=7\cdot5^2-2\cdot4^3=175-128=47.$

EXAMPLE 3 *When* $a=5,\ b=3,\ c=1$, *find the value of*

$$a^2\,\frac{a-b}{b+2c}-b^2\,\frac{a-c}{(a+c)^2}$$

$$\text{The expression}=5^2\times\frac{5-3}{3+(2\times1)}-3^2\times\frac{5-1}{(5+1)^2}$$
$$=25\times\frac{2}{5}-9\times\frac{4}{36}$$
$$=10-1=9$$

23 By Art 19 any term which contains a *zero factor* is itself zero, and may therefore be called a *zero term*

EXAMPLE 1 *If* $a=2,\ b=0,\ x=5,\ y=3$, *find the value of*

$$5a^3-ab^2+2x^2y+3bxy$$

$$\text{The expression}=(5\times2^3)-0+(2\times5^2\times3)+0$$
$$=40+150=190$$

NOTE. The two *zero terms* do not affect the result

EXAMPLE 2 *Find the values of the expression* $x^2-10x+21$ *when* x *has the values* 0, 2, 3, 7, 8

Here the following tabular arrangement will be found convenient

x	0	2	3	7	8
x^2	0	4	9	49	64
$10x$	0	20	30	70	80
$x^2-10x+21$	21	5	0	0	5

Thus the required values are 21, 5, 0, 0, and 5

24 In working examples the student should pay attention to the following hints

(i) Too much importance cannot be attached to neatness of style and arrangement The beginner should remember that neatness is in itself conducive to accuracy

(ii) The sign = should never be used except to connect quantities which are equal Beginners should be particularly careful not to employ the sign of equality in any vague and inexact sense

(iii) Unless the expressions are very short the signs of equality in the steps of the work should be placed one under the other

(iv) It should be clearly brought out how each step follows from the one before it, for this purpose it will sometimes be advisable to add short verbal explanations, the importance of this will be seen later

EXAMPLES I. e.

[*In this Exercise only a few of Examples* 1–24 *need be worked As soon as sufficient accuracy is secured pupils should pass on to Example* 25]

If $a=4$, $b=3$, $c=5$, $d=6$, $x=7$, $y=0$, find the value of

1. $2a+3b-c$ 2 $4b-2a+3y$ 3 $6a-3b-2d$

4. $3x-4y+2b$ 5. $6c-5d+2y$ 6 $4x-5c+7a$

7. $7a-4x-4y+2c-3b$ 8 $7c-4x+a-2b+9y$

9 $2dx-abc+4bc$ 10 $aby+bcy+dcy$

11. $5a-\frac{cd}{2}+\frac{abc}{3}$ 12 $\frac{ad}{8b}-\frac{2d}{ab}+\frac{xy}{4a}$

If $a=2$, $b=1$, $c=3$, $x=4$, $y=6$, $z=0$, find the value of

13 $x^2-a^2+c^2-z^2$ 14 $c^4-4b-3x+a^4$

15 $a^3+b^4+c^2+xyz$ 16 $3bcx-acx+3abx-y^2$

17 $-b+2b^2+3b^3-4b^4$ 18. $4a^4-3a^3+2a^2-a$

19 $5b^3+\frac{xyz}{4}-3abx+a^5$ 20. $\frac{4}{9}y^2-b^3-\frac{4}{27}c^2y+\frac{a^4}{x^2}$

21. $\frac{a^2}{b^3}+\frac{b^2}{a^2}-\frac{2y}{x^2}$ 22. $\frac{a^2}{b^2}\ c^2+\frac{a^2}{b^2}+c^2.$

23. $\frac{(a+y)^2}{(x-z)^3}-\frac{6(c^2-a)}{7(a^2+x)}$ 24 $\frac{a^2-b^2}{a^2b^2}-\frac{(a+b+z)^2}{(b+c-z)^2}$

25. Find the values of x^2+7x+2 when x has the values 0, 1, 3, 5, 7

26 When x has the values 0, 1, 2, 3, 4, find the values of the expression x^2-5x+8

27 Shew that $x^2-7x+12=0$ if $x=3$, and also if $x=4$ What is its value when $x=8$?

28 What are the values of the expression $\frac{x^2}{4}-\frac{3x}{2}+10$ when x has the values 2, 4, 6, 8?

29 When x has the values 1, 4, 7, 10, find the values of $16-x+x^2$

30 Shew, by substituting 9 for x and 3 for y, that the two expressions

$$4(x-y)+5(x+y), \quad 7(x+y)+2(x-3y)$$

are equal

Test their equality also when $x=10$, $y=0$

31. Shew that $x^3-9x^2+23x=15$ when x has the values 1, 3, or 5

32 By substituting 10 for p and 5 for q, shew that the expressions

$$8(p-q)+3(p+q), \quad 2(p+2q)+9(p-q)$$

are equal

Test their equality when $p=5$, $q=4$

33 Shew that y^3+19y is equal to $4(2y^2+3)$ when $y=1$, 3, or 4. Which expression is the greater when $y=2$?

CHAPTER II

NEGATIVE QUANTITIES ADDITION SIMPLE BRACKETS

25 HITHERTO in an expression involving a connected chain of additions and subtractions the sum of the subtractive terms has never been greater than the sum of the additive terms, and if an example were to reduce to a result such as $+4-9$ the subtraction could not be arithmetically performed As an algebraical result, however, such an expression can be explained, moreover a subtractive term, such as -5, can stand alone, and have an intelligible meaning given to it

26 We shall begin with some concrete illustrations suggesting extended meanings for the operations of addition and subtraction

(i) Suppose a merchant in the course of his business gains £100 and then loses £70, the result of his trading is a *gain* of £30 That is, as an algebraical statement we may say

$$+£100-£70=+£30$$

and $+£30$ denotes that he is £30 better off than before

But if he gained £70 and lost £70, the loss would exactly balance the gain,

that is, $$+£70-£70=£0$$

Thus he would be in the same position as when he began

If, however, he gained £70 and lost £100, the result of his trading would be a loss of £30 Now if we regard the loss of £100 as represented by one of £70 together with another of £30, the result of his trading may be stated as

$$+£70-£70-£30,$$

and since $+£70-£70=£0$, we see that

$$+£70-£100=-£30,$$

and $-£30$ denotes that he is £30 worse off than when he began

In other words $-£30$ denotes a *loss* or *debt* of £30

(ii) Again suppose a man to walk 5 miles due East and then to walk back 3 miles due West, his position relative to the starting point would be

$$+5 \text{ miles}-3 \text{ miles, or } +2 \text{ miles},$$

where $+2$ miles denotes the distance he was ultimately due East of his starting point

If he had walked 3 miles due East and then back 5 miles due West, his position relative to the starting point would be

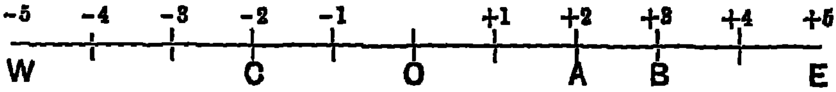
$$+3 \text{ miles} - 5 \text{ miles, or } -2 \text{ miles,}$$

where -2 miles denotes the distance he was ultimately due West of his starting point

Thus we see that -2 miles denotes a distance equal in magnitude but opposite in direction to that denoted by $+2$ miles

In general, if we denote a series of steps taken in one direction by quantities with the $+$ sign prefixed, we may represent a series of steps taken in the opposite direction by quantities with the $-$ sign prefixed

This may be illustrated graphically as follows



Let O be the starting point and let the line WOE be marked in centimetres, each centimetre representing 1 mile Also let the direction OE (from left to right) be considered as due East, and OW (from right to left) as due West Let the successive miles Eastwards be marked $+1, +2, +3,$, and those Westwards $-1, -2, -3,$.

Then 5 miles Eastwards followed by 3 miles Westwards will be represented by motion from O to E followed by motion in the opposite direction from E to A.

Thus $OA = OE - EA = +5$ cm -3 cm $= +2$ cm, which represents $+2$ miles

Again, 3 miles Eastwards followed by 5 miles Westwards will be represented by motion from O to B followed by motion in the opposite direction from B to C

Thus $OC = OB - BC = +3$ cm -5 cm $= -2$ cm, which represents -2 miles

(iii) On a Centigrade thermometer 15° C means 15° *above* the freezing point or *zero*, and -15° C means 15° *below* zero

27 From the experience of ordinary life, as shewn in the foregoing illustrations, we see that we frequently meet with concrete quantities which are capable of existing in two opposite states Hence in Algebra we find it convenient to use the $+$ and $-$ signs not only as indicating the actual operations of addition and subtraction, but as denoting a *quality* possessed by the quantities to which they are attached In this sense, since an algebraical quantity is *affected* by the sign which precedes it, the signs $+$ and $-$ are called *signs of affection*

28 Quantities preceded by the sign $+$ are said to be positive, those preceded by the sign $-$ are said to be negative. Quantities to which no sign is prefixed are counted as positive

29. When the value of a number is considered numerically without reference to the sign affecting it, it is called its **absolute** value

Thus $+5$ and -5 have the same absolute value, namely 5

30 Whatever quantities we happen to be dealing with, those with the $+$ sign attached may be considered as possessing a certain quality, while those with the $-$ sign attached may be considered as quantities related to the former but possessing the *opposite* quality

Thus, if the quantities are sums of money we may suppose $+5$ to denote a *gain*, and then -5 will denote a *loss* of the same absolute amount

Or we may suppose $+5$ to denote a *loss*, and then -5 will denote a *gain* of the same absolute amount

In other words, when the $+$ sign has been attached to either of the two related quantities we are thinking of, then the $-$ sign must belong to the other quantity

Thus in the language of Algebra a *fall* of 3° in the thermometer may be spoken of as a *rise* of $-3°$

Or, if two runners, A and B, are placed 10 yards *in front of* 'scratch' and 10 yards *behind* 'scratch' respectively, we may say that

A has a start of $+10$ yards,

B has a start of -10 yards

In racing language B would be said to *owe* 10 yards

EXAMPLES II a

(Most of these Examples may be taken orally)

1. What is a man worth in each of the following cases?

(i) If he has £6 in his purse, and owes £3

(ii) £10 and has unpaid bills of £8 and £2

2 How much is a man in debt

(i) if he has £5 in hand, and owes £7?

(ii) if he has £5 in hand, and owes £12?

How much is he *worth* in each of these cases?

3 Two places are respectively 12 miles and 7 miles due West of Bristol Cathedral How far are they apart?

4 What is the distance between two places which are a miles and b miles North of St Paul's, (i) when a is greater than b, (ii) when b is greater than a?

5. A ship sails due North from the Equator till her latitude is 40°; if she then sails due South till her latitude is $-20°$, how many degrees of latitude has she covered?

6 I have £a deposited in a bank, if I draw out £b, how much have I left? If $a=20$, and $b=25$, how is the answer to be interpreted?

7. How much must I pay in to my bank in order to bring my deposit of £p up to a sum of £q? Alter the wording of the question so as to make it apply to the case in which $p=15$, and $q=10$

8 Two cricket teams each play 24 matches, one wins 12, draws 4, and loses 8 matches, and the other wins 6, draws 8, and loses 10 Allowing one point for a win, nothing for a draw, and deducting one point for a loss, express the two results

9 A owes me £5, so that his debt is represented by $-$£5 If I cancel or take away the debt, how much better off is A than before? How much have I practically given him? Hence shew that $-(-5)$ is the same as $+5$

10. At 6 p m the temperature is 12° C, at 9 a m the following day it is -5° C, how far has the mercury fallen during the night?

11. At 3 a m the mercury stands at 4° C, and at 9 a m it is colder by 10° Between 9 a m and 1 p m the temperature rises 6° What are the readings of the thermometer at 9 a m and 1 p m?

12. On a Centigrade thermometer what is the rise between -5° and 20°?

If $-5+x=20$, what is x?

13. What is the rise between -16° and -4°?

If $-16+x=-4$, what is x?

14. What is the rise between 16° and -4°?

If $16+x=-4$, what is x?

15 Two men each fire 16 shots at a mark and agree to register 3 points for a hit and to deduct 2 points for a miss One hits the mark 9 times, the other 5 times Express their scores algebraically

16 Three boys, A, B, and C, each bowl 20 balls at a wicket, and agree to record 5 points for a hit and to deduct 2 points for a miss, A hits the wicket eight times, B five times, and C six times Place them with their respective scores in order of merit

17 From a point O two boys A and B run due E and W respectively for 30 seconds, and then due S and N respectively for 20 seconds If the East and North directions are regarded as positive, and A runs 8 yds every second, while B runs 42 yds every 5 seconds, express their final distances E and N of O Illustrate by a diagram

18 A has £p and B has £q A owes £a to B, and B owes £b to A How many pounds will each man have when their debts are paid?

Addition of Like Terms.

31 Definition When terms do not differ, or when they differ only in their numerical coefficients, they are called like, otherwise they are called unlike Thus, $3a$, $7a$, $5a^2b$, $2a^2b$, $3a^3b^2$, $-4a^3b^2$ are pairs of like terms, and $4a$, $3b$, $7a^2$, $9a^2b$ are pairs of unlike terms

32 Though we have now to consider the addition of positive and negative quantities generally (*i e* without direct reference to any concrete unit) the beginner will find it helpful to frequently re-assure himself by thinking out the different cases in connection with some concrete illustration For example, he will readily admit the truth of the following statements

(i) *The sum of two or more gains is a gain*

(ii) *The sum of two or more losses is a loss*

(iii) *A number of gains associated with a number of losses will result in a gain or a loss according as the sum of the gains is greater or less than the sum of the losses*

The rules for the addition of like terms are only a generalization of these principles

Example 1 *Find the value of* $8a+5a$

Whatever number a stands for, if we take it 8 times and then 5 times, we take it 13 times in all Hence

just as $$8 \text{ tens}+5 \text{ tens}=13 \text{ tens},$$

so $$8a+5a=13a,$$

that is, the coefficient of a in the result is the sum of the coefficients of a in the two like terms

Similarly $$8b+5b+b+2b+6b=22b,$$

by adding the coefficients of the several terms

Example 2 *To find the sum of* $-3x$, $-5x$, $-7x$, $-x$

Here we have to express, as one subtractive quantity, the *sum*, or total, of four subtractive quantities of like character To subtract in succession 3, 5, 7, 1 like things would have the same effect as to take away $3+5+7+1$, or 16, such things in one operation

Thus the sum of $-3x$, $-5x$, $-7x$, $-x$ is $-16x$

Hence the following rule

Rule I *The sum of a number of like terms*, all of the same sign, *is a single like term of the same sign And the numerical value of its coefficient is the sum of the numerical values of the several coefficients*

Example 3 *Find the sum of* $17x$ *and* $-8x$

A gain of 17 followed by a loss of 8 would result in a gain of 9, for the difference of 17 and 8 is 9, and the gain, or positive term, is the greater

In the same way the sum of $17x$ and $-8x=9x$

Example 4 *The sum of* $-17x$ *and* $8x=-9x$

Example 5 *Find the sum of* $8a$, $-9a$, $-a$, $3a$, $4a$, $-11a$, a

Adding up the *coefficients* from the left, taking each one with the sum of all which precede it, we have

$$-1, \ -2, \ +1, \ +5, \ -6, \ -5,$$

thus the required sum is $-5a$.

Or thus:

The sum of the coefficients of the positive terms is 16,

" " " negative " 21, *numerically*

The difference of these is 5, and the sign of the greater is negative, hence the required sum is $-5a$

Hence the following rule

Rule II *The sum of a number of like terms*, not all of the same sign, *is a single like term To obtain its coefficient add together the numerical values of the positive terms, and the numerical values of the negative terms Take the difference of these two results and prefix the sign of the greater*

33 From Example 5 we infer that the order in which the terms are taken does not affect the result so long as the terms preserve their positive or negative character We may adopt any order we find most convenient The process is called **collecting terms.**

EXAMPLE *What is the value of* $-3x^2y+5x^2y-7x^2y+x^2y$?

$$-3x^2y+5x^2y-7x^2y+x^2y=-10x^2y+6x^2y=-4x^2y$$

34 When quantities are connected by the signs $+$ and $-$, the resulting expression is called their **algebraical sum**

Thus $11a-27a+13a=-3a$ states that the algebraical sum of $11a$, $-27a$, $13a$ is equal to $-3a$

NOTE The sum of two quantities numerically equal but with opposite signs is zero Thus the sum of $5a$ and $-5a$ is 0

EXAMPLES II b

1. Add together the following pairs of numbers and read off the results

$+7, -5$, $-7, +5$, $-8, +6$, $-8, +13$, $+19, -15$,

$-27, -2$, $-8, -6$, $-15, -12$, $-16, -12$, $-20, -16$,

2. Read off the values of

(i) $+5-11$, (ii) $-18+9$, (iii) $-15-7$ (iv) $-16+25$;

(v) $+11-15+4$, (vi) $-15+3-8-6$, (vii) $-8-7-2+13$

Find the sum of

3 $3a, 2a, a, 5a, 9a$

4. $4x, x, 8x, 11x, 3x$

5 $7p, 11p, 5p, 3p, 9p$

6 $d, 10d, 100d, 1000d$

7 $-2y, -4y, -8y, -10y$

8 $-m, -9m, -11m, -19m$.

9 $-21z, -z, -17z, -8z$

10 $-25c, -7c, -c, -21c$

11. $-b, 2b, b, -6b, -2b$

12 $8x, -10x, 3x, -5x, -6x$

13. $2ab, -4ab, -6ab, 7ab$

14. $-xy, 5xy, -10xy, 6xy$

15. $-7cd, -3cd, 4cd, -5cd$

16. $pq, 3qp, -5pq, 7qp$.

Find the value of

17. $-7c^2+c^2+4c^2-8c^2$
18. $3b^3-2b^3-6b^3+4b^3$.
19. $-a^2b^2-4a^2b^2+9a^2b^2-3a^2b^2$
20. $-2x^2y+6x^2y+8x^2y-12x^2y$
21. $8y^3z^2+y^3z^2-12y^3z^2+2y^3z^2$
22. $-4c^4d+6c^4d-8c^4d+10c^4d$
23. $7abcd-9abcd-11abcd$
24. $6xyz-8yzx-24zxy$

25. If the sum of $2x$, $5x$, and $7x$ is 42, what is the value of x?
[*By collecting terms*, $14x=42$, *whence* $x=\frac{42}{14}=3$]

26. The algebraical sum of $15x$, $-8x$, and $3x$ is 100, what is the value of x?

27. What is the value of x if the sum of $4x$, x, $3x$, $7x$, and $9x=24$?

28. Find the value of y when $7y-11y+16y-3y$ is equal to 18

35 The addition of like terms may now be applied in solving easy problems

EXAMPLE 1 *One number is 5 times as great as another, and their difference is 48, what are they?*

Let x represent the smaller number, then $5x$ will represent the greater

Now the difference of the two numbers is 48;

$$5x-x=48,$$

that is,
$$4x=48$$

Dividing by 4, we get
$$x=\frac{48}{4}=12$$

Hence $5x=60$, and the required numbers are 60 and 12

EXAMPLE 2. *I think of a number, I double it, treble it, and multiply it by 5 I add these three results together and from the sum subtract four times the number first thought of If the result now comes to 66, what was the number thought of?*

Suppose x represents the number, then $2x$, $3x$, $5x$ have to be added together, and $4x$ subtracted from the sum This gives $2x+3x+5x-4x$

But by the question the result is 66,

$$2x+3x+5x-4x=66$$

Collecting the terms,
$$6x=66,$$
$$x=\frac{66}{6}=11$$

Thus the required number is 11

The answer may be checked or *verified* as follows

Twice $11=22$, three times $11=33$, five times $11=55$

The sum of these three numbers $=110$

Four times $11=44$ And $110-44=66$

All the examples in the following Exercise should be verified in a similar manner

EXAMPLES II. c.

1. There are two numbers one of which is 7 times as great as the other; if their sum is 96, what are the numbers?

2. Divide 70 into two parts so that one may be four times the other.

3. One number is 16 times as great as another, and their difference is 75; find them.

4. Find two numbers differing by 57 so that one of them is 20 times the other.

5. Find two numbers whose sum is 52, such that one of them is one twelfth of the other. [*Let* x *represent the* smaller *number.*]

6. Find two numbers whose difference is 88, such that one of them is one-ninth of the other.

7. Find three numbers whose sum is 56, such that the second is equal to twice the first, and the third equal to four times the first.

8. Divide 84 into three parts such that two of the parts are equal and the third is five times as great as either of the equal parts.

9. Having thought of a number, I multiply it successively by 4, 5, and 8, and add the results. I then subtract the double of the number thought of and find that the result comes to 45. What number did I think of?

EXAMPLE 3. *Divide £63 between A, B, and C so that B may have twice as much as A, and C three times as much as B.*

Let x be the *number of pounds* A has; then $2x$ will represent the number of pounds B has.

And since C has three times as much as B, he will have $6x$ pounds.

The sum of the three shares is £63;

$$x+2x+6x=63.$$

Collecting terms, $$9x=63,$$

$$x=\frac{63}{9}=7.$$

Hence A has £7, B has £14, C has £42.

EXAMPLE 4. *B's age is one third of A's, and one fifth of C's. Their combined ages are 72 years; find them.*

Let x years represent B's age; then since A is *three times* as old as B, his age will be $3x$ years. Similarly C's age will be $5x$ years.

But the sum of the ages is 72 years;

$$x+3x+5x=72.$$

Collecting terms, $$9x=72,$$

$$x=\frac{72}{9}=8.$$

Hence B is 8 years old. Therefore A's age is 24, and C's age is 40.

NOTE. It is very important to remember that the symbol x stands for a *number*; in Ex. 3 it is a *number* of *pounds*, and in Ex. 4 a *number* of *years*. The beginner is especially cautioned against the use of loose and inexact expressions such as "Let $x=A$'s share" or "let $x=B$'s age."

10 Divide £24 between A, B, and C so that A may have 5 times, and B 6 times as much as C

[*Let* x *denote the number of pounds C has*]

11 Divide £3 5s between A and B so that B may have 4s for every shilling that A has

[*Let* x *denote the number of* shillings *A has Express everything in shillings*]

12 Divide £2 5s between A, B, and C so that A and C may each have 7 times as much as B

[*Let* x *denote the number of* shillings *B has Express everything in shillings*]

13 In a month's business a man has 4 gains, each of the last three being double of the preceding one If his total gain is £150, what was the amount of his first gain?

14. A man who balances his accounts at the end of every quarter finds that he has three gains followed by a loss The third gain is 4 times the second, and the second is three times the first The loss is twice the first gain If on the whole he gains £112, find the amount of the loss

[*Let* x *denote the number of pounds in the* first *gain*]

15 Divide 48 shillings in wages for a man, a woman, and a boy, on the supposition that the man gets four times, and the woman three times as much as a boy

16 Divide 2 guineas between a man and a woman so that the man may have 4 shillings to every 3 shillings that the woman has

17 A man is four times as old as his daughter, and the difference between their ages is 36 years Find their ages

18 A man is twice as old as his son and 10 times as old as his grandson Their combined ages amount to 96 years, how old are they?

19 A father is nine times as old as his son and three times as old as his daughter Their combined ages make up 65 years, find them

[*Let* x *be the number of years in the* son's *age*]

36 Brackets ( ) are used to indicate that the terms enclosed within them are to be considered as one quantity The full use of brackets will be considered in Chapter VI, here we shall deal only with the simpler cases

The expression $8+(13+5)$ means that 13 and 5 are to be added and their sum added to 8 It is clear that 13 and 5 may be added to 8 separately or together without altering the result

Thus $$8+(13+5)=8+13+5=26$$

Similarly, if the letters a, b, c are used to represent any numbers, $a+(b+c)$ means that the sum of b and c is to be added to a, and since b and c may be added separately or together, it follows that

$$a+(b+c)=a+b+c \qquad (1)$$

Again, $8+(13-5)$ means that to 8 we are to add the excess of 13 over 5; now if we add 13 to 8 we have added too much by 5, and must therefore take 5 from the result.

Thus $$8+(13-5)=8+13-5=16$$

Similarly $a+(b-c)$ means that to a we are to add b, diminished by c. If therefore we add b we must afterwards subtract c.

Thus $$a+(b-c)=a+b-c. \quad \ldots \quad (2)$$

NOTE. It must be observed that in this last case we have assumed that b is not less than c. Otherwise we cannot subtract c from b arithmetically. We shall return to this point later.

By considering the results numbered (1) and (2) above we are led to the following rule:

Rule. *When an expression within brackets is preceded by the sign $+$, the brackets can be removed without making any change in the expression.*

Thus $$a-b-c+(d-e-f)=a-b-c+d-e-f.$$

37. The expression $9-(3+2)$ means that from 9 we are to subtract the sum of 3 and 2. To take the sum of 3 and 2 from 9 is clearly to obtain the same result as to subtract them *separately* from 9.

Thus $$9-(3+2)=9-3-2$$

Similarly $$a-(b+c)=a-b-c \quad . \quad (3)$$

NOTE. This last result assumes that a is not less than the sum of b and c.

Again $9-(3-2)$ means that from 9 we are to subtract the excess of 3 over 2. If we subtract 3 we shall have taken away too much by 2, and must therefore add 2 to obtain the correct result.

Thus $$9-(3-2)=9-3+2$$

Similarly $$a-(b-c)=a-b+c \quad . . (4)$$

NOTE. Here it is assumed for the present that b is not less than c, and a not less than b.

By considering the results numbered (3) and (4) we have the following rule.

Rule. *When an expression within brackets is preceded by the sign $-$, the brackets may be removed if the sign of every term within the brackets be changed.*

Thus by removing brackets $a+(b-c)-(d-f+e)$ may be written in the form
$$a+b-c-d+f-e$$

EXAMPLE 1. *Simplify* $7a-(3a-2a)$.
$$7a-(3a-2a)=7a-3a-2a=10a-2a=8a$$ *[sic: as printed]*

EXAMPLE 2. *Simplify* $6x^2-(2x^2-5x^2)$.
$$6x^2-(2x^2-5x^2)=6x^2-2x^2+5x^2=11x^2-2x^2=9x^2.$$

38 The rules for removing brackets have only been proved strictly for values of the symbols which make all the operations arithmetically possible But we may remark that whenever any new symbols or operations are introduced in Algebra we may give to them what meaning we like provided that we *always* employ such meaning, and that our meaning is not inconsistent with principles already established for arithmetical numbers Hence we shall *define*

$a+(b-c)$ as being equivalent to $a+b-c$,
$a-(b+c)$ „ „ $a-b-c$,
$a-(b-c)$ „ „ $a-b+c$,

no matter what numbers the symbols represent And we shall accept the conclusions to which these meanings lead us

The following special cases should be noted

$$+(+a)=+a, \qquad +(-a)=-a,$$
$$-(+a)=-a, \qquad -(-a)=+a$$

EXAMPLES II. d.

Simplify, by removing brackets and collecting like terms,

1. $27+(9-4)$ 2. $13-(9-4-3)$ 3 $9-(5-7+1)$
4. $-5x+(9x-3x)$ 5 $9x-(5x-2x)$
6. $(8a-5a)-10a$ 7. $12p-(5p+2p)$
8 $13y-(5y-2y+7y)$ 9. $xy+(xy-3xy+9xy)$
10 $7m^2+(2m^2+4m^2)-13m^2$ 11 $(8y^2-3y^2)+(7y^2-4y^2)$
12 $3y+9y+(2y-3y)+8y$ 13 $(4p^2+7p^2)-(6p^2-2p^2)$.
14 $(x^2+3x^2)-(5x^2+2x^2)+(x^2-9x^2)$
15 $4c^3-(7c^3+8c^3-c^3)-(2c^3-3c^3)$ 16 $-(6d^3-4d^3)-(2d^3-3d^3)+d^3$
17 $14x^4-(5x^4+3x^4)+(7x^4-9x^4)-(8x^4-11x^4)$

Addition of Unlike Terms.

39 When two or more *like* terms are to be added together we have seen that they may be collected and the result expressed as a *single* like term If, however, the terms are *unlike* they cannot be collected and expressed as one term

In Arithmetic the sum of 2 yds, 1 ft, and 11 in can be expressed only as 2 yds 1 ft 11 in, unless we reduce the three quantities to the same denomination Similarly, in Algebra, unless we know their numerical values, in finding the sum of two unlike quantities a and b, all that can be done is to connect them by the sign of addition and leave the result in the form $a+b$

Again, the sum of $3a$, $4b$, c, and $2d$ is $3a+4b+c+2d$, and cannot be written in any simpler form

We have now to consider the meaning of an expression like $a+(-b)$.

By the rule for removing brackets

$$a+(-b)=a-b$$

Hence to *add* $-b$ is the same thing as to *subtract* b

40 The wider use of the word *sum* in Algebra should be carefully noted In Arithmetic we use it to denote the addition of *positive* quantities only, whereas in Algebra we have seen that an expression such as $5-9$ means that the *negative* quantity -9 can be *added* to the positive quantity 5, and that the result of the addition is the negative quantity -4

Thus in Algebra $a-b$ means not only the *subtraction* of b from a but the *sum* of the two quantities a and $-b$, whatever may be the relative values of a and b

41 In finding an algebraical sum we may assume that the result is the same in whatever order the terms are taken From the nature of addition this must be so, just as in making up an account the several gains and losses may be taken in any order

42 We now proceed to find the sum of compound expressions

EXAMPLE 1 *Find the sum of* $3a-5b+2c$, $2a+3b-d$, $-4a+2b$

$$\begin{aligned}\text{The sum }&=(3a-5b+2c)+(2a+3b-d)+(-4a+2b)\\&=3a-5b+2c+2a+3b-d-4a+2b\\&=3a+2a-4a-5b+3b+2b+2c-d\\&=a+2c-d,\end{aligned}$$

by collecting like terms

When there are several expressions to be added together it is more convenient to use the following rule

Rule *Arrange the expressions in lines so that the like terms may be in the same vertical columns then add each column beginning with that on the left*

$$\begin{array}{llll}3a&-5b&+2c&\\2a&+3b&&-d\\-4a&+2b&&\\\hline a&&+2c&-d\end{array}$$

The algebraical sum of the terms in the first column is a, that of the terms in the second column is zero The single terms in the third and fourth columns are brought down without change

EXAMPLE 2 *Add together*

$-5ab+6bc-7ac$, $8ab+3ac-2ad$, $-2ab+4ac+5ad$, $bc-3ab+4ad$

$$\begin{array}{llll}-5ab&+6bc&-7ac&\\8ab&&+3ac&-2ad\\-2ab&&+4ac&+5ad\\-3ab&+bc&&+4ad\\\hline -2ab&+7bc&&+7ad\end{array}$$

Here we first re-arrange the expressions so that like terms are in the same vertical columns, and then add up each column separately

EXAMPLES II. e.

Find the sum of the following expressions:

1. $2a+3b-c$, $-a+b+2c$, $3a-b+c$.
2. $2x-y+3z$, $x+4y-z$, $4x+2y-2z$.
3. $p+3q-4r$, $3p-q+r$, $5p+q-2r$.
4. $5a-4b-c$, $-2a+b+2c$, $-3a+3b+c$.
5. $6x+y-2z$, $-5x-y+z$, $-x+3y-z$.
6. $8l-2m+5n$, $-6l+7m+4n$, $-l-m-8n$.
7. $7x-5y-7z$, $x-3y-3z$, $5x-3y+2z$.
8. $3c-4d+5e$, $-2c+8d-5e$, $c-4d+3e$.
9. $4a-7b+5x$, $2a+5b-3x$, $a-2b+4x$.
10. $6l-8m+10n$, $-3l+5m-8n$, $2l-m-n$.
11. $-16a-10b+5c$, $3a+5b-c$, $10a+5b+c$.
12. $-15a+5b+8c$, $a+15b-4c$, $14a-19b+4c$.

Add together the following expressions:

13. $6ab+3bc-ca$, $2ab-4bc$, $-ab+ca$.
14. $-xy+yz+zx$, $-3xy-2yz+3zx$, $xy+yz-zx$.
15. $a-2b+3c-d$, $b-3c$, $2c+3d$, $2a-b+d$.
16. $4x-y+z+l$, $y-z+3l$, $z-5l$, $x+2z$.
17. $pq-3rp+4qr$, $3qr-pq$, $2pq-3qr+4rp$.
18. $17ab-13kl-5xy$, $7xy$, $12kl-5ab$, $3xy-4kl$.
19. $5-a-b$, $7+2a$, $3b-2c$, $-4+a-2b$.
20. $p-q+7r+3$, $2q-3r+5$, $3r+2p$, $p-8-7r$.
21. $6-x-2y$, $4+3x$, $2x-5y$, $-3+x-5y$.
22. $7xy-5yz$, $-2yz+3zx$, $4xy-3yz+6zx$.
23. $3a-2b+6c+2$, $4a-3b+5c+8$, $3b-5c-5$.

Dimension and Degree.

Ascending and Descending Powers.

43. Each of the letters composing a term is called a **dimension** of the term, and the number of letters involved is called the **degree** of the term. Thus the product abc is said to be *of three dimensions*, or *of the third degree*; and ax^4 is said to be *of five dimensions*, or *of the fifth degree*.

A numerical coefficient is not counted. Thus $8a^2b^5$ and a^2b^5 are each of *seven* dimensions.

44 **The degree of an expression** is the degree of the term of highest dimensions contained in it, thus a^4-8a^3+3a-5 is *an expression of the fourth degree*, and $a^2x-7b^2x^3$ is *an expression of the fifth degree* But it is sometimes useful to speak of the dimensions of an expression with regard to some one of the letters it involves For instance the expression ax^3-bx^2+cx-d is said to be of *three dimensions in* x

45 A compound expression is said to be **homogeneous** when all its terms are of the same dimensions Thus $8a^6-a^4b^2+9ab^5$ is a *homogeneous expression of six dimensions*

46 *Different* powers of the *same* letter are *unlike* terms; thus the result of adding together $2x^3$ and $3x^2$ cannot be expressed by a single term, but must be left in the form $2x^3+3x^2$

Similarly the algebraical sum of $5a^2b^2$, $-3ab^3$, and $-b^4$ is $5a^2b^2-3ab^3-b^4$ This expression is in its simplest form and cannot be written in any shorter way

47 In adding together several algebraical expressions containing terms with different powers of the same letter, it is convenient to arrange all expressions in *descending* or *ascending* powers of that letter This will be made clear by the following examples

EXAMPLE 1 *Add together the following expressions*

$3x^3-7+6x-5x^2,\ 2x^2-8-9x,\ 4x-2x^3+3x^2,\ 3x^3-9x-x^2,\ x-x^2-x^3+4$

$$\begin{array}{r}
3x^3-5x^2+6x-7 \\
2x^2-9x-8 \\
-2x^3-3x^2+4x \\
3x^3-x^2-9x \\
-x^3-x^2-x+4 \\
\hline
3x^3-2x^2-7x+3
\end{array}$$

The highest power of x in the first expression is x^3 We write the term $3x^3$ first, then that involving x^2, then that involving x, and then that which does not contain x The terms are said to be arranged in *descending order* The other expressions are arranged in the same way, so that in each column we have *like powers of the same letter*.

EXAMPLE 2 *Find the sum of*

$3ab^2-2b^3+a^3,\ 5a^2b-ab^2-3a^3,\ 8a^3+5b^3,\ 9a^2b-2a^3+ab^2$

$$\begin{array}{rrrr}
-2b^3 & -3ab^2 & & +a^3 \\
 & -ab^2 & +5a^2b & -3a^3 \\
5b^3 & & & +8a^3 \\
 & ab^2 & +9a^2b & -2a^3 \\
\hline
3b^3 & +3ab^2 & +14a^2b & +4a^3
\end{array}$$

Here each expression contains powers of two letters, and is arranged according to *descending* powers of b, and *ascending* powers of a

EXAMPLES II. f.

Add together the following expressions

1. a^2-ab+b^2, $2a^2+ab-b^2$, $a^2-ab+2b^2$.
2. $2x^2-x+3$, x^2+2x-5, $3x^2-x-3$
3. $-c^2+3c-1$, $-2c^2-5c+6$, $3c^2+c-8$
4. $3p^2-pq+q^2$, $-p^2+pq-2q^2$, $2p^2-3pq+q^2$.
5. $5a^2+2a-3$, $3a^2-4a+5$, $2a^2+3a+1$
6. $-2m^2+4m+2$, m^2-6m+3, $3m^2+3m-5$
7. $4x^2-3$, $2x+8$, $-3x^2-7$, x^2+5x
8. x^3-2x^2+x-7, $3x^2+5x+2$, x^3+2x^2-6x
9. $2-4a+a^2-3a^3$, $1-2a^2+a^3$, $3a+a^2-5a^3$
10. $5-b+b^3$, $2+3b^2+5b^3$, $-6+4b+7b^2$

Find the sum of the following expressions

11. y^3+3y^2, $6y^2-7y$; $4y-8$, y^3+y^2+1
12. $x^4+3x^3+x^2-4x$, $-2x^3-3x^2+x$, $5x^4+2x^2+3x$
13. $6a^4+3a^3$, $-2a^3+7a^2$, $-5a^2-2a$, a^4-a^3
14. $3b^3-b$, $4b^2+3b$, b^4-8b^3, $7b^4-2b^2$
15. $-2a^3+2a^2b-2ab^2-b^3$, $-3a^2b+b^3$, $-4a^2b+6ab^2$
16. $x^4-2x^2y^2+y^4$, $-3x^3y+2x^2y^2$, $6x^3y-y^4$, xy^3-x^4
17. $-m^3+1$, m^4-m^2-5, m^5+3m^2-4, $-2m+3+m^3$
18. $7-2a+3a^3$, $-2+a^3+a^4$, $-5+2a-a^3-a^4+a^5$
19. $6c^3+8$, $3c^2+11$, $-4c^3-2c^2$, c^4-c^2-19
20. p^4+12p^3, $-8p^3+4p^2$, $-5p^2-3p$, $4p-3$
21. If $a=3p^2-2p+7$, $b=8-2p^2$, $c=4p+p^2-15$, find the value of $a+b+c$
22. Distinguish between *like* and *unlike* terms Find the algebraical sum of the like terms in the expression
$$7x^3-3x^2y+2y^3+6xy^2-5x^2y+xy^4+10x^2y$$
23. What is the *degree* of a term in an algebraical expression? In the expression $5a^4b-8a^3b^4+b^6-9a^2b^7$, what is the degree of each negative term?
24. In the expression
$$a^4-3a^2b^3+7ab^4-a^2b^2c-8a^2b^3,$$
which terms are *like*, and which are *homogeneous*?

CHAPTER III.

SUBTRACTION .

Subtraction of Simple Expressions.

48 THE simplest cases of Subtraction have already come unde the head of addition of *like* terms, of which some are negative [Art 32]

Thus $\quad 5a-3a=2a, \quad 3a-7a=-4a, \quad -3a-6a=-9a$

In these three examples we have *added negative* quantities, or what is the same in effect, *subtracted positive* quantities

Also by the rule for removing brackets [Art 37],

$$3a-(-8a)=\ 3a+8a=11a,$$

and

$$-3a-(-8a)=-3a+8a=\ 5a$$

Again,

$$a-(+b)=a-b,$$

and

$$a-(-b)=a+b$$

Thus we see that to *subtract* a *positive* quantity we must *add* a *negative* quantity of the same absolute value, and to *subtract* a *negative* quantity we must *add* a *positive* quantity of the same absolute value

Rule *Change the sign of the term to be subtracted and add to the other term*

49 In expressing the difference between two numbers in Arithmetic, we place the greater number first and the smaller number follows the $-$ sign If however we are using symbols whose values are not known, another symbol $\sim$ is used

Thus $a \sim b$ means the arithmetical difference of a and b without specifying whether a or b is the greater

The expression $a-b$ in Algebra always means "b subtracted from a" *whatever values* a *and* b *may have*

50 If $a-b$ is *positive*, a is said to be algebraically *greater* than b

If $a-b$ is *negative*, a is said to be algebraically *less* than b

The sign $>$ is used for the words "is *greater* than"

" $\quad <$ "is *less* than"

Thus $4>-5$, because $4-(-5)$, or $4+5$ is positive,

and $-8<-3$, " $-8-(-3)$, or $-8+3$ is negative

Generally, $-a<-b$ if $-a-(-b)$, or $-a+b$ is negative, that is if the absolute value of a is greater than that of b

EXAMPLES III. a

From

1 2 take 4	2 2 take -4	3 -2 take 4.
4 -2 take -4	5 $7a$ take $3a$	6 $7a$ take $-3a$
7. $-7a$ take $3a$	8 $-7a$ take $-3a$	9. $13y$ take $9y$
10 $13y$ take $-9y$	11. $-13y$ take $-9y$	12 $-13y$ take $+9y$
13 $2x$ take $-2x$	14 $-2x$ take $-4x$	15 $-3z$ take $-26z$
16. $-9y$ take $45y$	17. $-ab$ take ab	18 $-ab$ take $-ab$
19 $12cd$ take $7cd$	20 $12cd$ take $-7cd$	21 $-3x^2$ take $6x^2$
22. $-4y^2$ take $-9y^2$	23 $5x^2y$ take $8x^2y$	24 yz^4 take $-2yz^4$

Subtract

25 $-a^2b$ from $3a^2b$	26 $5x^2z$ from x^2z	27 5 from $2c^2$
28. $2c^2$ from 5	29 -5 from $2c^2$	30 b^2 from a^2
31 $-b^2$ from a^2	32. $-x^2z^2$ from z^3	33 $-7xy$ from $9xy$

Subtraction of Compound Expressions

51 In dealing with expressions which contain unlike terms we may proceed as in the following examples

EXAMPLE 1 *Subtract* $3a-2b-c$ *from* $4a-3b+5c$

The difference

$$=4a-3b+5c-(3a-2b-c)$$
$$=4a-3b+5c-3a+2b+c$$
$$=4a-3a-3b+2b+5c+c$$
$$=a-b+6c$$

The expression to be subtracted is first enclosed in brackets with a minus sign prefixed, then on removal of the brackets the like terms are combined by the rules already explained in Art 32

It is, however, more convenient to arrange the work as follows, the signs of all the terms in the lower line being changed

$$\begin{array}{rr} & 4a-3b+5c \\ & -3a+2b-c \\ \hline \textit{by addition,} & a-b+6c \end{array}$$

The like terms are written in the same vertical column, and each column is treated separately

Rule *Change the sign of every term in the expression to be subtracted, and add to the other expression*

NOTE It is not necessary that in the expression to be subtracted the signs should be *actually* changed, the operation of changing signs ought to be performed mentally

EXAMPLE 2 *From* $5x^2+xy$ *take* $2x^2+8xy-7y^2$

$$\begin{array}{r} 5x^2+xy \\ 2x^2+8xy-7y^2 \\ \hline 3x^2-7xy+7y^2 \end{array}$$

In the first column we combine mentally $5x^2$ and $-2x^2$, the algebraic sum of which is $3x^2$ Similarly in the second column we combine xy and $-8xy$ In the last column the sign of the term $-7y^2$ has to be changed before it is put down in the result

Terms containing *different* powers of the *same* letter being *unlike* must stand in different columns

EXAMPLE 3 *Subtract* $3x^2-2x$ *from* $1-x^3$

$$\begin{array}{r} -x^3 \qquad\quad +1 \\ \underline{3x^2-2x \quad} \\ -x^3-3x^2+2x+1 \end{array}$$

In the first and last columns, as there is nothing to be subtracted, the terms are put down without change of sign In the second and third columns each sign has to be changed

The rearrangement of terms in the first line is not *necessary*, but it is convenient, because it gives the result of subtraction in descending powers of x

EXAMPLES III. b.

From

1. $a+b$ take $2b$
2. $a-b$ take $-2a$
3. $a+b$ take $a-b$
4. $2x-3y$ take $x+4y$
5. $7c+5d$ take $-3c-2d$
6. $-m+5n$ take $-4m-3n$
7. $b+3c-d$ take $2b+2c+d$
8. $3x+y-z$ take $5x-4y+2z$
9. $4p-9q+2r$ take $3p-2q+5r$
10. $-x+y-5c$ take $-x-y-5c$
11. $a+2b-3c$ take $-2a-3b$
12. $7a-5b$ take $-a+9b-4c$
13. $x-2y-2z$ take $-x-2y$
14. $-2m+4n$ take $2n-2p$
15. $2ab-2cd+2ac$ take $-ab+2cd-3ac$
16. $5xy-3zx$ take $xy-2yz-4zx$
17. $-7np$ take $-mn+7np-5pm$.
18. a^3+1 take a^2+a-1
19. a^2+x^2 take $a^2-2ax+x^2+3$
20. x^3+x^2+x+1 take x^4+x^2+1
21. x^2y+3xy^2-xyz take $2x^2y-xy^2+3xyz$
22. a^3+3a^2+a take $-a^4-a^3+3a$
23. m^4+m^2+1 take m^3-2m^2+m
24. $1-a^3$ take $2a^3-3a^2b+2ab^2-b^3$

Subtract

25. x^3-3xy^2 from $x^3+3x^2y+3xy^2+y^3$
26. x^3-y^3 from $x^3-3x^2y+3xy^2+y^3$
27. $4-x+x^2+x^3$ from $6+x-x^2$
28. $m^3-3n^3-4mn^2-m^2n$ from $4mn^2+n^3-9m^2n$
29. $d^4-1+d-d^3$ from $1-d+d^5-d^4-d^3$
30. $5x^3y+11xy^3-x^4-6y^4-3x^2y^2$ from $8xy^3+2x^3y-5y^4$

What must be added to

31. $c-d$ to give c?
32. $x+y$ to give y?
33. $a-b-c$ to give $a+c$?
34. $2x^2+y^2-z^2$ to give $3x^2-2y^2+4z^2$?
35. x^4-1 to give $x^3y-x^2y^2+xy^3$?
36. a^2b-ab^2 to give a^3-b^3?

37 Take c^2 from 1, and the result from c^3+c^2-1

38 From $3a+4c$ take $2a-3b+7c$, and from the result subtract $-2a+3b-c$

39 Take the sum of $m^4-2m^3n+m^2n^2$ and $m^3n+2m^2n^2-mn^3$ from $m^4-2m^2n^2-n^4$, and give the numerical value if $m=0$, $n=1$

40 Take the sum of $3x^2y+x^3+y^3$, $x^2y-xy^2-2y^3$, and $3y^3+2xy^2-4x^3$ from $-2x^3+y^3+x^2y+2xy^2$

52 The following equivalents have been established in Art 38

$$a+b-c=a+(b-c),\quad a-b-c=a-(b+c),\quad a-b+c=a-(b-c)$$

Hence the rules for removing brackets may be stated conversely

Rule I *Any part of an expression may be enclosed within brackets and the sign + prefixed the sign of every term within the brackets remaining unaltered*

Rule II *Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed*

EXAMPLES (i) $a+b-c+d-e=a+(b-c+d-e)$
$=a+b-c+(d-e)$

(ii) $a-b+c+d-e=a-(+b-c-d+e)$
$=a-b+c-(-d+e)$

53 The following results in connection with addition and subtraction have now been established

I *Additions and subtractions may be made in any order*
[See Art 41]

Thus $a+b-c+d-e-f=a-c+b-e+d-f$
$=a+b+d-c-e-f$

This is known as the **Commutative Law for Addition and Subtraction**

II *The terms of an expression may be grouped in any manner*

Thus $a+b-c+d-e-f=(a+b)-c+(d-e)-f$
$=a+(b-c)+(d-e)-f=a+b-(c-d)-(e+f)$

This is known as the **Associative Law for Addition and Subtraction.**

MISCELLANEOUS EXAMPLES I.

Exercises for Revision

A

1. Simplify (i) $4x-2x^2-(2x-3x^2)$,
(ii) $3a-4b-(3b+a)-(5a-8b)$

2. To the sum of $2a-3b-2c$ and $2b-a+7c$ add the sum of $a-4c+7b$ and $c-6b$

3. When $x=3$, $y=2$, $z=0$, find the value of

$$\text{(i) } x^2+\frac{3}{2}y^3-xyz^3, \qquad \text{(ii) } \frac{1}{4}x^3y^4+\frac{5z^2}{6}$$

4. Define *index*, *coefficient* In the expressions $4x^2+3x$, $2x^3+x^2$, x^2+7x, find (i) the sum of the indices, (ii) the sum of the coefficients

5. From $5x^3+3x-1$ take the sum of $2x-5+7x^2$ and $3x^2+4-2x^3+x$

6. If $2x+3x=25$, find the value of $2x^3-3x^2$

B

7. Distinguish between *like* and *unlike* terms Pick out the like terms in the expression

$$a^3-3ab+b^2-2a^3-a^2+3b^2+5ab+7a^2$$

8. Write down in as many ways as possible the result of adding together x, y, and z

9. Subtract $5x^2+3x-1$ from $2x^3$, and add the result to $3x^2+3x-1$

10. Express x lbs in ounces, y cwt in stones, and z lbs in cwt If $x=3$, $y=5$, and $z=784$, what are the answers?

11. Write down in algebraical symbols the result of diminishing $2a$ by the sum of $3b$ and $5c$

12. When $x=1$, $y=2$, $z=3$, find the value of the sum of $5x^2$, $-2x^3z$, $3y^4$ Also find the value of $2x^y-3y^x$

C

13. Add the sum of $2y-3y^2$ and $1-5y^3$ to the remainder left when $1-2y^2+y$ is subtracted from $5y^3$

14. Explain clearly why $x-(y-z)=x-y+z$

15. If $x=4$, $y=3$, $z=2$, $a=0$, find the value of $3x^2-2yz-ax+5ax^2y$

16. Simplify $2a-b-(3a-2b)+(2a-3b)-(a-2b)$

17. Find the algebraical sum of the like terms in the expression

$$5a^3-4a^2b+b^3+6a^3b+7ab^2-3a^2b+4ab^3+8a^2b$$

18. A boy works $x+y$ sums, of which only $y-2z$ are right, how many are wrong?

D

19 In the expression $3a^3-7a^2b+b^4$, point out the highest power, the lowest power, the positive terms, and the coefficient of a^2

20 Take x^3-y^3 from $3xy-4y^2$, and add the remainder to the sum of $4xy-x^2-3y^2$ and $2x^2+6y^2$

21 A man who can row 5 miles an hour in still water rows for one hour *against* a stream flowing at the rate of 2 miles an hour He then turns, and, using the same force, rows *with* the stream for one hour Illustrate by a diagram, and express his final distance from his starting point with the proper sign

22 What is the *degree* of a term in an algebraical expression? In the expression $4x^6-3x^5a^3+a^8$, what is the degree of the negative term?

23 Find the sum of $5a-7b+c$ and $3b-9a$, and subtract the result from $c-4b$ What is the value of the answer when $a=5$, $b=3$, $c=15$?

24 If $x=3$, $y=4$, $p=8$, $q=10$, find the value of

$$xyp\div\frac{2y}{p-y}+2q$$

25. Express the sum of a pounds, b shillings, and c pence in pence. What is the answer if $a=3$, $b=11$, and $c=8$?

26 What are the meanings of y^3, $3y$, $\frac{y}{3}$? What is the numerical value of each if $y=3$?

E

27 If x represents the date 10 A D what will $-3x$ stand for?

28 Add together $3x^2-7x+5$ and $2x^3+5x-3$, and diminish the result by $3x^2+2$

29 In the expression

$$4a^2b^3-b^4+3a^3b^2+5b^5-ab^3x+2x^3ab+abx^4-a^2b^3,$$

point out which terms are *like*, and which are *homogeneous* What is the degree of the expression?

30 A man starts on a journey of x miles and walks at the rate of a miles per hour for b hours How far has he still to go? If $x=40$, $a=4$, $b=7$, what is the answer?

31 A man walks $2a-b$ miles due North from a fixed point O, and then walks a distance $3a+2b$ miles due South, what is his final position with regard to O?

32 In a train there are $3x+4y-z$ passengers, of these $x-y+z$ are first class, $2x+3y-z$ are third class Give the algebraic expression for the number of second class passengers What are the numbers in each class if $x=45$, $y=36$, $z=5$?

33. From the square of m take the square of n, and subtract $2mn+n^2$ from the result

34 A man has two sons, one of whom is twice as old as the other If the man's age is six times that of the younger, and if the sum of the three ages is 63 years, what is the age of each?

CHAPTER IV

MULTIPLICATION

Simple Expressions or Monomials.

54 IN Arithmetic we know that the factors of a product may be taken in any order

Thus $3\times4=4\times3$, $7\times5=5\times7$, and so on

Again,
$$\frac{3}{7}\times\frac{4}{5}=\frac{3\times4}{7\times5}=\frac{4\times3}{5\times7}$$
$$=\frac{4}{5}\times\frac{3}{7}$$

Hence also in Algebra *as long as* a *and* b *denote any* positive *quantities, whole or fractional,*

$$a\times b=b\times a, \text{ or } ab=ba$$

55 Multiplication by a negative quantity Rule of Signs Multiplication in its primary sense is repeated addition

$$3\times4=3 \text{ taken 4 times}$$
$$=3+3+3+3$$
$$=12 \qquad (i)$$

Similarly
$$(-3)\times4=-3 \text{ taken 4 times}$$
$$=-3-3-3-3$$
$$=-12 \qquad (ii)$$

At present a result such as $4\times(-3)$ has no arithmetical meaning In accordance with the principle laid down in Art 38, we shall assume that the general law expressed by $a\times b=b\times a$ is universally true, and we shall accept the conclusions so derived

Hence we have
$$4\times(-3)=(-3)\times4$$
$$=-12, \text{ by (ii)}$$
$$=-(4\times3)$$

Hence *when the multiplier is negative we first multiply by its absolute value, and then change the sign of the resulting product*

Thus
$$3\times(-4)=-(3\times4)=-12 \qquad (iii)$$

To get the value of $(-3)\times(-4)$ we first multiply (-3) by 4 and then change the sign of the result The first operation gives -12, and the second $+12$

Hence
$$(-3)\times(-4)=+12 \qquad (iv)$$

From the results numbered (i) to (iv) we see that the product is *positive* when the two factors have *like* signs, and negative when the two factors have *unlike* signs Hence, using general symbols, with the full signs attached, the above results may be grouped as follows

$$(+a)\times(+b)=+ab,\quad (-a)\times(+b)=-ab,$$

$$(-a)\times(-b)=+ab,\quad (+a)\times(-b)=-ab$$

In other words, *when two algebraical quantities are multiplied together,*

like signs give $+$, *unlike signs give* $-$

This is known as the **Rule of Signs**

We also see that the above four algebraical products have the same *absolute* value, the only difference being in the *signs* Hence, *multiplication by a negative quantity indicates that we are to proceed just as if the multiplier were positive, and then change the sign of the product*

Thus we have learnt (i) that the absolute value of a product is not affected by the signs of its factors, and (ii) that the sign of the product does not depend upon the order of its factors

56 Since the result of Art 54 is universally true,

that is, $$a\times b=b\times a,$$

for *all* values of a and b,

hence also, *for all values* of a, b, and c,

$$\begin{aligned} abc&=(ab)\times c & bac&=b\times(ac) & bac&=(ba)\times c\\ &=(ba)\times c & &=b\times ca & &=c\times(ba)\\ &=bac, & &=bca, & &=cba, \end{aligned}$$

and so on, whence it follows that *the factors of a product may be taken in any order*

This is the **Commutative Law for Multiplication**

EXAMPLE $2a\times 3b\times c=2\times 3\times a\times b\times c=6abc$

57 Again, *the factors of a product may be grouped in any way we please*

Thus $$abcd=a\times b\times c\times d$$

$$=(ab)\times(cd)=a\times(bc)\times d=a\times(bcd)$$

This is the **Associative Law for Multiplication.**

EXAMPLE 1 *Multiply* $4a$ *by* $-3b$

By the rule of signs the product is negative;

also $$4a\times 3b=4\times 3\times a\times b=12ab$$

$$4a\times(-3b)=-12ab$$

EXAMPLE 2 $-7ab\times(-8cd)=56abcd$

EXAMPLES IV. a. (*Oral*)

Multiply together

1. $-5, 4$	2. $3, -7$	3 $-6, -9$	4 $8a, 7$
5 $3, 2b$	6 $9x, -4$	7 $-12y, 5$	8 $-9m, -12$
9. $5c, 7d$	10 $-6m, -5n$	11. $-8x, 13y$	
12 $15a, -7b$	13 $ab, 2x$	14. $3mn, -p$	
15 $-ab, -7c$	16 $-9xy, z$	17. $4ab, 5cd$	
18. $-3xyz, d$	19 $-4lm, -3ln$	20 $12ay, 13bx$	

58 To further illustrate the rule of signs, we add a few examples in substitution where some of the symbols denote negative quantities

EXAMPLE 1 *If* $a=-4$, *find the value of* a^3, $-a^3$, *and* $(-a)^3$

Here $a^3=(-4)^3=(-4)\times(-4)\times(-4)$

$=(+16)\times(-4)=-64,$

and $-a^3=-(-4)^3=-(-64)=64$

Also $(-a)^3=(+4)^3=4\times4\times4=64$

By repeated applications of the rule of signs it may easily be shewn that any *odd* power of a negative quantity is *negative*, and any *even* power of a negative quantity is *positive*

EXAMPLE 2 *If* $a=-1$, $b=3$, $c=-2$, *find the value of* $-3a^4bc^3$

Here $-3a^4bc^3=-3\times(-1)^4\times3\times(-2)^3$

$=-3\times(+1)\times3\times(-8)$

$=72$

We write down at once $(-1)^4=+1$, and $(-2)^3=-8$

EXAMPLES IV. b

If $a=-1$, $b=-2$, $c=-3$, $x=0$, $y=1$, find the value of

1. $5b$	2 $-4c$	3 ay	4. a^2
5 $(-c)^2$	6 $-5b^2$	7. $(-b)^2$	8 $-b^2$
9 ax	10 a^3	11 $(-c)^3$	12. $-c^3$
13 $4a^4y$	14 $-5a^2b$	15 $2a^3c^2$	16. $-a^2y^2$
17 $-b^4$	18 $(-b)^5$	19 $-3a^3bc^2$	20 $6a^3xy^4$

If $x=-2$, $y=1$, $z=0$, $m=-3$, $n=-1$, find the value of

21. $2x+5y-3n$	22 $-5y-6z+9n$	23 $-4x-2m+n$
24 $xy-3mn-ny$	25. n^3-y^3	26 $n^2+x^2+y^2$
27. $3x^2n-3z+n$	28. $-ny+mn^2-n^3$	29 $xyz-4x^2-x^4$

30. When x has the values 0, -1, -3, find and tabulate the values of the expression x^2+2x+3 [See Art 23, Ex 2]

31 Tabulate the values of the expression $x^2-7x+10$ when x has the values -1, -3, -5, -7

32. Tabulate the values of x^3-3x when $x=-3, -2, 1, 2, 3$

59 The Law of Indices.

Since, by definition, $a^3 = aaa$, and $a^5 = aaaaa$,

$$a^3 \times a^5 = aaa \times aaaaa = aaa\ aaaaa = a^8 = a^{3+5}$$

More generally, if m and n are any positive whole numbers,

$$a^m = a\ a\ a \ldots \text{ to } m \text{ factors},$$
$$a^n = a\ a\ a \ldots \text{ to } n \text{ factors},$$
$$a^m \times a^n = (a\ a\ a \ldots \text{ to } m \text{ factors})(a\ a\ a \ldots \text{ to } n \text{ factors})$$
$$= a\ a\ a \ldots \text{ to } m+n \text{ factors}$$
$$= a^{m+n}, \text{ by definition},$$

that is, the index of a in the product is found by adding the indices of a in the factors of the product This is the Law of Indices

EXAMPLE 1 *Find the product of* $5a^2$ *and* $7a^3$

$$5a^2 \times 7a^3 = 5 \times 7 \times a^2 \times a^3 = 35a^{2+3} = 35a^5$$

The Index Law may be extended to cases in which more than two expressions are to be multiplied together

EXAMPLE 2 *Find the product of* x^2, x^3, *and* x^8

The product $= x^2 \times x^3 \times x^8 = x^{2+3} \times x^8 = x^5 \times x^8 = x^{13}$

60 When the expressions to be multiplied together contain powers of *different* letters, a similar method is used

EXAMPLE $5a^3b^2 \times 8a^2bx^3 = 5 \times 8a^{3+2}\ b^{2+1}\ x^3$

$$= 40a^5b^3x^3$$

NOTE The beginner must be careful to observe that in this process of multiplication *the indices of one letter cannot combine in any way with those of another* Thus the expression $40a^5b^3x^3$ cannot be written in any shorter form

61 Rule *To multiply two simple expressions together, multiply the coefficients together and prefix their product, with its proper sign, to the product of the different letters, giving to each letter an index equal to the* **sum** *of the indices that letter has in the separate factors*

The product of three or more expressions is called **the continued product**

EXAMPLE *Find the continued product of* $5x^2y$, $-8y^2z^5$, *and* $3z^4$

The product $= 5x^2y \times (-8y^2z^5) \times 3z^4 = -120x^2y^3z^9$

NOTE 1 The product of any number of negative factors is *positive* or *negative* according as the number of factors is *even* or *odd*

NOTE 2 $(a^3)^4$ must be clearly distinguished from $a^3 \times a^4$.

We have seen that $a^3 \times a^4 = a^{3+4} = a^7$

Whereas

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3$$
$$= a^{3+3+3+3}$$
$$= a^{12} = a^{3\times 4}.$$

EXAMPLES IV. c.

Multiply together

1. b^3, b^2	2 x^5, x	3 $-5z, 6z^2$	4 $9y^3, 5y^4$
5 $8c, -7c^5$	6. $-3m^2, 2m^4$	7. $-4d^2, -d^3$	
8. $p, -p^9$	9 $3ax, 4a^2$	10. $4c^2, -2d^3$	
11 $3ab, 3ab$	12 $4ac, -7ad$	13 $-c^2d^3, -c^2d^3$	
14 $5x^4y^3, -4y^2z^2$	15. m^3n^5, m^5n^3	16. $a^3b^4c, -ab^3c^2$	

Find the continued product of

17 $3a, 4b, -5$	18 $-x, -y, -z$	19 $-4x, 5y, -3z$
20 $7x, -2yz, -1$	21. $ab, cd, -x$	22 $-3y, 4a, -7x$
23. $a^2, ab, 3ab^2$	24. $4a^2, -3b^2, 6c^2$	25. xy^2, yz^2, x^2z
26. $-xy^2, -2x^3y, 6xy^4$	27 a^2bc^3, b^4c, ac^3d^5	28. $-3a^2, 4ab^3, -5$

29 Write down the values of $(b^2)^4$, $(x^5)^3$, $(-y^3)^4$, $(-a^2b^2)^3$

30 Write down the third power of $-ab^4$, x^2y^5, $-p^2qr^4$

Multiplication of a Compound Expression by a Simple Expression.

62 By definition,

$$\begin{aligned}(a+b)\times 10 &= (a+b) \text{ taken 10 times}\\ &= a \text{ taken 10 times together with } b \text{ taken 10 times}\\ &= 10a+10b\end{aligned}$$

In like manner, when m is a positive whole number,

$$\begin{aligned}(a+b)\times m &= (a+b) \text{ taken } m \text{ times}\\ &= (a+a+a+ \quad \text{taken } m \text{ times}),\\ \text{together with} \quad &\quad (b+b+b+ \quad \text{taken } m \text{ times})\\ &= ma+mb = am+bm\end{aligned}$$ [Art 56]

$(a+b)\times m$ may also be written $m(a+b)$

Thus $m(a+b)=ma+mb$

Again, if a is greater than b, and m is a positive whole number,

$$\begin{aligned}(a-b)\times m &= (a-b) \text{ taken } m \text{ times}\\ &= (a+a+a+ \quad \text{taken } m \text{ times}),\\ \text{diminished by} \quad &\quad (b+b+b+ \quad \text{taken } m \text{ times})\\ &= ma-mb = am-bm\end{aligned}$$

NOTE When a, b, and m are not restricted in value, in accordance with Art 38, we *define* $(a-b)m$ as being equivalent to $am-bm$

Similarly $(a-b+c)m=am-bm+cm$

Thus it appears that *the product of a compound expression by a single factor is the algebraic sum of the partial products of each term of the compound expression by that factor*

This is known as the **Distributive Law for Multiplication.**

EXAMPLE 1 $3(4a+5b)=3\times 4a+3\times 5b=12a+15b$

EXAMPLE 2 *Find the value of* $(x-2y+3z)\times ab$

Here we have to take the three products

$$x\times ab, \quad (-2y)\times ab, \quad 3z\times ab,$$

and form the algebraical sum thus

$$(x-2y+3z)\times ab=abx-2aby+3abz$$

EXAMPLE 3 *Multiply* $6a^3-5a^2b-4ab^2$ *by* $(-3ab^2)$

The product is the algebraical sum of the three products obtained by multiplying each term of the compound expression by $-3ab^2$.

Thus $(6a^3-5a^2b-4ab^2)\times(-3ab^2)=-18a^4b^2+15a^3b^3+12a^2b^4$

The product may usually be written down at once

EXAMPLES IV. d

Multiply

1. $a+3b-5c$ by 4
2. $ax+a^2x^2-a^3x^3$ by a^3
3. a^2b-ab^2 by $4ab$
4. $-x^2y+xy^2$ by $-x^3y^2$
5. $3x^2-7y^3$ by $-2x^2y$
6. $c^4d^3-c^2d^3+1$ by $-cd^3$
7. $xy-yz-zx$ by xy
8. $-x^3y-x^2y^2+y^3$ by $-x^2y$
9. $-2c^3+3cd-5d^2$ by $3c^4d^3$
10. $5a^2-3b^2c-8$ by $-4ab^3c^2$
11. $-a^3b+a^2b^2-ab^3$ by a^3b^4
12. x^3-4+2x^2a by $3a^3bx$
13. $ab^2c-abc^2+a^2bc$ by $-a^2bc^3$
14. $1-a^2+2b^2-3c^2$ by $-abc$
15. $2y^2-3xyz+x^2z^2$ by $-3xyz$
16. $3x^2y^2-6bxy-2ax$ by $-2ax$.

Multiplication of Compound Expressions.

63 *To find the product of* a+b *and* c+d.

From Art 62, $(a+b)m=ma+mb$,

replacing m by $c+d$, we have

$$(a+b)(c+d)=(c+d)a+(c+d)b$$
$$=ac+ad+bc+bd$$

Similarly it may be shewn that

$$(a-b)(c+d)=ac+ad-bc-bd,$$
$$(a+b)(c-d)=ac-ad+bc-bd,$$
$$(a-b)(c-d)=ac-ad-bc+bd$$

NOTE These results are subject in the first instance to the condition that $a-b$ and $c-d$ are positive quantities, but the restrictions may be removed as in Art 38, the results may henceforth be regarded as true for all values of a, b, c, and d

64 When the expressions to be multiplied together contain more than two terms, a similar method may be used

For instance, $(a-b+c)m=am-bm+cm$,

replacing m by $x-y$, we have

$$\begin{aligned}(a-b+c)(x-y)&=a(x-y)-b(x-y)+c(x-y)\\&=ax-ay-bx+by+cx-cy\end{aligned}$$

65 The preceding results enable us to state the general rule for multiplying together any two compound expressions

Rule. *Multiply each term of the first expression by each term of the second When the terms multiplied together have like signs, prefix to the product the sign* +, *when unlike prefix* −, *the algebraical sum of the partial products so formed gives the complete product*

This process is called **Distributing or Expanding the Product.**

EXAMPLE 1 *Multiply* $x+8$ *by* $x+7$

The product $=(x+8)(x+7)$

$=x^2+8x+7x+56$

$=x^2+15x+56$

The operation is more conveniently arranged as follows

$$\begin{array}{rl} & x+8 \\ & \underline{x+7} \\ & x^2+8x \\ & \underline{\quad\;\; +7x+56} \\ \text{by addition,} & x^2+15x+56\end{array}$$

We begin on the left and work to the right, placing the second result one place to the right, so that like terms may stand in the same vertical column

Since $(x+8)(x+7)=x^2+15x+56$ for *all* values of x, the result should be true for any value we choose to give to x

For example, if $x=2$,

$(x+8)(x+7)=10\times 9=90$,

also $x^2+15x+56=4+30+56=90$

Beginners should learn to check their work by tests of this kind

EXAMPLE 2 *Multiply* $2x^2-3y$ *by* $4x^2-7y$

$$\begin{array}{rl} & 2x^2-3y \\ & \underline{4x^2-7y} \\ & 8x^4-12x^2y \\ & \underline{\qquad\; -14x^2y+21y^2} \\ \text{by addition,} & 8x^4-26x^2y+21y^2\end{array}$$

[Check by putting $x=1$, $y=2$

$2x^2-3y=2-6=-4$,

$4x^2-7y=4-14=-10$,

product $=40$

$8x^4-26x^2y+21y^2$

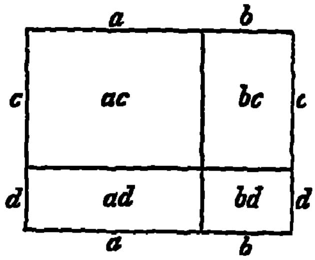
$=8-26\cdot 2+21\cdot 4$

$=40$]

NOTE In applying numerical tests, care must be taken not to choose values which reduce either of the two factors to zero

66 The results of Art 63 may be illustrated graphically

In the diagram below we have a rectangle whose adjacent sides are $a+b$ and $c+d$ units respectively



The whole area $=(a+b)(c+d)$ square units, and it is made up of four smaller rectangles whose areas are ac, ad, bc, bd square units respectively

Thus $(a+b)(c+d)=ac+ad+bc+bd$

EXAMPLES IV e

Find the products of the following pairs of binomials, and check the results in Examples 1–24

1. $x+3,\ x+4$	2 $x-3,\ x+9$	3. $c-5,\ c-7$
4. $x-6,\ x+6$	5 $d+7,\ d-12$	6. $m-3,\ m+4$
7 $x+5,\ x-5$	8 $f-11,\ f-7$	9 $a+11,\ a-7$
10 $s-11,\ s+7$	11. $a-9,\ a-1$	12 $z-1,\ z+1$
13 $a+1,\ a+1$	14. $c-1,\ c-1$	15 $y+9,\ y+9$
16. $d-4,\ 4-d$	17. $m-7,\ m-7$	18 $-x+4,\ -x-4$
19 $x-10,\ x+10$	20 $-a-5,\ -a-5$	21 $6+e,\ -6+e$
22. $3c-7,\ 2c+3$	23 $d-9,\ 5d+4$	24 $3m-2,\ 3m+2$
25 $-4+5x,\ 3-2x$	26 $2x-8,\ 5x-7$	27. $x+a,\ 2x-a$
28 $a-2b,\ 2a+3b$	29 $4c-5d,\ 5c+4d$	30 $3x-2y,\ x^2+2xy$
31 $x^2+3y,\ x^2-3y$	32 $5p^3+q,\ 5p^3-2q$	33 $z^2-2a^2,\ z^2-2a^2$
34. $4a-b^3,\ 3a+2b^3$	35. $3x^2-y^2,\ x^2+y^2$	36 $b^2-c,\ 3b^2c+c^2$

37. Illustrate graphically, as in Art 66,

(i) $m(a+b)=ma+mb$

(ii) $(x+5)(x+4)=x^2+9x+20$

(iii) $(a+b)^2=a^2+2ab+b^2$

(iv) $(a-b)(c+d)=ac-bc+ad-bd$

38. Find the value of $(x+5)(x+2)+(x-3)(x-4)$ in its simplest form What is the numerical value when $x=-6$?

39. Simplify $(x+2)(x+10)-(x-5)(x-4)$ Find the numerical value of this expression when $x=-3$

40 Find the value of $ma+mb$ when $a=67\ 3$, $b=32\ 7$, $m=2\ 43$ [*Remember that* ma+mb=m(a+b)]

41 Find the value of $ax-ay$ when $x=31\ 35$, $y=25\ 55$, $a=12$

42. Shew that the two expressions

$$m(x-y)+n(x+y), \quad x(m+n)+y(n-m)$$

are equal Find their numerical value when $m=n=25$, and $x=4$

67 **Products written down by inspection** Although the result of multiplying together two binomial factors, such as $x+8$ and $x-7$, can always be obtained by the methods already explained, it is of the utmost importance to learn to write down the product rapidly *by inspection*

This is done by observing in what way the coefficients of the terms in the product arise, and noticing that they result from the combination of the numerical coefficients in the two binomials which are multiplied together

Thus

$$\begin{aligned}(x+8)(x+7)&=x^2+8x+7x+56\\&=x^2+15x+56\\(x-8)(x-7)&=x^2-8x-7x+56\\&=x^2-15x+56\\(x+8)(x-7)&=x^2+8x-7x-56\\&=x^2+x-56\\(x-8)(x+7)&=x^2-8x+7x-56\\&=x^2-x-56\end{aligned}$$

In each of these results we notice that

(i) The product consists of *three* terms

(ii) The *first* term is the product of the first terms of the two binomial expressions

(iii) The *third* term is the product of the second terms of the two binomial expressions

(iv) The *middle* term has for its coefficient the sum of the numerical quantities (taken with their proper signs) in the second terms of the two binomial expressions

The intermediate step in the work may be omitted, and the products written down at once, as in the following examples

$$\begin{aligned}(x+2)(x+3)&=x^2+5x+6\\(x-3)(x+4)&=x^2+x-12\\(x+6)(x-9)&=x^2-3x-54\\(x^2-11)(x^2+10)&=x^4-x^2-110\\(x-4y)(x-10y)&=x^2-14xy+40y^2\\(x^2-6y^2)(x^2+4y^2)&=x^4-2x^2y^2-24y^4\end{aligned}$$

By an easy extension of these principles we may write down the product of *any* two binomials

Thus
$$(2x+3y)(x-y)=2x^2+3xy-2xy-3y^2$$
$$=2x^2+\ xy-3y^2$$
$$(3x-4y)(2x+y)=6x^2-5xy-4y^2$$
$$(x+4)(x-4)=x^2+4x-4x-16$$
$$=x^2-16$$
$$(2x+5y)(2x-5y)=4x^2-25y^2$$

EXAMPLES IV. f.

Write down the values of the following *by inspection*

1. $(a+1)(a-2)$	2 $(a-5)(a-6)$	3 $(c-6)(c+7)$
4 $(x-7)(x+6)$	5. $(d-3)(d+1)$	6 $(x-1)(x+1)$
7. $(y+5)(y-4)$	8 $(p+7)(p+6)$	9 $(y+11)(y-10)$
10 $(x-9)(x+9)$	11 $(c-9)(c-9)$	12 $(a+9)(a+9)$
13. $(a-x)(a-x)$	14. $(c+z)(c+z)$	15 $(x+y)(x-y)$
16 $(2x-1)(2x+3)$	17. $(4-3c)(3+4c)$	18 $(5x+2)(5x+2)$.
19. $(1-7y)(1+7y)$.	20. $(a+2x)(a-3x)$	21. $(m-3n)(m-3n)$.
22 $(2x+3y)(3x+y)$	23. $(5c-3d)(5c+3d)$	24 $(7a-3b)(7a+b)$
25. $(3x+2a)(2x-3a)$	26 $(a+b)(-a+b)$	27 $(a^2+3)(a^2-6)$
28 $(4a^2+b^2)(a^2-2b^2)$	29 $(ac-1)(ac+3)$	30 $(1-2a)(1-10a)$
31 $(1+7b)(1-6b)$	32 $(2+3x)(1-4x)$	33 $(x^2+y^2)(x^2-y^2)$
34 $(3a^2+b^2)(3a^2-b^2)$	35 $(4m-n)(m+3n)$	36 $(3+ab)(9-ab)$

37 Write down the cost in pounds of $3x+2y$ things at $4x-y$ pounds each

38 How many square feet are there in a rectangle which has adjacent sides measuring $2p+q$, $3p-4q$ feet respectively?

39. Write down the distance a train will go in $5a+2b$ hours at $5a-2b$ miles per hour

40. A horse eats $3p-q$ bushels of corn in a week, how many bushels does he eat in $p+q$ weeks?

*** *Further cases of Multiplication will be discussed in Chapters* VII. *and* XV

CHAPTER V

DIVISION

68 THE object of division in Algebra, as in Arithmetic, is to find out the **quotient**, that is the quantity by which the **divisor** must be multiplied so as to produce the **dividend**

For example, $\qquad 4\times 7=28, \qquad \therefore \frac{28}{7}=4$

$$a\times b=ab, \qquad \therefore \frac{ab}{b}=a$$

The operation of dividing a by b is denoted by $a\div b$, $\frac{a}{b}$, or a/b, in each of these modes of expression a is called the **dividend** and b the **divisor.**

Division is thus the inverse of multiplication and

$$(a\div b)\times b=a$$

This statement may also be verbally expressed as follows

$$\textit{quotient}\times\textit{divisor}=\textit{dividend}$$

69 Since Division is the inverse of Multiplication, it follows that the Laws of Commutation, Association, and Distribution, which have been established for Multiplication, hold for Division

Division of Simple Expressions or Monomials.

70 The method is shewn in the following examples

EXAMPLE 1 Since the product of 4 and x is $4x$, it follows that when $4x$ is divided by x the quotient is 4,

or otherwise, $\qquad 4x\div x=4$

EXAMPLE 2 *Divide* $27a^5$ by $9a^3$

$$\text{The quotient}=\frac{27a^5}{9a^3}=\frac{27aaaaa}{9aaa}$$
$$=3aa=3a^2$$

We remove from the divisor and dividend the factors common to both, just as in Arithmetic

Therefore $\qquad 27a^5\div 9a^3=3a^2$

EXAMPLE 3 *Divide* $35a^3b^2c^3$ *by* $7ab^2c^2$

$$\text{The quotient}=\frac{35aaa\ bb\ ccc}{7a\ bb\ cc}=5aa.c=5a^2c$$

We see, in each case, that *the index of any letter in the quotient is the* difference *of the indices of that letter in the dividend and divisor* This is called the **Index Law for Division.**

71 It is easy to prove that the Rule of Signs holds for division

Thus
$$ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = b$$
$$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b$$
$$ab \div (-a) = \frac{ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b$$
$$-ab \div (-a) = \frac{-ab}{-a} = \frac{(-a) \times b}{-a} = b$$

Hence in division as well as multiplication

like signs produce $+$, *unlike signs produce* $-$

Rule *To divide one simple expression by another, obtain the index of each letter in the quotient by* subtracting *the index of that letter in the divisor from that in the dividend*

To the result so obtained prefix, with its proper sign, the quotient of the coefficient of the dividend by that of the divisor

EXAMPLE 1 *Divide* $45a^6b^2x^4$ *by* $-9a^3bx^2$

The quotient $= (-5) \times a^{6-3}b^{2-1}x^{4-2} = -5a^3bx^2$

EXAMPLE 2 $-21a^2b^3 \div (-7a^2b^2) = 3b$

NOTE If we apply the rule to divide any power of a letter by the same power of the letter, we are led to a curious conclusion

Thus, by the rule, $a^3 \div a^3 = a^{3-3} = a^0$,

but also $a^3 \div a^3 = \dfrac{a^3}{a^3} = 1$

$$\therefore a^0 = 1$$

This result will be better understood by the pupil when he has read the chapter on the Theory of Indices

Division of a Compound Expression by a Simple Expression.

72 **Rule** *To divide a compound expression by a single factor, divide each term separately by that factor, and take the algebraic sum of the partial quotients so obtained*

This follows at once from the Distributive Law, Art 69

EXAMPLES (1) $(9x - 12y + 3z) \div -3 = -3x + 4y - z$

(2) $(36a^3b^2 - 24a^2b^5 - 20a^4b^2) \div 4a^2b = 9ab - 6b^4 - 5a^2b$

EXAMPLES V. a.

Read off the quotients in Examples 1 to 3

1. $7x-7$, $9y-y$, $12m-2m$, $18a^2-6a^2$, $10bc-5c$
2. $-x^3-x$, $-8x^3-2x$, $-7a^3-(-7)$, $6m^2-(-3m)$
3. $3b^2/3$, $16a^3/2a$, $2pq/q$, $5x^4/x^3$, $6b^5/3b^2$

Divide

4. $16y^7$ by $8y^2$	5. x^2y^3 by x^2y	6. $15ay^5$ by $5y^4$
7. $21x^3y^2$ by $3xy$	8. $9a^4b^3$ by $3a^2b$	9. $2p^3q^5$ by q^4
10. $-4x^6y^3$ by $2x^5y$	11. $6m^3n^4$ by $-2mn$	12. $8b^3c^5$ by $-4b$
13. $48pq^2r$ by $-6pq$	14. $-9l^2m^3n$ by $-ln$	15. $-x^4y^5z^3$ by $-xy^2z$.
16. $-81k^{11}$ by $27k^4$	17. $28pq^3$ by $-28pq^2$	18. $-32l^5m$ by $8l^3$

19. $63x^4y^2z^7$ by $9x^4y^2$	20. $45abc^3$ by $-5ac$
21. $-36a^4b^3c$ by $-4ac$	22. $-27ab^3c^3x^4$ by $3ac^2x^2$
23. $6a^2x^3-3ax^4$ by $3ax^2$	24. $5x^4y^3+xy^6$ by xy^3
25. $-24a^4-32a^3$ by $-8a^3$	26. $34m^3n^5-51mn^2$ by $17mn$
27. $x^5-5x^4+3x^2$ by x^2	28. $3x^6-6x^4-3x^3$ by $3x^3$
29. $2a^2-ab-3ac$ by $-a$	30. $a^3-a^2b^2+a^4b$ by a^2
31. $3m^3-9m^2n+12mn^3$ by $-3m$	32. $4p^3-36p^2q^3-16p^4$ by $-4p^2$

Division by a Compound Expression

73 The Division of one Compound Expression by another follows the arrangement of 'Long Division' in Arithmetic

Consider the division of 992 by 31

(i)

$$\begin{array}{r|l} 31\,) & 992\,(\,32 \\ & \underline{93} \\ & 62 \\ & \underline{62} \end{array}$$

(ii)

$$\begin{array}{rl} 3\cdot 10+1\,) & 9\cdot 10^2+9\cdot 10+2\,(\,3\cdot 10+2 \\ & \underline{9\cdot 10^2+3\cdot 10} \\ & \qquad\qquad 6\cdot 10+2 \\ & \qquad\qquad \underline{6\cdot 10+2} \end{array}$$

In (i) we have the usual compact arrangement of Arithmetic, and in (ii) we have the same work set forth in full when every number is expressed algebraically

EXAMPLE 1 *Divide* $9x^2+9x+2$ *by* $3x+1$

If in (ii) above we replace 10 by x, we have

$$\begin{array}{rl} 3x+1\,) & 9x^2+9x+2\,(\,3x+2 \\ & \underline{9x^2+3x} \\ & \qquad\quad 6x+2 \\ & \qquad\quad \underline{6x+2} \end{array}$$

Explanation The first term of the dividend is divided by the first term of the divisor Thus $9x^2 \div 3x = 3x$ This gives the first term of the quotient The *whole* divisor is multiplied by $3x$, and the result subtracted from the dividend. Thus we have the remainder $6x+2$ We treat this as a new dividend and divide its first term by the first term of the divisor Thus $6x \div 3x = 2$ This is the second term of the quotient On multiplying the divisor by 2, and subtracting the result from $6x+2$, there is no remainder Hence the complete quotient is the sum of the partial quotients that is $3x+2$

The process succeeds because it separates the dividend into parts each of which is divisible by the divisor, and the complete quotient is found by taking the sum of the partial quotients Thus $9x^2+9x+2$ is divided into two parts, namely $9x^2+3x$ and $6x+2$ Each of these is divided by $3x+1$, giving quotients $3x$ and 2 Thus the complete quotient is $3x+2$

EXAMPLE 2 *Divide* $24x^2-65xy+21y^2$ *by* $8x-3y$

$$
\begin{array}{r|l}
8x-3y\,)\,24x^2-65xy+21y^2\,(\,3x-7y \\
\underline{24x^2-\ \ 9xy} \qquad\qquad\quad \\
-56xy+21y^2 \qquad \\
\underline{-56xy+21y^2} \qquad
\end{array}
$$

Divide $24x^2$ by $8x$, this gives $3x$, the first term of the quotient Multiply the whole divisor by $3x$, and place the result under the dividend By subtraction we obtain $-56xy+21y^2$ Divide the first term of this by $8x$, and so obtain $-7y$, the second term of the quotient

EXAMPLE 3 *Divide* $16a^3+9+9a-34a^2$ *by* $3+8a$

Here we shall arrange divisor and dividend in *ascending* powers of a

$$
\begin{array}{l}
3+8a\,)\,9+9a-34a^2+16a^3\,(\,3-5a+2a^2 \\
\qquad\quad\ \underline{9+24a} \\
\qquad\qquad -15a-34a^2 \\
\qquad\qquad \underline{-15a-40a^2} \\
\qquad\qquad\qquad\quad 6a^2+16a^3 \\
\qquad\qquad\qquad\quad \underline{6a^2+16a^3}
\end{array}
$$

The pupil should work this example also in *descending* powers of a, and compare the steps of his work with that given above

74 It will be now seen that the process of division is embodied in the following rule

Rule 1 *Arrange divisor and dividend in ascending or descending powers of some common letter The terms of the quotient will preserve the same order*

2 *Divide the term on the left of the dividend by the term on the left of the divisor The result is the first term of the quotient*

3 *Multiply the* WHOLE *divisor by this quotient and subtract the product from the dividend*

4 *Bring down from the dividend as many terms as may be necessary to form a new dividend, and repeat these operations till all the terms of the dividend have been used, and there is no remainder The complete quotient is the sum of the partial quotients obtained in the several steps of the division*

EXAMPLES V. b.

[In some of examples 31–44 a rearrangement of terms will be necessary before division.]

Divide

1. a^2+3a+2 by $a+1$
2. b^2+4b+3 by $b+1$
3. x^2+4x+4 by $x+2$
4. x^2+5x+6 by $x+3$
5. $b^2+13b+42$ by $b+7$
6. $z^2+15z+44$ by $z+11$
7. $x^2-15x+54$ by $x-6$
8. $y^2-13y+36$ by $y-4$
9. $p^2-8p-65$ by $p-13$
10. $b^2+10b-39$ by $b-3$
11. $2x^2+9x+4$ by $2x+1$
12. $3y^2+y-2$ by $y+1$
13. $3x^2+8x+4$ by $3x+2$
14. $3x^2+10x+3$ by $3x+1$.
15. $4x^2+23x+15$ by $x+5$
16. $5a^2+16a+3$ by $a+3$.
17. $4m^2-4m-3$ by $2m-3$
18. $6m^2-7m-3$ by $3m+1$.
19. $6c^2-7c+2$ by $3c-2$
20. $12l^2-17l+6$ by $4l-3$
21. $28c^2+c-15$ by $7c-5$
22. $15b^2-14b-16$ by $5b-8$
23. $5p^2-17p+6$ by $5p-2$
24. $6a^2-13a+6$ by $3a-2$
25. $-15x^2+17x+4$ by $5x+1$
26. $-21y^2+58y-21$ by $3y-7$.
27. $-21x^2+x+10$ by $-7x+5$
28. $16x^2-9y^2$ by $4x+3y$
29. $49y^2-4z^2$ by $7y+2z$
30. $-x^2+81$ by $x-9$
31. $36c^2-2d^2-6cd$ by $-6c+2d$
32. $9c^2-25d^2$ by $3c+5d$
33. $3x^2+x^3-13x-15$ by $x-3$
34. $24+9x^2+26x+x^3$ by $4+x$
35. $a^3-a^2-41a+105$ by $a-5$
36. $6a^3+7a^2-a-2$ by $3a+2$
37. $6+b^2-19b+6b^3$ by $2+b$
38. $9m^2+27m^3-3m-10$ by $3m-2$.
39. $4a^3-16a^2x+21ax^2-9x^3$ by $a-x$
40. $2a^4x-7a^3bx^2+9ab^3x^4$ by $2a^2-3abx$
41. Divide $14a(a-1)-15(a+1)$ by $2a-5$
42. Divide $3(2x^2+3x-1)-2(16-x)$ by $3x-5$
43. Divide the sum of $10(p^2-2)-(p+1)$ and $5(p^2-1)+2(p-1)$ by $5p+7$
44. Simplify $2(a+2b)(2a-3b)-(a-2b)(a+3b)-8ab$, and divide the result by $3a+2b$

⁂ Further cases of Division will be discussed in Chapters VII. *and* XV

CHAPTER VI.

BRACKETS

Removal of Brackets

75 BRACKETS are used to enclose quantities that are to be operated upon in the same way Thus in the expression $2a-3b-(4a-2b)$, the brackets indicate that the expression $4a-2b$, *treated as a whole*, has to be subtracted from $2a-3b$

In removing brackets we apply the rules given in Arts 36 and 37.

EXAMPLE 1 *Simplify, by removing brackets, the expression*

$$(2a-3b)-(3a+4b)-(b-2a)$$

$$\text{The expression} = 2a-3b-3a-4b-b+2a$$
$$= a-8b, \text{ by collecting like terms}$$

76 A coefficient placed before any bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient

EXAMPLE 1 $2x+3(x-4)=2x+3x-12=5x-12$

EXAMPLE 2. $5(3m+2)-3(5+2m)=15m+10-15-6m=9m-5$

EXAMPLE 3 $a(b+c)-a(b-c)=ab+ac-ab+ac=2ac$

EXAMPLES VI a

Simplify by removing brackets and collecting like terms

1 $x+3y+(2x-2y)$
2. $x+3y-(2x-2y)$
3 $m-3-(4-2m)$
4. $m-3+(4-2m)$
5 $2a-3b+(2b-3a)$
6 $4c+3d-(2c+3d)$
7. $y-(2x-5y)-(4y+x)$
8 $x-(y+3x)-(2x-y)$.
9 $a+b+(a-b)$
10 $(a+b)-(a-b)$
11. $a^2-(2b^2-3a^2)+(a^2-b^2)-(2a^2+5b^2)$
12 $m+(p-n)-(3n+2m-2p)-(n-m+2p)$
13 $2c^2-(3d^2-c^2)-(c^2-4d^2)$
14. $a^2-3b^2-(3b^2-4a^2)-(3a^2-6b^2)$
15 $4x-(5y+3x)-(3y+5x)-(2x-7y)$
16 $(m+n)-(n-2m)+(2m-3n)-(4m+n)$
17. $a-b+c-(c-a+b)+(a+b+c)-(b-c+a)$
18 $3x-4y-(2z-4x-2y)-(5x-3y+z)+(2x+y-8z)$
19. $x(x+y-z)+y(y+z-x)+z(z+x-y)$
20. $2mn(xy+yz)-(my-yz)2nx$

When $a=2$, $b=-4$, $c=-3$, $d=0$, find the value of

21 $3(a-2b)+5(b-2c)-4a-(c-2a)$

22 $2(a^2-b^2)-(a^2-2ab+b^2)-(a^2-2ab-b^2)$

23. $3c^2-2c(c-d)-3d(c-2d)$

24 $3ab^3-b^2(2a^2+3b)+2a^2(2a+b^2-3)-3b^3(a-1)$

77 Sometimes it is convenient to enclose within brackets part of an expression already enclosed within brackets For this purpose it is usual to employ brackets of different forms The brackets in common use are ( ), { }, [ ]

78 When there are two or more pairs of brackets, a beginner will find it simplest to remove the innermost pair first In dealing with each pair in succession we apply the rules already given in Arts 36 and 37

EXAMPLE *Simplify, by removing brackets, the expression*

$$a-2b-[4a-6b-\{3a-c+(2a-4b+c)\}]$$

Removing the brackets one by one, beginning from within,

$$\begin{aligned}\text{the expression} &= a-2b-[4a-6b-\{3a-c+2a-4b+c\}]\\ &= a-2b-[4a-6b-3a+c-2a+4b-c]\\ &= a-2b-4a+6b+3a-c+2a-4b+c\\ &= 2a, \text{ by collecting like terms}\end{aligned}$$

79 When there are two or more pairs of brackets to be considered, a prefixed coefficient must be used as a multiplier for every term within its own pair

EXAMPLE *Simplify* $5a-4[10a+3\{x-a-2(a+x)\}]$

The expression

$$\begin{aligned}&=5a-4[10a+3\{x-a-2a-2x\}]\\ &=5a-4[10a+3\{-x-3a\}]\\ &=5a-4[10a-3x-9a]\\ &=5a-4[a-3x]\\ &=5a-4a+12x\\ &=a+12x\end{aligned}$$

On removing the innermost brackets each term is multiplied by -2 Then before multiplying by 3, the expression within its brackets is simplified The other steps will be easily seen

If we were to begin with the outermost brackets we should have

$$\begin{aligned}\text{the expression} &= 5a-40a-12\{x-a-2(a+x)\}\\ &= 5a-40a-12x+12a+24a+24x\\ &= a+12x\end{aligned}$$

It will be noticed that we have fewer lines of work but larger coefficients to deal with

80 Sometimes a line called a vinculum is drawn over the symbols to be connected, thus $a-\overline{b+c}$ is used with the same meaning as $a-(b+c)$, and hence $a-\overline{b+c}=a-b-c$.

NOTE The line between the numerator and denominator of a fraction is a kind of vinculum Thus $\frac{x-5}{3}$ is equivalent to $\frac{1}{3}(x-5)$

EXAMPLE *Find the value of*

$$84-7[-11x-4\{-17x+3(8-\overline{9-5x})\}]$$

$$\begin{aligned}\text{The expression} &= 84-7[-11x-4\{-17x+3(8-9+5x)\}]\\ &= 84-7[-11x-4\{-17x+3(5x-1)\}]\\ &= 84-7[-11x-4\{-17x+15x-3\}]\\ &= 84-7[-11x-4\{-2x-3\}]\\ &= 84-7[-11x+8x+12]\\ &= 84-7[-3x+12]\\ &= 84+21x-84\\ &= 21x\end{aligned}$$

After a little practice the number of steps may be considerably diminished

NOTE If we had begun with the outermost brackets, the first two lines of work would have given

$$84+77x+28\{-17x+3(8-\overline{9-5x})\},$$
$$84+77x-476x+84(8-\overline{9-5x}),$$

thus the coefficients become inconveniently large

EXAMPLES VI b

Simplify

1 $7x-(6z-9y)-\overline{6x+7y-3z}$

2 $5x-\{3x+(4x-2x)\}$

3 $7a-\{4a-(3a+6a)\}$

4 $c^2-2c+\{5c^2-(3c-4c^2)\}$

5 $l-2m-(l-2n)-\{2m-l-(2n+l)\}$

6 $a+3b-(b-3a)-\{a+2b-(2a-b)\}$

7 $x-y-\{x-y-(x+y)-\overline{x-y}\}$

8 $x-2(y+z)-\{x+y-z-4(y-2z)\}$

9 $m-\{m+(m-\overline{m+1})\}$

10 $2a-3\{4a-(3+5a)\}$

11 $2c^2-d(3c+d)-\{c^2-d(4c-d)\}+\{2d^2-c(c+d)\}$

12 $-3(1-x^2)-2\{x^2-(3-2x^2)\}$

13 $x-[x-\overline{x-y}-\{x-(x-y)\}]$

14 $5\{c^2-(c+1)\}-3c(2-3c)-8\{4-c(1-c)\}$

15 $3x^2-2(y^2-\overline{x^2-z^2})-3\{(x^2-y^2+z^2)-\overline{z^2-y^2}\}$

16 $4a-b-[a-(3b-c)-\{2a-2(b-c)\}]$

17 $\{3d-(d-\overline{d+e})\}-[2d-\{3d-(d-2e)\}]$

18 $-c-\{a+2(d-\overline{e+a-c})\}$

19 $2\{p-3(q+\overline{p-2q})\}$

20 $n-[n-\overline{m-n}-\{n-(n-\overline{m-n})\}]$

21 $c^2-[a^2-\{a^2-(c^2-\overline{a^2-b^2})-b^2\}-b^2]$

22 $3x-2\{2x-(x-y-3)\}+4\{3x-2(y-2+x)\}$

Simplify

23. $2a-2[2a-\{2(a-b)-b\}]$ 24. $a^2x-2\{ax-3x(2x^2-a)\}$

25 $1-a-(1-\overline{a+a^2})-\{1-(a-\overline{a^2+a^3})\}-[1-\{a-(a^2-\overline{a^3+a^4})\}]$

26. Simplify $-x-[3-(x-\overline{3-x})+\{x+(3-\overline{x+3})\}]$ If the value of the expression is 10, what must the value of x be?

27. Find the value of x when $1-[1-\{1-(1-\overline{1+x})\}]=11$

28 Find the value of x when $1+2\{x+4-3[x+5-4(x+1)]\}=23$

Insertion of Brackets.

81 The rules for the insertion of brackets are the converse of those for the removal of brackets

It is convenient to quote them again here

Rule I *Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered*

Rule II *Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed*

82 The terms of an expression can be bracketed in various ways.

EXAMPLE. The expression $ax-bx+cx-ay+by-cy$

may be written $(ax-bx)+(cx-ay)+(by-cy)$,

or $(ax-bx+cx)-(ay-by+cy)$,

or $(ax-ay)-(bx-by)+(cx-cy)$

83 When two or more terms of an expression are divisible by a common factor, the expression may often be written in a more useful form if we divide each of such terms by this factor, enclose the quotient within brackets, and place the common factor outside as a coefficient

Thus $3x-21=3(x-7)$, $2x^3-6x^2y=2x^2(x-3y)$,

$$x^2-2ax+2bx=x^2-2x(a-b)$$

EXAMPLE *In the expression*

$$ax^3-cx+7-dx^2+bx-c-dx^3+bx^2-2x$$

bracket together the powers of x *so as to have the signs before the brackets alternately + and −*

Writing the terms in descending powers of x, we have the expression

$$=ax^3-dx^3-dx^2+bx^2+bx-cx-2x-c+7$$
$$=(ax^3-dx^3)-(dx^2-bx^2)+(bx-cx-2x)-(c-7)$$
$$=x^3(a-d)-x^2(d-b)+x(b-c-2)-(c-7)$$
$$=(a-d)x^3-(d-b)x^2+(b-c-2)x-(c-7)$$

In this last result the compound expressions $a-d$, $d-b$, $b-c-2$ are regarded as the coefficients of x^3, x^2, and x respectively

EXAMPLES VI. c

Collect in brackets, placing the common factors outside

1. $3x+6y$ 2. $7a-21b$ 3. $5a^2+10b^2$
4. $2x^2-4xy+2y^2$ 5. $7c^2-21d^2+28e^2$ 6. $2a^2x^2-6b^2y^2$
7. $ax-bx$ 8. $ad-d^2$ 9. a^2x+ax^2
10. $5c^2d^2-10cd$ 11. $3a^2b-3ab^2$ 12. $3a^3-6a^2b+3ab^2$

Collect in brackets the coefficients of like powers of x, y, z

13. $x^2+ax+bx$ 14. $y^2-ay-by$ 15. $z^2+az-bz$
16. $ax-bx^2-ax^2$ 17. $y^2-2ay^3-5by^3$ 18. $z^3-3a^2z^2+3bz^3$
19. $px^2-2ax+a^2y+qx^2-2bx+b^2y$ 20. $c^2x^2+2cx+b^2z^2-d^2x^2-dx-a^2z^2$
21. $ax-ay-az-bx-by+bz-cx+cy-cz$

In the following expressions bracket like powers of x so that the signs before the brackets may be (1) positive, (2) negative

22. $3x^3-cx^3+5x^2-c^2x^2$ 23. $a^2x^4-b^2x^4+b^2x^2-c^2x^2$
24. $x+x^3-2xa^2-2x^3b^2$ 25. $2x^4+px^3-qx^4+rx^3-3x^3$
26. $ax^2+ax^3+bx^2-bx^3-cx^2+cx^3$
27. $mx-2mx^5-2nx+nx^3-2px^3+px^5$
28. $ax^3+5x^2-bx+2x-cx^2-x^3$

MISCELLANEOUS EXAMPLES II

EXERCISES FOR REVISION

A

1. Simplify $3\{a-2(b-\overline{c+d})\}$

2. I had x shillings and lost y of them (i) How many *pence* had I left? (ii) How many pounds?

3. Divide $27a^3b^3-1$ by $3ab-1$

4. Subtract $4x^2+xy+z$ from $x^2+2xy+3z-1$

5. Express algebraically the multiplication of the sum of a and b by their product

6. Out of a collection of foreign stamps, $3ax+a^2$ in all, $ax-5a^2$ are found to be forgeries if the others are divided equally among $2a$ boys, express the share of each

B

7. If $a=-5$ give the numerical values of

(i) a^3, (ii) $-a^3$, (iii) $(-a)^3$, (iv) $3(-a)$

8. Subtract the product of $2a-3b$ and $2a+3b$ from $4a^2+9b^2$

9. If n represents a number, what are the next higher and lower numbers? Write down three consecutive numbers of which p is the middle one

10 If $x=3$, $y=2$, $z=1$, find the value of $\frac{1}{x^2}-\frac{1}{y^2}+\frac{1}{z^2}$

11. Simplify $a-[b-c+a-\{b-(a-b-\overline{c+a-b}+c)\}]$

12. Thirty articles cost $8a$ *pence* each, and were sold for a total of b *pounds*. Express the gain in pounds

C

13. Find the value of $ma+mb-mc$ when $m=7$, $a=29\cdot4$, $b=40\cdot8$, $c=20\cdot2$

14 Take three times the sum of x and y from four times the excess of x over y

15 In a school there are p scholars in the first class, $3p-10$ in the second, and $62-3p$ in the other classes. Express the total number of scholars. If this number is 80, what is the value of p?

16 Shew that the expressions y^3-9y^2 and $24-26y$ are equal when $y=2$, 3, or 4

17. Express in the simplest form the difference between the products $(4p-q)(p+2q)$ and $(2p-3q)(2p+3q)$

18 Simplify $5x-[3y-\{4x-(5y-\overline{6x-7y})\}]$

Find the value of the expression when $x=219$, $y=69$

D

19 Find the square of (i) $a+b$, (ii) $a-b$

20. Express in words the difference in meaning of $(a+b)^2$ and a^2+b^2.

21 Simplify $3a-2(b-c)-\{2(a-b)-\overline{3c+a}\}$

22 When x has the values 0, 1, 3, 4, 5 find the values of x^2-4x+3

23. Distinguish between *coefficient* and *index*. Express m (i) as the coefficient, (ii) as the index of x. What is the difference in the values of the two expressions when $m=3$, $x=4$?

24 Subtract $5x-2x^2$ from unity, x^2-1-2x from zero, and add the results

E

25. Divide x^3+a^3 by $x+a$, and x^3-1 by $x-1$

26 What must be subtracted from m that the result may be $m+n$?

27. A horse can eat $8p+3$ bushels of corn in a week, how many weeks will he be in eating $24p^2-55p-24$ bushels?

28. *Write down* the values of the following products

(i) $(x-2)(x-7)$, (ii) $(x+2)(x+3)$, (iii) $(2x-5)(x-4)$

Subtract the last from the sum of the first two

29 What number must be subtracted from $4x+15$ in order to obtain $4x$? If $4x+15=35$, what is the value of x?

30 What is the cost in pence of x things at y shillings each? Find the cost in pence of x feet y inches of gold wire at x shillings y pence per inch

F

31. What do you mean by a *Monomial Expression*? Write down two such expressions involving the letters a and b, using 5 as a *coefficient*, and 3 and 4 as *indices*.

32. If $a=5x-y$, $b=-3x+2y$, $c=-x+5y$, and $d=4x+3y$, find the value of $a-b-c+d$.

33. Divide a^3+b^3 by $a+b$, and a^3-b^3 by $a-b$. Express in words the sum of the two quotients.

34. From a piece of wood $(m+3n)$ yards long a piece $3(m-5n)$ feet in length is cut off; how many yards are left?

35. Simplify, by removing brackets,

$$\text{(i)}\quad a^2+2d^2-(2e^2-b^2)-\{(d^2-c^2-e^2)+(d^2-e^2)\},$$

$$\text{(ii)}\quad 7a-4b-\{5a-3[b-2(a-b)]\}.$$

36. The digits of a two-figure number are x and y; how is the number expressed? If a new number is formed by reversing the order of the digits, shew that the product of the two numbers is expressed by $10x^2+101xy+10y^2$. *Test the truth of this statement in the case of the* number 23.

G

37. What law is illustrated by the statement $p(a+b+c)=pa+pb+pc$? If $a=47.3$, $b=35.9$, $c=38.6$, $p=8.4$, what is the value of $pa-pb+pc$?

38. Find the cost in shillings of m tons n cwt. at m pounds n shillings per cwt.

39. A farmer has to pack $(10p^2-29pq+10q^2)$ eggs equally into $(2p-5q)$ boxes: how many does he put in each?

40. Simplify $10a-[4\{5x-3(x-1)\}-3\{4x-3(x+1)\}+2a]$.

41. Take $x^2(x-2y)-x(3-y)$ from $3x(y-1)-x^4$.

42. A school of 600 boys is separated into upper, middle, and lower divisions containing $4(p-5)$, $5(p+6)$, and $3p-10$ boys respectively. Find the value of p and the number of boys in each division.

H

43. Add the sum of $2y-3y^2$ and $1-5y^3$ to the remainder left when $1-2y^3+y$ is subtracted from $5y^3$.

44. Add together the following products

$$(x+1)(x-3),\quad (x+2)(x+4),\quad (x+3)(x-1),\quad (x+1)(x-2).$$

45. Simplify $3(a^2-b^2)-2[a^2-\{b^2+ab+b(b-\overline{a-b})\}]$.

46. Divide $16x^2+2xy-5y^2$ by $3x-5\{y-(x+2y)\}$.

47. When x has the values 2, 3, 4, 5, or 6, find the values of $x^2-9x+24$.

48. Simplify the following expression by removing brackets, and then bracket together the coefficients of like powers of x:

$$ax^3-x\{b(x^2+x)+c(x-1)+a\}+x(x^2-3x-1).$$

CHAPTER VII

Revision of Elementary Rules

[*If preferred*, Chapters VIII–X *may be taken before this chapter, but the pupil should read* Arts 89–93, *and work* Examples VII e. 10–21 *before attempting* Examples VIII c 16–31]

Important Cases in Multiplication and Division

84 The following results in Multiplication are very important and should be committed to memory

$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2, \qquad (1)$$
$$(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2, \qquad (2)$$
$$(a+b)(a-b)=a^2-b^2 \qquad (3)$$

These statements are true for all values of a and b, hence they may be regarded as *general theorems* applicable to *any* two numbers or algebraical quantities

Thus, if in (1) we put $5m$ for a and $3n$ for b, we have

$$\begin{aligned}(5m+3n)^2&=(5m)^2+2 \cdot 5m \cdot 3n+(3n)^2\\&=25m^2+30mn+9n^2\end{aligned}$$

Again, suppose we require the square of 97

Put $a=100$, $b=3$ in (2), then

$$\begin{aligned}97^2=(100-3)^2&=100^2-2 \cdot 100 \cdot 3+3^2\\&=10000-600+9=9409\end{aligned}$$

A general theorem thus briefly expressed by means of symbols is called a formula

85 In practice it is convenient to apply the above formulæ by means of the following verbal rules

Rule 1. *The square of the* SUM *of two quantities is equal to the sum of their squares* INCREASED *by twice their product*

Rule 2 *The square of the* DIFFERENCE *of two quantities is equal to the sum of their squares* DIMINISHED *by twice their product*

Rule 3 *The product of the sum and difference of two quantities is equal to the difference of their squares*

EXAMPLE 1
$$\begin{aligned}(x+2y)^2&=x^2+2 \cdot x \cdot 2y+(2y)^2\\&=x^2+4xy+4y^2\end{aligned}$$

EXAMPLE 2 $\quad (2a^3-3b^2)^2=(2a^3)^2-2\cdot 2a^3\cdot 3b^2+(3b^2)^2$
$$=4a^6-12a^3b^2+9b^4$$

EXAMPLE 3 $\quad (3m+2n)(3m-2n)=(3m)^2-(2n)^2$
$$=9m^2-4n^2$$

EXAMPLE 4 *Find the product of* 2025 *and* 1975

$$2025\times1975=(2000+25)(2000-25)$$
$$=(2000)^2-(25)^2=4000000-625$$
$$=3999375$$

EXAMPLES VII a

Write down the squares of the following expressions

1 $x+5$	2 $d-2$	3 $\frac{4}{5}-y$	4 $c+1$.
5 $1+2a$	6 $x-9$	7 $1-7b$	8 x^2-1
9. $3x+y$	10 $x-2y$	11. $9+4z$	12 $a+b^2$
13 $x-yz$	14 $ab+c$	15 y^2-2z.	16 a^3+a
17. $5a+4b$	18. $1-y^2$	19. $ad-d^3$	20 $xy-xz$
21 $a-2cd$	22 $3a^2-4ab$	23 $5x^3+x^2y$	24 $3c^2-2c^4$

Without actual multiplication find the value of

25 112^2	26. 199^2	27 $(9\cdot6)^2$	28 $(49\cdot9)^2$

Write down the values of the following products

29 $(a+c)(a-c)$	30 $(a-1)(a+1)$	31. $(1-2x)(1+2x)$
32 $(2a+b)(2a-b)$	33 $(3x-2y)(3x+2y)$	34. $(4x^2+1)(4x^2-1)$
35. $(ab-3)(ab+3)$	36 $(2x^2y-1)(2x^2y+1)$	37 $(m^2+n^3)(m^2-n^3)$

38 $(5x^3-4x)(5x^3+4x)$ $\qquad$ 39 $(7ax^2-4a^2x)(7ax^2+4a^2x)$

40. 103×97	41 115×85	42 475×525	43 $200\cdot5\times199\cdot5$

Find the value of

44 121^2-120^2	45 339^2-319^2	46 287^2-213^2
47 2731^2-269^2	48 $(11\cdot3)^2-(8\cdot7)^2$	49 $(87\cdot2)^2-(12\cdot8)^2$

86 By actual division we have

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2, \qquad (1)$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2 \qquad (2)$$

These two formulæ shew that (i) the sum of the cubes of any two quantities is exactly divisible by the sum of the two quantities, and (ii) the difference of the cubes of two quantities is divisible by the difference of the two quantities

In each case the quotient is made up of the same three terms, namely, the squares of the two quantities and their product, the only difference being in the sign of the product term It is to be noticed that the sign of this term is in each case *opposite* to the sign which separates the symbols in the dividend and divisor

87 The above formulæ may also be quoted as follows

$$a^3+b^3=(a+b)(a^2-ab+b^2),$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

EXAMPLES (i) $\dfrac{x^3+1}{x+1}=x^2-x+1$, (ii) $\dfrac{x^3-8}{x-2}=\dfrac{x^3-2^3}{x-2}=x^2+2x+4$,

(iii) $\dfrac{1-27a^3}{1-3a}=\dfrac{1-(3a)^3}{1-3a}=1+3a+(3a)^2=1+3a+9a^2$,

(iv) $64a^6+27=(4a^2)^3+3^3$
$=(4a^2+3)(16a^4-12a^2+9)$

EXAMPLES VII. a (*Continued*)

Without actual division write down the quotients in the following cases

50. $\dfrac{x^3-1}{x-1}$. 51. $\dfrac{c^3+1}{c+1}$. 52 $\dfrac{x^3+8y^3}{x+2y}$ 53 $\dfrac{27-x^3}{3-x}$.

54. $\dfrac{a^3+8c^3}{a+2c}$ 55 $\dfrac{x^6-y^6}{x^2-y^2}$ 56 $\dfrac{a^6+64}{a^2+4}$. 57 $\dfrac{8a^6+1}{2a^2+1}$

Without multiplication write down the products in the following cases

58 $(p+q)(p^2-pq+q^2)$ 59 $(1-m)(1+m+m^2)$

60. $(3-b)(9+3b+b^2)$ 61. $(x+2y)(x^2-2xy+4y^2)$

62. $(c^2-1)(c^4+c^2+1)$ 63 $(4+3d)(16-12d+9d^2)$

Simplify

64. $(2x+3)^2+(x-6)^2-(2x+5)(2x-5)$

65 $\dfrac{x^2-144}{x-12}+\dfrac{64-x^2}{8+x}+\dfrac{x^2-625}{x+25}$ 66. $\dfrac{2(a^3-b^3)}{a-b}-(a+b)^2-(a-b)^2$.

67. $\dfrac{a^3+27b^3}{a+3b}+\dfrac{8a^3-b^3}{2a-b}-\dfrac{a^3-64b^3}{a-4b}$

68 $(4-x)(16+4x+x^2)+(x-3)(x^2+3x+9)+(x+2)(x^2-2x+4)$

Fractional Coefficients

*88 The rules which have been already explained in the case of coefficients which are *integral*, or whole numbers, will still apply when the coefficients are *fractional*

Example 1 *From the sum of* $\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2$, $-x^2-\frac{2}{3}xy+2y^2$, *and* $\frac{2}{3}x^2-xy-\frac{5}{4}y^2$ *take* $\frac{1}{3}x^2-xy-y^2$

$$\begin{array}{r}
\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2 \\
-\ x^2-\frac{2}{3}xy+2y^2 \\
\frac{2}{3}x^2-\ xy-\frac{5}{4}y^2 \\
\hline
\frac{1}{3}x^2-\frac{4}{3}xy+\frac{1}{2}y^2 \\
\frac{1}{3}x^2-\ xy-\ y^2 \\
\hline
-\frac{1}{3}xy+\frac{3}{2}y^2
\end{array}$$

(i) In adding the first three lines the coefficient of x^2 is the algebraical sum of $\frac{2}{3}$, -1, and $\frac{2}{3}$, which is $\frac{1}{3}$ The coefficients in the second and third columns are treated in the same way

(ii) In the subtraction, after mentally changing the signs of the coefficients in the lower line, we find the algebraical sum of the coefficients in each column

Example 2 (i) *Multiply* $\frac{1}{2}a-\frac{2}{3}b$ *by* $\frac{1}{3}a+b$

(ii) *Divide* $\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3$ *by* $\frac{1}{2}x+\frac{1}{3}y$

(i)

$$\begin{array}{l}
\frac{1}{2}a\ -\frac{2}{3}b \\
\frac{1}{3}a\ +\ b \\
\hline
\frac{1}{6}a^2-\frac{2}{9}ab \\
\qquad +\frac{1}{2}ab-\frac{2}{3}b^2 \\
\hline
\frac{1}{6}a^2+\frac{5}{18}ab-\frac{2}{3}b^2
\end{array}$$

(ii)

$$\begin{array}{l}
\frac{1}{2}x+\frac{1}{3}y\ )\ \frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3\ (\ \frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2 \\
\qquad\qquad\ \ \underline{\frac{1}{4}x^3-\frac{1}{6}x^2y} \\
\qquad\qquad\qquad -\frac{1}{6}x^2y+\frac{1}{72}xy^2 \\
\qquad\qquad\qquad \underline{-\frac{1}{6}x^2y-\frac{1}{9}xy^2} \\
\qquad\qquad\qquad\qquad\quad \frac{1}{8}xy^2+\frac{1}{12}y^3 \\
\qquad\qquad\qquad\qquad\quad \underline{\frac{1}{8}xy^2+\frac{1}{12}y^3}
\end{array}$$

Here again the coefficients of the several terms are dealt with by the ordinary rules of Arithmetic

89 Any symbol with a fractional coefficient may be written in two ways

Thus $\frac{2}{5}x$ and $\frac{2x}{5}$ have the same meaning

Again, $\frac{1}{9}(x+y)$ has the same meaning as $\frac{x+y}{9}$ [Art 80, Note]

Now just as in Arithmetic $\frac{5+2}{9}$ may be written $\frac{5}{9}+\frac{2}{9}$,

so in Algebra $\frac{x+y}{9}$ " " $\frac{x}{9}+\frac{y}{9}$

Thus $\frac{1}{9}(x+y)$, $\frac{x+y}{9}$, $\frac{x}{9}+\frac{y}{9}$, $\frac{1}{9}x+\frac{1}{9}y$ are different ways of writing the same expression The pupil may verify this by giving any numerical values to x and y

Note Since the line separating numerator and denominator of a fraction is a vinculum with the same effect as a bracket, special care must be taken with expressions like $-\frac{x-3}{5}$, $-\frac{x+y}{7}$

Thus $-\frac{x-3}{5}=-\frac{1}{5}(x-3)=-\frac{1}{5}x+\frac{3}{5}$ And $-\frac{x+y}{7}=-\frac{x}{7}-\frac{y}{7}$

*EXAMPLES VII. b.

1. Add together $\frac{2}{3}x-\frac{1}{2}y$, $\frac{1}{4}x+\frac{1}{3}y$, $-x+y$

2. Find the sum of $m-\frac{1}{3}n$, $\frac{3}{4}m+\frac{1}{2}n$, $-\frac{1}{2}m-\frac{1}{6}n$

3. From $\frac{1}{2}a-\frac{1}{3}b$ take $-a+\frac{2}{3}b$

4. Find the sum of $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{4}c$, $\frac{1}{4}b-\frac{5}{12}c$, $\frac{2}{3}a-\frac{3}{4}b+\frac{5}{6}c$, and $-\frac{1}{6}a+\frac{1}{2}b$.

5. Take $-x^2-\frac{2}{3}xy+2y^2$ from $\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2$.

6. From the sum of $\frac{1}{2}a-\frac{1}{3}b$, $-a+\frac{2}{3}b$, $\frac{3}{4}a-b$ take $\frac{1}{4}a-\frac{5}{3}b$

7. Subtract $-\frac{3}{2}m^2+mn-n^2$ from $\frac{1}{2}m^2-\frac{1}{3}mn-\frac{3}{2}n^2$

8. Add together $\frac{3}{8}c^2-\frac{5}{7}cd-7d^2$, $\frac{2}{3}cd+\frac{18}{5}d^2$, $-\frac{5}{8}c^2+4d^2$

9. From $\frac{5}{8}a-\frac{4}{5}b-\frac{15}{4}c$ take the sum of

$$-\frac{13}{8}a-\frac{11}{4}c,\ 2a-3b,\ \text{and}\ \frac{11}{5}b-c$$

Find the product of

10. $a-3b+6c$ and $-\frac{2}{3}a$

11. $-2x+5y-3$ and $-\frac{3}{15}y$

12. $-\frac{9}{2}xy$ and $-4x^2+\frac{2}{9}xy$

13. $-\frac{3}{2}m^3n^2$ and $-\frac{1}{3}m^2+2n^2$

Multiply

14. $x-\frac{1}{2}y$ by $\frac{1}{3}x-\frac{1}{4}y$

15. $\frac{5}{6}a+\frac{1}{4}b$ by $\frac{4}{5}a-\frac{2}{15}b$

16. $2x^2-\dfrac{x}{3}$ by $2x^2+\dfrac{x}{3}$

17. $\frac{1}{6}x^2+\frac{1}{8}y^2$ by $\frac{1}{6}x^2-\frac{1}{8}y^2$

Write down the squares of the following expressions:

18. $x^4+\frac{1}{2}$

19. $\dfrac{m^3}{4}-1$

20. $2ac-\dfrac{d}{4}$

21. $\dfrac{m^2n^3}{2}-\dfrac{m^3n^2}{3}$

Divide

22. $-3a^2+\frac{9}{2}ab-6ac$ by $-\frac{3}{2}a$

23. $-\frac{5}{2}x^3+\frac{5}{3}xy+\frac{10}{3}x$ by $-\frac{5}{6}x$

24. $\frac{1}{2}x^5y^2-3x^3y^4$ by $-\frac{3}{2}x^3y^2$

25. $\frac{21}{2}x^3y-x^2y^2$ by $\frac{7}{2}xy$

26. $\frac{1}{3}x^2+3xy-30y^2$ by $\frac{2}{5}x+6y$

27. $\dfrac{2a^2}{3}-\dfrac{17ab}{18}+\dfrac{b^2}{3}$ by $\dfrac{2a}{3}-\dfrac{b}{2}$

28. $\dfrac{m^3}{6}+\dfrac{m^2n}{72}-\dfrac{n^3}{18}$ by $\dfrac{m}{2}-\dfrac{n}{3}$

29. $\dfrac{a^3}{4}+\dfrac{a}{72}-\dfrac{1}{12}$ by $\dfrac{a}{2}-\dfrac{1}{3}$

30. Simplify $\frac{1}{4}(2x-3y)-\frac{1}{3}(3x+2y)+\frac{1}{12}(7x-5y)$

31. Simplify by removing brackets

$$8\left(\frac{m}{4}-\frac{n}{2}\right)+5\left\{2m-3\left(m-\frac{n}{3}\right)\right\}$$

32. Find the sum of $(x-\frac{1}{2}y)(\frac{1}{3}x+y)$ and $(2x-\frac{1}{3}y)(\frac{1}{2}x-y)$.

Compound Terms and Coefficients

90 In an expression such as

$$2(b-x)-3(d-y)+az,$$

the brackets may be removed by the rules already given, and it will then be found to consist of *five unlike terms* But it is sometimes more convenient *not to remove the brackets*, in which case the expression may be regarded as consisting of *three terms only*, namely the two *compound* terms $2(b-x)$, $3(d-y)$, and the simple term az

91 When compound terms have to be added to, or subtracted from, other like terms, it is usually most convenient to retain the brackets and deal with the compound terms as if they were simple

EXAMPLE *Add together*

$$\tfrac{3}{5}(a+x)-\tfrac{1}{3}(a-x), \quad -(a+x)+\tfrac{1}{9}(a-x), \quad \tfrac{7}{5}(a+x)+\tfrac{2}{3}(a-x),$$

and express the result in the simplest form

$$\begin{array}{r} \tfrac{3}{5}(a+x)-\tfrac{1}{3}(a-x) \\ -(a+x)+\tfrac{1}{9}(a-x) \\ \tfrac{7}{5}(a+x)+\tfrac{2}{3}(a-x) \\ \hline (a+x)+\tfrac{4}{9}(a-x) \end{array}$$

Now
$$\begin{aligned}(a+x)+\tfrac{4}{9}(a-x)&=a+x+\tfrac{4}{9}a-\tfrac{4}{9}x \\ &=\tfrac{13}{9}a+\tfrac{5}{9}x\end{aligned}$$

If we had begun by removing brackets the work would have been far less simple, for we should have had to collect the fractional coefficients of a and x in each expression as a first step Thus we should have had *six* fractional operations before beginning the process of addition

92 *Compound coefficients* may also be dealt with as they stand without removal of brackets

EXAMPLE *From* $(a+b)x-3(b+c)y+4(c-2a)z$
take $5(a+b)x-4(b+c)y-2(c-2a)z$

Noticing the coefficients of x, y, and z in the two expressions, we retain the brackets throughout the work, and proceed as follows

$$\begin{array}{r} (a+b)x-3(b+c)y+4(c-2a)z \\ 5(a+b)x-4(b+c)y-2(c-2a)z \\ \hline -4(a+b)x+(b+c)y+6(c-2a)z \end{array}$$

93 The full treatment of fractional expressions involving symbols will be given later, but there is one form of simplification which may conveniently be explained here to facilitate the reduction of a certain class of equations which occur in Chap VIII

EXAMPLE *Simplify* $\frac{x+1}{2}+\frac{8x-5}{12}-\frac{x-2}{8}-\frac{2x}{3}$ (1)

Here we might begin as follows

$$\text{The expression}=\frac{1}{2}(x+1)+\frac{1}{12}(8x-5)-\frac{1}{8}(x-2)-\frac{2x}{3} \qquad (2)$$

$$=\frac{1}{2}x+\frac{1}{2}+\frac{2}{3}x-\frac{5}{12}-\frac{1}{8}x+\frac{1}{4}-\frac{2x}{3},$$

and then complete the simplification by collecting like terms

But in such a case it is better to consider the expression as consisting of only four fractional terms that is we retain the vinculum in each term of (1) (or the equivalent brackets in (2)), and bring the fractions to a common denominator exactly as in Arithmetic

The L C M of the denominators is 24, thus the successive multipliers of the several numerators in (1) are 12, 2, 3, and 8 Hence

$$\text{The expression}=\frac{12(x+1)+2(8x-5)-3(x-2)-2x\times 8}{24}$$

$$=\frac{12x+12+16x-10-3x+6-16x}{24}$$

$$=\frac{9x+8}{24}, \text{ by collecting like terms}$$

Beginners are recommended to use brackets in the first line of work, otherwise they will be very liable to mistakes of sign in dealing with a term like $-\frac{x-2}{8}$

EXAMPLES VII c.

1 Add together

$5(a+x)-9(a-x)$, $-(a+x)+3(a-x)$, $7(a+x)+2(a-x)$, and express the result in the simplest form

Find in the simplest form the sum of

2 $6(x+y)-3(x-y)$, $-7(x+y)+9(x-y)$, $3(x+y)-7(x-y)$

3 $\frac{4}{5}(m-2n)+\frac{3}{4}(m+n)$, $-(m-2n)-\frac{1}{2}(m+n)$, $\frac{1}{5}(m-2n)+\frac{1}{4}(m+n)$

4. $\frac{1}{2}(2b+\frac{2}{3}a)+\frac{1}{3}(3b-a)$, $\frac{1}{3}(2b+\frac{2}{3}a)+\frac{3}{4}(3b-a)$, $\frac{1}{6}(2b+\frac{2}{3}a)-\frac{5}{12}(3b-a)$

Subtract

5 $\frac{1}{3}(2x+3y)-\frac{5}{18}(6x+y)$ from $\frac{5}{6}(2x+3y)-\frac{4}{9}(6x+y)$

6 $\frac{1}{9}(12a-18b)-\frac{7}{12}(16a+4b)$ from $\frac{1}{4}(12a-18b)-\frac{11}{24}(16a+4b)$

7 From $6(a^2+b^2)-5(a+b)-2$ take $4(a^2+b^2)+3(a+b)+2$, and find the value of the result when $a=\frac{3}{2}$, $b=-\frac{4}{2}$

8. Find the sum of $3(a+b)x-2(a-b)y$, $-2(a+b)x+9(a-b)y$, and $7(a+b)x-6(a-b)y$, and find the value of the result when

$$a=b=\frac{1}{16}$$

9. Subtract $-5x^2(a+b)-y^2(a-b)$ from $-2x^2(a+b)+3y^2(a-b)$

Simplify

10 $\dfrac{x-2}{2}+\dfrac{x+10}{9}$ 11 $\dfrac{x-8}{7}+\dfrac{x-3}{3}$ 12 $\dfrac{7+2x}{8}-\dfrac{x-3}{6}$

13. $\dfrac{x+3}{3}+\dfrac{5-x}{6}+\dfrac{3x-1}{12}$ 14. $\dfrac{1}{2}(x-1)+\dfrac{1}{5}(x+3)-\dfrac{1}{10}(2x-5)$.

15. $\dfrac{2x+1}{3}-\dfrac{x-4}{5}-\dfrac{4}{15}$ 16 $\dfrac{2a-3}{9}-\dfrac{a+2}{6}+\dfrac{5a+8}{12}$

17 $\dfrac{1}{6}(y+4)-\dfrac{y}{3}+\dfrac{1}{12}(y-3)$ 18. $\dfrac{1}{6}(4-3c)+\dfrac{1}{15}(5-2c)+\dfrac{1}{20}(c+7)$

19 Find in its simplest form the value of

$$\left(\frac{5x-1}{8}-\frac{3x-2}{7}+\frac{x-5}{4}+\frac{1}{2}\right)\times 56$$

20. Multiply $\frac{3}{16}(5x-2)-\frac{1}{6}(4x-3)-\frac{1}{3}(x+7)$ by 48, and simplify the result

21 Multiply $\dfrac{5x-6}{12}+\dfrac{3x+8}{9}-\dfrac{x-7}{3}+\dfrac{5}{18}$ by the smallest number which will remove the denominators, and express the result in its simplest form

*Roots Substitutions

***94** The square root of any proposed expression is that factor whose square, or second power, is equal to the given expression

Thus the *square root* of 4 is 2 because $2^2=4$
,, ,, ,, 9 ,, 3 ,, $3^2=9$
,, ,, ,, a^4 ,, a^2 ,, $(a^2)^2=a^4$

Similarly

the **cube root** is that factor whose *cube* gives the quantity,
,, **fourth root** ,, ,, *fourth power* ,,
,, **fifth root** ,, ,, *fifth power* ,,

and so on

Thus the *cube root* of 8 is 2 because $2^3=8$
,, *fourth root* of 81 ,, 3 ,, $3^4=81$
,, *fifth root* of 32 ,, 2 ,, $2^5=32$

The symbol $\sqrt{\ }$, called the **radical sign**, is used to denote the root of a quantity

The square root is denoted by $\sqrt[2]{\ }$, or more simply $\sqrt{\ }$
,, cube ,, ,, $\sqrt[3]{\ }$,
,, fourth ,, ,, $\sqrt[4]{\ }$,

and so on

EXAMPLE 1. *If $a=4$ find the value of $7\sqrt{a^3}$*

Here $7\sqrt{a^3}=7\times\sqrt{(4^3)}=7\times\sqrt{64}=7\times 8=56$

EXAMPLE 2 *Find the value of* $5\sqrt{(6a^3b^4c)}$, *when* $a=3$, $b=1$, $c=8$

$$5\sqrt{(6a^3b^4c)}=5\times\sqrt{(6\times3^3\times1^4\times8)}$$
$$(\sqrt{6\times27\times8})$$
$$=5\times\sqrt{(3\times27)\times(2\times8)}$$
$$=5\times\sqrt{(9\times9)\times(4\times4)}$$
$$=5\times9\times4=180$$

NOTE An expression of the form $\sqrt{(6a^3b^4c)}$ is often written $\sqrt{6a^3b^4c}$, the line above being used as a vinculum indicating the square root of the expression *taken as a whole*

*EXAMPLES VII. d

If $a=3$, $b=2$, $c=4$, $d=6$, $f=1$, find the value of

1. $\sqrt{a^4}$	2. $\sqrt{b^8}$	3. $\sqrt{c^3}$	4. $\sqrt{f^4}$
5 $\sqrt[3]{4b}$	6 $\sqrt{3bd}$	7 $\sqrt{a^2b^2}$	8 $\sqrt{12ab^2}$
9 $\sqrt{ab^3d}$	10 $\sqrt{a^3bd}$	11. $\sqrt{9cd^2}$	12 $\sqrt[3]{6a^2b^2}$
13 $5\sqrt{2bcf}$	14 $-\sqrt{25b^4}$	15. $-3b^3\sqrt{c^4f^8}$	16. $\sqrt{abcd^3}$

*95 Since $$(-a)^2=(-a)\times(-a)=a^2,$$
it appears that the square of $-a$ is exactly the same as the square of $+a$, and, conversely, it follows that every positive quantity has two square roots equal in value, but opposite in sign.

Thus $$\sqrt{25}=+5, \text{ or } -5,$$
$$\sqrt{a^4}=+a^2, \text{ or } -a^2$$

For the present the pupil will be required to deal with the *positive* value only

*96 Any odd power of a quantity gives an expression of the same sign as the quantity itself

Thus
$$(+5)^3=(+5)(+5)(+5)=+125, \quad (-5)^3=(-5)(-5)(-5)=-125$$

Therefore an *odd root* of a quantity will have the same sign as the quantity itself

Thus $$\sqrt[5]{32}=+2, \qquad \sqrt[5]{-32}=-2$$

EXAMPLE 1 *If* $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, *find the value of*
$$7\sqrt[3]{(a^2cx)}-3\sqrt{b^4c^2}+5\sqrt{(f^2x)}$$

$$\text{The expression}=7\sqrt[3]{(-4)^2(-1)4}-3\sqrt{(-3)^4(-1)^2}+5\sqrt{(0)^2\,4}$$
$$=7\sqrt[3]{(16)(-1)4}-3\sqrt{(81)(1)}+0$$
$$=7\sqrt[3]{-64}-3\sqrt{81}$$
$$=7\times(-4)-3\times9$$
$$=-55$$

NOTE. Because any *power* of 0 is 0 any *root* of 0 must also be 0

To denote the root of a fraction one radical sign only is generally used, thus

$$\sqrt{\frac{4}{9}} \text{ is the same as } \frac{\sqrt{4}}{\sqrt{9}}, \text{ that is } \frac{2}{3}$$

EXAMPLE 2 *When* x=4, y=25, *find the value of* $\sqrt[3]{\frac{2x}{5y}} - \sqrt{\frac{x}{4y}}$

$$\sqrt[3]{\frac{2x}{5y}} - \sqrt{\frac{x}{4y}} = \sqrt[3]{\frac{2}{5} \cdot \frac{4}{25}} - \sqrt{\frac{4}{4 \cdot 25}} = \sqrt[3]{\frac{8}{125}} - \sqrt{\frac{1}{25}}$$

$$= \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$$

*EXAMPLES VII e

If $a=-4$, $b=-3$, $c=2$, $x=-1$, find the value of

1. $\sqrt{ac^2x}$ 2 $\sqrt{3ab^3}$ 3 $\sqrt{25ax^3}$. 4 $\sqrt{a^3c^2x}$

5. $\sqrt{b^4c^4x^4}$ 6 $x\sqrt{a^2c^4}$ 7. $\sqrt[3]{6ab^2}$ 8 $\sqrt[3]{27b^3}$

9 $b^2\sqrt{a^3x^5}$ 10 $-b^3\sqrt{c^6x^4}$ 11 $\sqrt[3]{2ax^2}$ 12 $\sqrt[3]{a^2c^3x}$

If $a=16$, $b=27$, $c=64$, find the value of

13. $\sqrt{\frac{3b}{a}} - \sqrt[3]{\frac{b}{c}} + \sqrt[4]{\frac{b}{27a}}$ 14 $\sqrt[3]{4a} - \sqrt{\frac{25a}{12b}} + \sqrt[3]{\frac{a}{2b}}$

15. $\sqrt[3]{\frac{a}{2c}} - \sqrt[4]{\frac{3b}{4c}} + \sqrt{\frac{a}{4c}}$ 16. $\frac{3}{8}a - \sqrt{\frac{4a}{3b}} - \sqrt[3]{\frac{4ab}{b^2}}$

*97 A radical sign before a compound expression which is enclosed within brackets, or is placed under a vinculum, indicates that the root of the compound expression *taken as a whole* is to be found.

Thus $\sqrt{x}+y$ denotes that the square root of the quantity x is to be found and the result added to y, but $\sqrt{(x+y)}$ or $\sqrt{x+y}$ denotes the square root of the sum of the quantities x and y

If $x=16$, $y=9$,

$$\sqrt{x}+y=\sqrt{16}+9=4+9=13,$$

$$\sqrt{x+y}=\sqrt{16+9}=\sqrt{25}=5$$

EXAMPLE *If* a=6, b=4, z=3, *find the value of*

$$\sqrt{(a^2+b^3)} - \sqrt[3]{a^2b-b^2-z}$$

$$\text{The expression} = \sqrt{(6^2+4^3)} - \sqrt[3]{6^2 \cdot 4-4^2-3}$$

$$= \sqrt{(36+64)} - \sqrt[3]{144-16-3}$$

$$= \sqrt{100} - \sqrt[3]{125} = 10-5=5$$

*EXAMPLES VII. f.

If $a=3$, $b=2$, $c=4$, $d=6$, $f=1$, find the value of

1 $\sqrt[4]{2(b+d)}$

2 $2\sqrt{a^2+c^2}$

3 $a^3-\sqrt{cd-f}$

4 $\sqrt[3]{a^4-2ab+b^4}$

5 $\sqrt{c^3}-\sqrt{c^2+d^2}$

6 $\sqrt{a^2+2b^2+2c^2}-\sqrt[3]{a^2b^2d}$

7. $\sqrt{a^2+4ad}-\sqrt{2a^2-3af}$

8 $\sqrt{a^2+b^2+c^2+2ab+2bc+2ca}$

If $a=-4$, $b=-3$, $c=2$, $x=-1$, find the value of

9 $\sqrt{15c-2b}$

10 $\sqrt{80+a^3}$.

11. $\sqrt[3]{a^3+b^3+17cx}$.

12 $\sqrt{4ab+13ax}$

13. $\sqrt[3]{b^3c^6}+\sqrt{a^2+b^2}$

14. $\sqrt[4]{3b^2c-b^3}\ \sqrt{a^2+2ac+c^2}$

15 $\sqrt{\dfrac{2x^3-2bc-b^2}{a^2+4b}}$

If $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, $y=1$, find the value of

16. $2\sqrt{(ac)}-3\sqrt{(xy)}+\sqrt{(b^2c^4)}$

17 $3\sqrt{(acx)}-2\sqrt{(b^2y)}-6\sqrt{(c^2y)}$

18 $7\sqrt{a^2x}-3\sqrt{b^4c^2}+5\sqrt{f^2}$

19 $3c\sqrt{3bc}-5\sqrt{4c^2}-2c\sqrt{3bc}$

20. If $x=-2$, $y=3$, $4a=-1$, find the value of $3xy+4a^2+\sqrt{10-xy}$

21. If $a=10$, $b=-\frac{1}{4}$, $c=-\frac{1}{5}$, find the value of $a^4b^2c^3\sqrt{b^2-c^2}$

22. If $x=1$, $y=-3$, $z=1$, find the value of

$$\sqrt{(x^2+y^3+z)(x+y+z)}-\sqrt[3]{xy^3z^3}$$

23. When $a=1$, $b=-1$, $c=2$, find the value of

$$\sqrt{3a^3(b-c)+3b^3(c-a)+3c^3(a-b)}$$

24 Find the value of

$$\sqrt{(x^3+y^3+z)(x-y-3z)}-\sqrt{-2x-2y+z},$$

when $x=-1$, $y=-3$, $z=1$

CHAPTER VIII

SIMPLE EQUATIONS

98 AN equation is a statement that two algebraical expressions are equal

If the two expressions are *always* equal for *any* values we give to the symbols, the equation is called an **identical equation**, or more simply an **identity**.

Thus (i) $x+3+x+4=2x+7$,

(ii) $a^2-b^2=(a+b)(a-b)$

are identities

99 If the two expressions are only equal for a particular value or values of the symbols, the equation is called an **equation of condition**

Unless otherwise specified an 'equation' is always taken to mean an equation of condition

Thus the statement $4x+2=14$, which will be found to be true only when x has the value 3, is an equation in the ordinary sense of the term

100 To distinguish *identical equality* from *conditional equality* the symbol $\equiv$ is sometimes used instead of $=$

Thus we may write $a^2-b^2\equiv(a+b)(a-b)$

101 The parts of an equation separated by the sign of equality are called **members** or **sides** of the equation, and are distinguished as the *right side* and the *left side* The abbreviations R S and L S will sometimes be found useful

102 In the equation $4x+2=14$, the value 3, which when substituted for x makes both sides equal, is said to **satisfy** the equation The object of the present chapter is to shew how to find the values which satisfy equations of the simpler kinds

103 The symbol whose value it is required to find in any equation is called the **unknown quantity**, or briefly, the **unknown**. The process of finding its value is called **solving the equation**. The value so found is called the **root** or **solution** of the equation

104 An equation which, when reduced to a simple form, involves no power of the unknown higher than the first is called a **simple equation**. It is usual to denote the unknown by x

105 Many of the easier types of equations may be solved by inspection

Thus if $x+4=7$, x must stand for 3,

if $7x=14$, x „ „ 2,

if $\frac{x}{2}=5$, x „ „ 10

In fact the pupil has already had instances of such examples without any knowledge of the formal definition of an equation.

[See Examples I a 28–44, Examples II b 25–28, and Examples II c]

The following recapitulatory Exercise may be here taken orally

EXAMPLES VIII. a (*Oral*)

Find the values of x which satisfy the following equations

1 $2x=6$ 2. $3x=9$ 3 $6x=12$ 4 $5x=15$

5. $4x=20$ 6 $7x=21$ 7. $8x=40$ 8. $9x=-9$

9. $2x=0$ 10 $-2x=0$ 11. $-3x=6$ 12 $11x=55$

13 $7x=42$ 14. $7x=-42$ 15 $12x=60$ 16 $12x=-60$

17. $x+2=5$ 18. $x+9=15$ 19 $x-2=5$ 20. $x-3=7$

21 $x-3=1$ 22 $x-3=0$ 23. $x+4=4$ 24. $x+11=20$

25. $\frac{x}{2}=1$ 26 $\frac{x}{2}=3$ 27 $\frac{x}{2}=-3$ 28. $\frac{x}{3}=0$

29. $2x=1$ 30 $3x=1$ 31. $5x=2$ 32. $6x=-1$.

33. $\frac{x}{7}=-2$ 34. $2x=\frac{1}{3}$ 35 $\frac{x}{3}=\frac{1}{6}$ 36 $\frac{x}{2}=-\frac{1}{4}$

37. $2x+3x=10$ 38. $3x+4x=21$ 39. $7x-2x=5$

40 $15x-12x=6$ 41 $-2x+9x=28$ 42 $-3x+5x=10$

43. $7x-2x-x=24$ 44. $x+2x+6x=-9$ 45 $4x+3x=5+2$

46. $7x-3x=27-11$ 47 $8x-5x=24-15$ 48. $4x+2x=27-15$

106 Beginners are very apt to treat the solution of equations in a mechanical and unintelligent way, without keeping the object in view clearly before them It must be remembered that an equation is *a statement of conditional equality*, that is, it is true only for some particular value, or values, of the unknown In solving the equation we are seeking the value, or values, of the unknown which will make the two sides of the equation equal The process consists of changing the form of the equation, step by step, until it assumes the form "x=some known quantity" It will be found that the solution of a simple equation ultimately depends only on the following axioms

1 If to equals we add equals the sums are equal

Thus if $x=a$, $x+2=a+2$

2 If from equals we take equals the remainders are equal

Thus if $x=a$, $x-3=a-3$

3 If equals are multiplied by equals the products are equal.

Thus if $x=a$, $x\times 7=a\times 7$

4. If equals are divided by equals the quotients are equal

Thus if $x=a$, $x-3=a-3$

EXAMPLE 1 *Find the value of* x *which satisfies the equation*

$$6x-8-3x=2x+12-x$$

Collecting like terms on each side, we have

$$3x-8=x+12$$

Subtracting x from both sides, we obtain

$$3x-x-8=12 \qquad \text{[Axiom 2]}$$

Adding 8 to both sides,

$$3x-x=12+8, \qquad \text{[Axiom 1]}$$

$$2x=20$$

Dividing by 2, $x=\frac{20}{2}=10$ [Axiom 4]

107 It is useful to verify, that is, prove the correctness of the solution, by substituting in both sides the value obtained for the unknown

Thus in the equation $6x-8-3x=2x+12-x$, if $x=10$,

$$\text{L S } =60-8-30=22$$

$$\text{R S } =20+12-10=22$$

Since these two results are equal the solution is correct

Beginners should verify every solution in this way

EXAMPLE 2 *Solve the equation* $\frac{4x}{5}-\frac{3}{10}=\frac{x}{5}+\frac{x}{4}$

Here it is convenient to begin by clearing the equation of fractional coefficients This can be done by multiplying every term on each side by the L C M of the denominators

Hence, multiplying throughout by 20,

$$\frac{4x}{5}\times 20-\frac{3}{10}\times 20=\frac{x}{5}\times 20+\frac{x}{4}\times 20$$

that is, $16x-6=4x+5x$

Subtracting $9x$ from each side,

$$7x-6=0$$

Adding 6 to each side, $7x=6$

Dividing by 7, $x=\frac{6}{7}$

[*Verification* When $x=\frac{6}{7}$,

$$\text{L S} = \frac{4}{5}\times\frac{6}{7}-\frac{3}{10}=\frac{48-21}{70}=\frac{27}{70}$$

$$\text{R S} = \frac{1}{5}\times\frac{6}{7}+\frac{1}{4}\times\frac{6}{7}=\frac{6\times4+6\times5}{5\times7\times4}=\frac{27}{70}$$

Thus the solution is correct]

The following example illustrates types of equations of very frequent use

EXAMPLE 3 *Find the value of the unknown quantity which satisfies*

$$\text{(i) } \frac{x}{7}=1\tfrac{3}{5}, \qquad \text{(ii) } \frac{4}{3}=\frac{5y}{6}, \qquad \text{(iii) } 1\tfrac{2}{5}=\frac{14}{3z}$$

Our object is to detach the unknown quantity from the fraction in which it occurs This we may do by Axioms 3 and 4

(i) In $\frac{x}{7}=1\frac{3}{5}$, we multiply both sides by 7,

thus $$\frac{x}{7}\times 7=1\tfrac{3}{5}\times 7,$$

or $$x=7\tfrac{21}{5}=11\tfrac{1}{5}$$

(ii) In $\frac{4}{3}=\frac{5y}{6}$, we divide both sides by 5,

thus $$\frac{4}{15}=\frac{y}{6}$$

If now we multiply both sides by 6, we have

$$\frac{4\times6}{15}=y, \qquad \text{or} \quad y=\frac{8}{5}=1\tfrac{3}{5}$$

These steps may easily be taken together and performed mentally

(iii) In $1\frac{2}{5}=\frac{14}{3z}$, we multiply both sides by $3z$;

thus $$\frac{7}{5}\times 3z=14$$

Then in one step divide both sides by 7×3, and multiply by 5,

and we have $$z=\frac{14\times5}{7\times3}=\frac{10}{3}=3\tfrac{1}{3}$$

Each of these solutions should be verified by the pupil

108 The preceding examples have been worked out very fully in every detail for the purpose of emphasising the importance of shewing clearly the meaning of every step of the work in solving simple equations Each step should occupy a separate line, and each successive process should be a direct application of one of the fundamental axioms

Orderly arrangement should be studied throughout, and the signs of equality in the several lines should be written neatly in column Beginners are particularly cautioned against placing a meaningless sign of equality at the beginning of a line

In order to furnish the requisite practice in *method* and *arrangement*, we shall now give an exercise containing easy equations which are free from difficulty in the way of reduction, and which involve little actual work

EXAMPLES VIII. b.

Find the value of the unknown quantity which satisfies each of the following equations, and in each case verify the solution

1. $7x-4=17$ 2 $3x-5=10$ 3 $2x+15=23$

4. $5x-9=21$ 5 $7x=18-2x$ 6. $3x=25-2x$

7. $4x-3=2x+1$ 8 $5x+2=6x-1$ 9 $3x+2=4x-3$

10 $4x-3=3x+4$ 11 $8x-9=33-4x$ 12 $5x+3=15-x$.

13 $2x+15=27-4x$ 14 $7x+11=3x+27$

15 $15-5x=24-8x$ 16 $9x+21-4x=46$

17 $5x+7+4x+11+3x=24$ 18 $0=9-6x-19+10x$

19 $7-3y=5+4y+11-16y$ 20 $-3y-5=-7y+1$

21 $6y+7-19=7y+13-3y-21$

22 $3y+4+10y-17=14-23y+16-7y$

23 $\frac{x}{2}+\frac{x}{3}=5$ 24. $\frac{x}{3}-\frac{x}{4}=1$ 25. $z=\frac{z}{4}+6$

26 $x-5=\frac{3}{4}x$ 27 $\frac{1}{2}x+\frac{1}{3}x=x-3$ 28 $\frac{y}{2}-3=\frac{y}{4}+\frac{y}{5}$

29 $\frac{1}{2}x-\frac{1}{4}x=x-9$ 30 $\frac{k}{3}-\frac{1}{2}=\frac{k}{5}+1\frac{1}{2}$ 31. $\frac{k}{3}-2\frac{1}{2}=\frac{4k}{9}-\frac{2k}{3}$

32 $\frac{x}{9}+2\frac{2}{9}=6-\frac{3x}{7}$. 33 $\frac{x}{3}+1\frac{1}{2}=\frac{2x}{9}-\frac{x}{6}+4$ 34 $5\frac{1}{2}+\frac{p}{2}=\frac{3p}{4}+\frac{2p}{3}$.

35 $\frac{x}{3}=\frac{5}{6}$ 36. $\frac{x}{5}=\frac{4}{3}$ 37 $\frac{2z}{3}=\frac{5}{12}$ 38 $\frac{4x}{5}=\frac{7}{15}$.

39 $\frac{7x}{6}=\frac{4}{9}$ 40 $\frac{3x}{8}=\frac{5}{9}$ 41 $\frac{6}{7}=\frac{3x}{2}$ 42 $\frac{8}{45}=\frac{2y}{15}$.

43. $\frac{3}{8x}=\frac{15}{7}$. 44 $\frac{25}{4x}=\frac{5}{2}$ 45. $\frac{2}{5}=\frac{3}{x}$ 46. $\frac{10}{3}=\frac{5}{2k}$

47 From the condition $\frac{2}{3}$ of $\frac{6}{7}$ of $x=1\frac{3}{5}$, find x

48 Find p from the condition $p\times\frac{3}{7}=2\frac{4}{5}$ of $1\frac{1}{4}$

Shew that $x=5$ satisfies the equations

49. $5x-11x+29=2x-11$ 50. $9x-41-13x=24-17x$

109 After enough practice to enforce the reasons for the several steps, the solutions may be presented in a shorter form

When any term is brought over from one side of an equation to the other it is said to be **transposed.**

We shall now shew that any term may be transposed from one side of an equation to the other by simply writing it down on the opposite side *with its sign changed*

Consider the equation $3x-8=x+12$

Subtracting x from each side, we get $3x-x-8=12$

Adding 8 to each side, we have $3x-x=12+8$

Thus we see that $+x$ has been removed from one side, and appears as $-x$ on the other, and -8 has been removed from one side and appears as $+8$ on the other.

Similar steps may be employed in all cases

It appears from this that *we may change the sign of every term in an equation*, for this is equivalent to transposing *all* the terms, and then making the two sides change places

For example, consider the equation $-3x-12=x-24$

Transposing, $-x+24=3x+12$,

or $3x+12=-x+24$,

which is the original equation with the sign of every term changed

EXAMPLE 1 *Solve* $11x-5(2x-1)=3(6-x)+1$

Removing brackets, we have

$$11x-10x+5=18-3x+1;$$

and by transposing, $11x-10x+3x=18+1-5$

Collecting like terms, $4x=14$,

$$x=\frac{14}{4}=3\tfrac{1}{2}$$

[*Verification* When $x=3\frac{1}{2}$,

$$\text{L S}=11\times\tfrac{7}{2}-5(7-1)=\tfrac{77}{2}-30=8\tfrac{1}{2}$$

$$\text{R S}=3\left(6-\tfrac{7}{2}\right)+1=3\times\tfrac{5}{2}+1=8\tfrac{1}{2}\ ]$$

In subsequent examples we shall leave the verification as an exercise for the pupil.

EXAMPLE 2 *Solve* $\frac{1}{4}(x-2)-\frac{1}{6}(2x-5)-1+\frac{3x}{20}=0$

First clear of fractions by multiplying each side by 60, which is the L C M of the denominators

Thus, $15(x-2)-10(2x-5)-60+9x=0$,

that is, $15x-30-20x+50-60+9x=0$

Transposing, $15x-20x+9x=60+30-50$;

$$4x=40,$$

$$\therefore x=10$$

NOTE 1 The above equation might have been written

$$\frac{x-2}{4}-\frac{2x-5}{6}-1+\frac{3x}{20}=0$$

When fractional equations are given in this form care must be taken in dealing with a term like $-\frac{2x-5}{6}$ It must be remembered that $-\frac{2x-5}{6}$ and $-\frac{1}{6}(2x-5)$ have exactly the same meaning

NOTE 2 Observe that on multiplying by 60, we still have 0 on the right side, for $0\times 60=0$

110 When the coefficients involve decimals, we may express the decimals as common fractions and proceed as before, but it is often simpler to work entirely in decimals Useful simplification can sometimes be effected by multiplying each term of the equation by a suitable power of 10

EXAMPLE *Solve* $.375x-1.875=.12x+1.185$

Transposing, $.375x-.12x=1.185+1.875$,

collecting terms, $(.375-.12)x=3.06$,

that is, $.255x=3.06$,

$$x=\frac{3.06}{.255}$$

$$=12$$

EXAMPLES VIII. c

[*It is recommended that Nos* 1–20 *of the following examples should be solved in full detail, explaining every step In the rest of the Exercise the solutions may be shortened by transposition of terms*]

Solve the following equations and verify the solutions

1 $19(1+x)=16x-11$

2 $5(x-3)=3(x-1)$

3 $18-5(x+1)=3(x-1)$

4 $5-4(x-3)=x-2(x-1)$

5 $3(x-7)+5(x-4)=15$

6. $6(x-3)-13(x-2)=1$

7. $6(x-1)-(3x+11)+7=0$

8 $21-7(2x-9+3x)=0$

9 $2(4-x)-3(x-7)-1=16x$

10 $6-\{2x-(3x-4)-1\}=0$

11 $4(x-3)-3(3-x)=5(x+2)-9(8-x)+20$

12. $4(3+x)-3(2x-5)=6-x-2(3-x)$

13. $2x-5\{7-(x-6)+3x\}-28=39$

14. $20(7x+4)-18(3x+4)-5=25(x+5)$

15. $3[15-2\{x-2(x-5)\}]-\overline{5x-20}=0$

16 $\frac{7x+2}{5}=\frac{4x-1}{2}$

17 $\frac{3x-13}{7}+\frac{11-4x}{3}=0$

Solve the following equations and verify the solutions

18. $2x-\frac{1}{3}(x+27)=16$

19. $\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3}$

20. $\frac{3x-1}{3}+\frac{5}{12}=\frac{x}{4}+\frac{2x+1}{5}$

21. $\frac{7x-4}{15}+\frac{x-1}{3}=\frac{3x-1}{5}-\frac{7+x}{10}$

22. $\frac{2x-1}{3}-\frac{3x-2}{4}=\frac{5x-4}{6}-\frac{7x+6}{12}$

23. $6-\frac{x-1}{2}-\frac{x-2}{3}=\frac{3-x}{4}$

24. $13-\frac{9-x}{11}=\frac{3x}{22}+12\frac{1}{2}$

25. $\frac{1}{3}(1-2x)-\frac{1}{6}(4-5x)+\frac{13}{42}=0.$

26. $\frac{1}{12}(9x-2)-\frac{1}{15}(x-1)=4$

27. $\frac{1}{3}(x+1)+\frac{1}{4}(x+3)=\frac{1}{5}(x+4)+16$

28. $\frac{3-4x}{5}-\frac{4+5x}{9}+\frac{7x+11}{15}=0$

29. $\frac{2x-7}{11}-\frac{x-2}{7}=\frac{5x-3}{7}-6$

30. $\frac{3}{2}(x-1)-\frac{2}{3}(x+2)+\frac{1}{4}(x-3)=4$

31. $\frac{2x-5}{6}+\frac{6x+3}{4}=5x-17\frac{1}{2}$

32. $5x-3= 25x+ 2x$

33. $1\ 3= 7+ 2x$

34. $3x-1\ 8+ 2x=\cdot 7$

35. $04x- 07= 11.$

36. $4x=1\ 3- 2x-1$

37. $5x+\frac{x}{3}=x-3.$

38. $2(x-1)+ 5(x-9)=3$

39. $2\ 25x- 125=3x+3\ 75$

40. $\frac{75-x}{3}+\frac{47+2x}{5}=\frac{4\ 4x}{1\ 5}$

41. $\frac{x+ 25}{15}-\frac{x- 35}{45}=18$

111 We shall now give some examples which require more simplification before any terms can be transposed

EXAMPLE *Solve* $\frac{x}{3}-\frac{(4x-7)(3x-5)}{15}=\frac{2}{5}-\frac{(4x-9)(x-1)}{5}$

Clear of fractions by multiplying every term by 15, thus

$$5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$$

Here the products $(4x-7)(3x-5)$ and $(4x-9)(x-1)$ must be multiplied out (or written down by inspection as in Art 67) before any further reduction can be made

Forming the products, we have

$$5x-(12x^2-41x+35)=6-3(4x^2-13x+9),$$

and by removing brackets,

$$5x-12x^2+41x-35=6-12x^2+39x-27$$

The term $-12x^2$ may be removed from each side without altering the equality, thus

$$5x+41x-35=6+39x-27$$

Transposing, $\quad 5x+41x-39x=6-27+35,$

collecting terms, $\quad 7x=14,$

$$x=2$$

NOTE 1 Since the minus sign before a bracket affects every term within it, we do not remove the brackets until we have formed the products

NOTE 2 The terms involving x^2 on each side destroy each other If this were not so the equation would not be a *simple* equation [Art 104]

112 When there are fractional expressions within brackets the brackets should be removed before clearing of fractions

EXAMPLE *Solve* $\frac{2}{3}\left(6-\frac{x}{3}\right)=3\frac{3}{4}-\frac{3}{4}\left(\frac{2x}{3}+\frac{16}{27}\right)$

Removing brackets, we get

$$4-\frac{2x}{9}=\frac{15}{4}-\frac{x}{2}-\frac{4}{9}$$

Multiplying by 36, the L C M of the denominators,

$$144-8x=135-18x-16,$$

or

$$-8x+18x=135-16-144,$$

$$10x=-25,$$

$$x=-\frac{25}{10}=-2\frac{1}{2}$$

113 From the foregoing examples it will now be seen that any simple equation with one unknown may be solved by the following general rule

Rule *If necessary, clear of fractions and remove brackets Transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other Collect the terms on each side, divide both sides by the coefficient of the unknown quantity and the value required is obtained*

EXAMPLES VIII d

Solve the following equations

1 $(x+1)(x+2)=x(x+7)-6$

2 $2(x-1)(x+1)=x(2x-6)+16$

3 $15-x(8-x)=(x-5)^2$

4 $(x+1)^2+(x-2)^2=2x^2-5$

5 $3x(2x+1)-11x=6(x+7)(x-8)+320$

6 $(x-3)(x-4)-2x(x-3)=x(11-x)$

7 $(x-5)^2-4(3-x)=8x+(x+2)^2$

8 $(3x+4)(4x-1)-(7x-2)(x+1)=(5x-3)(x-2)-1$

9 $(3x-2)(3x+3)-(3-4x)(3+4x)=(5x-3)(5x+3)$

10 $(5x+1)(x-2)-(4x-3)(3x-1)=10-(7x+2)(x+1)$.

11. $3(x+1)(x+3)-2(x+1)(x-1)=(x-1)^2+3(5x+1)$

Solve the following equations

12. $(3x-2)(2x-3)-(2x-1)(x-2)=(2x-3)^2-6x$

13. $\frac{(3x-4)(3x+1)}{3}-\frac{(8x-11)(x+1)}{4}=\frac{(6x-1)(2x-3)}{12}$.

14 $3+\frac{(2x-1)(3x-2)}{9}-\frac{x^2}{3}=\frac{x^2-2}{3}$.

15 $\frac{x(2x+1)}{14}-\frac{(x+2)(x-4)}{7}=1\frac{1}{2}$

16. $\frac{1}{6}(2x+9)-\frac{1}{10}(x^2-1)=\frac{3x}{20}-\frac{1}{10}(x-5)(x+3)$

17 $\frac{(3x-2)(x-1)}{21}=1\frac{2}{7}+\frac{(x-3)^2}{7}$

18. $\frac{(x+2)(x-3)}{5}-\frac{3x^2}{10}=\frac{3}{5}(x-1)-\frac{1}{10}(x-4)(x+3)$

19 $\frac{1}{3}\left(x-\frac{5}{2}\right)-\frac{3}{5}\left(x+\frac{4}{3}\right)+\frac{7}{2}=0$

20 $\frac{4x-5}{3}+\frac{1}{2}=\frac{1}{10}\left(\frac{7x}{2}+8\right)$

21 $\frac{1}{3}\left(x-\frac{1}{2}\right)+\frac{1}{2}\left(x+\frac{1}{3}\right)=\frac{1}{4}(x+1)$

22 $\frac{8x+13}{9}=\frac{6x+1}{5}+\frac{2}{3}\left(6-\frac{3x}{2}\right)$

23 $3+\frac{x}{4}=\frac{1}{2}\left(4-\frac{x}{3}\right)-\frac{5}{6}+\frac{1}{3}\left(11-\frac{x}{2}\right)$

24. $\frac{2x}{15}+\frac{x-6}{12}=\frac{3}{10}\left(\frac{x}{2}-5\right)$

25. $3x-4-\frac{4(7x-9)}{15}=\frac{4}{5}\left(6+\frac{x-1}{3}\right)$

26. $\frac{2}{5}\left(\frac{3x}{4}-\frac{2}{3}\right)=\frac{5}{7}\left(\frac{12x}{25}-\frac{1}{75}\right)$.

27. Find the value of x which makes the two expressions

$$(9x-19)(x+2), \quad (3x+1)(3x-2)$$

equal to each other

28. Shew that the following equations are *identities* [Art 98].

(i) $(2x+3)(x-7)-2(x+8)(x-2)=11-23x$,

(ii) $\frac{3}{5}(2x-7)-\frac{2}{3}(x-8)=\frac{4x+1}{15}+4+\frac{4}{15}(x-11)$

29. Shew that the equation

$$7x-3-(7-5x)=3-3x-(5x+8)+5(4x-1)$$

is satisfied by any and every value of x

*** *Harder types of Simple Equations will be discussed in Chapter* XXIII

CHAPTER IX

SYMBOLICAL EXPRESSION FORMULÆ.

114 THE principal use of equations in Algebra is for solving problems, some examples of which will be given in the next chapter In attempting such problems, the first step is always to express the conditions of the question in algebraical language so as to form an equation, which, on solution, gives the answer to the problem

The requisite facility in expressing the conditions of a problem by means of symbols can only be acquired by constant and varied practice, accordingly we shall here give a large number of examples in symbolical expression in continuation of the easy cases already discussed in Chap I These will prepare the way for the problems in Chap X

EXAMPLE 1 *If* a *is one factor of* x, *what is the other factor?*

If 5 is one factor of 75, the other is $\frac{75}{5}$, or 15

So if a is one factor of x the other is $\frac{x}{a}$

EXAMPLE 2 *In* m *years a man will be* n *years old, what is his present age?*

By taking a numerical instance it is easily seen that the required result is obtained by subtracting m from n

Thus the required age is $(n-m)$ years

EXAMPLE 3 *If £20 is divided equally among* p *men, what is the share of each?*

The share of each is the total sum divided by the number of men, or £$\frac{20}{p}$.

EXAMPLE 4 *How far can a man walk in* m *hours at 4 miles an hour?*

In one hour he walks 4 miles

In m hours he walks m times as far, that is, $4m$ miles

EXAMPLE 5 *Out of a purse containing £*x *and* y *florins a man spends* z *shillings, express in pence the sum left*

$$£x = 20x \text{ shillings},$$

and $$y \text{ florins} = 2y \text{ shillings},$$

$$\text{the purse contained } (20x+2y) \text{ shillings},$$

$$\text{the sum left} = (20x+2y-z) \text{ shillings}$$

$$= 12(20x+2y-z) \text{ pence}$$

EXAMPLES IX. a.

(*Many of these examples may be taken orally*)

1. By how much is x less than 15? By how much does $5x$ exceed 5?

2. One factor of a is b, what is the other?

3. What must be taken from p to obtain q?

4. If 27 is less than x by 10, what is x?

5. If 13 is greater than y by 7, what is y?

6. What dividend gives 7 as the quotient when x is the divisor?

7. What divisor gives 5 as the quotient when y is the dividend?

8. The difference between two numbers is c, and the greater of them is 16, what is the other?

9. A boy is a years old, how old was he 5 years ago? p years ago?

10. A boy will be 15 years old in b years, how old is he now?

11. How old will a boy be in p years if he is q years old now?

12. The sum of three numbers is 24, if one of them is 9 and another is p, what is the third?

13. The product of two factors is c, and one of them is d, what is the other?

14. How many times is b contained in $3x$?

15. If a book cost x pence, how many can be bought for y shillings?

16. If a book costs eighteenpence, how many shillings will $3x$ books cost?

17. How many pounds would be spent in buying x books at z shillings each?

18. If there are l numbers each equal to x, what is their sum?

19. If there are 4 numbers each equal to d, what is their product?

20. A man weighs x stones in his clothes, and y lbs when stripped, how many lbs do his clothes weigh?

21. A cart loaded with coal weighs a tons, if it holds b cwts, what is the weight of the cart in lbs?

22. A boat's crew can pull in still water at the rate of a miles per hour. If they row on a river whose stream flows at the rate of b miles per hour, how far will they go in c hours against the stream?

23. If I give away x shillings out of a purse containing m sovereigns and n florins, how many shillings had I left?

24. How many square yards of carpet will be required for a room which is x feet long and y feet broad?

25. How many days must a man work in order to earn £5 at the rate of c shillings a day?

26. How many hours will it take to walk n miles at 3 miles an hour?

27. How far can I walk in x hours at y miles an hour?

28. In x days a man walks y miles, what is his rate per day?

29. How many miles is it between two places if a train travelling p miles an hour takes 5 hours to perform the journey?

30 What is the velocity in feet per second of a train which travels 30 miles in x hours?

31 Out of a purse containing x pounds and y shillings a man spends z pence, express in pence the sum left

32 If in every dozen oranges only m are good, how many good ones are there (i) in 96, (ii) in n oranges?

33 A box contains 10 dozen oranges of which 24 are bad How many good ones may be expected (i) in 80, (ii) in x oranges?

34. When eggs are sixpence a dozen, find (i) the cost in shillings of x eggs, (ii) the cost in pence of $6y$ eggs, (iii) how many can be bought for x pence

35 When buns are sold at 16 for a shilling, what is the cost in pence of x buns? How many buns can be bought for y pence?

115 EXAMPLE 1 *If a number* N *is divided by a divisor* D, *giving quotient* Q *and remainder* R, *shew that* $N = Q \times D + R$

If a number 17 is divided by 6 with quotient 2 and remainder 5, we know that $17 = 2 \times 6 + 5$

or $$\textit{number} = \textit{quotient} \times \textit{divisor} + \textit{remainder}$$

Hence also $$N = Q \times D + R$$

EXAMPLE 2 *A rectangular room is* l *feet long,* b *feet broad, and* h *feet high, how many square yards of paper will be required for the walls?*

To find the perimeter of the room we must add *twice* the length to *twice* the breadth

Thus $$\text{perimeter} = 2(l+b) \text{ feet},$$

and the $$\text{height} = h \text{ feet}$$

Hence the area of the walls $= 2h(l+b)$ square feet,

$$\text{number of square } \textit{yards} \text{ required} = \frac{2h(l+b)}{9}$$

EXAMPLE 3 *The digits of a number beginning from the left are* a, b, c, *what is the number?*

Here c is the digit in the units' place, b standing in the tens' place represents b tens; similarly a represents a hundreds

The number is therefore equal to a hundreds $+ b$ tens $+ c$ units

$$= 100a + 10b + c$$

If the digits of the number are inverted, a new number is formed which is symbolically expressed by

$$100c + 10b + a$$

EXAMPLE 4 *How many men will be required to do in* p *hours what* q *men do in* np *hours?*

np hours is the time occupied by q men;

1 hour ,, ,, $q \times np$ men,

that is, p hours ,, ,, $\dfrac{q \times np}{p}$ men

Therefore the required number of men is qn.

EXAMPLE 5 *What is (i) the sum, (ii) the product of three consecutive numbers of which the least is* n?

The numbers consecutive to n are $n+1$, $n+2$,

$$\text{the sum}=n+(n+1)+(n+2)$$
$$=3n+3$$

And the product$=n(n+1)(n+2)$

NOTE Any *even* number may be denoted by $2n$ where n is *any* positive whole number, for this expression is exactly divisible by 2

Similarly, any *odd* number may be denoted by $2n+1$, for this expression divided by 2 leaves remainder 1

Thus three *consecutive even* numbers may conveniently be represented by $2n$, $2n+2$, and $2n+4$, and three *consecutive odd* numbers by $2n+1$, $2n+3$, $2n+5$

EXAMPLES IX. b

1 Write down the product of four consecutive numbers of which m is the least

2 Write down the sum of three consecutive numbers of which n is the greatest

3 Write down five consecutive numbers of which k is the middle one

4 Write down the product of three consecutive odd numbers of which the middle one is $2p+1$

5 The product of three consecutive even numbers, of which the middle one is $2n$, is equal to d, express this by an equation

6 How old will a boy be in 12 years if he was x years old 3 years ago?

7 How old is a man who in n years will be twice as old as his son now aged 9 years?

8 In 5 years a boy will be y years old, what is the present age of his father if he is twice as old as his son?

9 Write down a number which when divided by l gives a quotient m and remainder n

10 What is the remainder if x divided by y gives a quotient z?

11 What is the quotient if when a is divided by b there is a remainder c?

12 A room is a yards in length, and b feet in breadth, how many square feet are there in the area of the floor?

13 A square room measures m feet each way how many square yards of carpet will be required to cover the floor?

14 A room is p feet long and q yards in width how many yards of carpet 2 ft. wide will be required for the floor?

15 What is the cost in pounds of carpeting a room x yards long, y feet broad, with carpet costing z shillings a square yard?

16 How many miles can a man walk in 50 minutes if he walks 1 mile in p minutes?

17. How many miles can a man walk in 1 hour if he walks a miles in b minutes?

18 How long will it take a man to walk m miles if he walks 18 miles in n hours?

19 How far can a pigeon fly in p hours at the rate of 2 miles in 5 minutes?

20 A man travels a miles by coach and b miles by train, if the coach goes at the rate of 7 miles an hour, and the train at the rate of 35 miles an hour, how long does the journey take?

21 How would you express the number whose digits in order from left to right are m, n, and r? Why may not such a number be expressed by mnr?

22 Write down any two numbers whose digits are a, b, c (by taking the digits in different orders) Shew that the difference between two such numbers is always divisible by 9.

23 A train is running at a speed of m feet per second, how many miles will it travel in n hours?

24 A man has £x in his purse, he pays away 25 shillings, and receives y pence express in shillings the sum he has left

25. If a men do a work in $5a$ hours, how many men will be required to do the same work in b hours?

116 The following examples are added to assist the pupil in stating the conditions of a problem in the form of an equation

EXAMPLE 1 *If* y *is the product of three consecutive numbers, of which the greatest is* p, *express this fact by an equation*

If p is the greatest, the three numbers are p, $p-1$, and $p-2$

$$\text{the product} = p(p-1)(p-2)$$

But the product also equals y, thus the required equation is

$$p(p-1)(p-2)=y$$

EXAMPLE 2 *A man is* x *years older than his son, whose present age is* m *years five years hence the father's age will be twice that of the son, express this statement in algebraical symbols*

The father's present age is $(m+x)$ years, and 5 years hence his age will be $(m+x+5)$ years

The son's age 5 years hence will be $(m+5)$ years But the father's age will then be equal to twice the son's age

Thus the required equation is $(m+x+5)=2(m+5)$

EXAMPLE 3 *A has £p and B has q shillings; A hands £x to B, and finds that he then has three times as much as B; express this fact by an equation.*

B's money has been *increased* by the same amount that *A*'s has been *decreased*.

A has $(p-x)$ pounds, that is, $20(p-x)$ shillings.

B has q shillings $+\,x$ pounds, that is $(q+20x)$ shillings.

Since *A*'s money is now three times *B*'s, the required equation is

$$20(p-x)=3(q+20x).$$

NOTE. It must be carefully observed that the sign of equality connects two expressions that are *numerically* equal; hence, both sides of the equation must be expressed in the *same denomination*. *Shillings* have here been selected to avoid a fractional expression.

EXAMPLES IX. b. (*Continued.*)

26. If a is increased by b, the sum is equal to x; express this algebraically.

27. The product of x and y is equal to five times the excess of c over d; express this by an equation.

28. If m is divided by n, the quotient is equal to 12 less than the sum of p and q; express this in algebraical symbols.

29. A man who is x years old has a son whose age is y years; seven years ago the father was six times as old as the son: express this in algebraical symbols.

30. If x is divided by $4a$, the quotient is equal to 9 less than the product of $2m$ and $3n$; express this by an equation.

31. A man is x years older than his son, whose present age is a years; five years hence the father's age will be twice that of the son; express this in algebraical symbols. If the son is now 15, what is the father's age? If the father is now 53, how old is the son?

32. A man, whose present age is a years, has a son aged c years; five years hence the son's age will equal one-half of the age of his father two years hence: express this algebraically.

33. A has x sheep and B has y times as many; if C, whose sheep are z in number, has as many as A and B together, express this by an equation.

34. A has x marbles and B has y; after they have played and A has won four of *B*'s, he finds that B and he then have the same number: express this in algebraical symbols.

35. Two ladies go shopping. One has a pounds in her purse, the other b shillings; if each of them spends c pounds, and they then have equal sums left, express this equality in algebraical symbols.

36. C buys m horses at x pounds each, and D buys n lambs at y shillings each; express algebraically that the money C has spent is equal to that which D has spent.

37. A walks c miles an hour for x hours, and B walks d miles an hour for y hours, and finds that his walk is 9 miles less than *A*'s; express this fact in algebraical symbols.

Use of Formulæ.

117 Some examples in the use of formulæ have been given in Chap VII Other cases have occurred in Examples 1 and 2 of Art 115 in the present chapter

Thus in Ex 1 we proved

$$N = Q \times D + R,$$

a result which gives in a single statement a general relation expressing the connection between a number, its divisor, and the resulting quotient and remainder

118 A formula, it must be observed, includes all particular cases in one general statement, and may be defined as *a general relation established among certain quantities, any one of which may in turn be regarded as the unknown*

Thus in the formula above mentioned, if Q, R, and D are given quantities, we have an equation to find the corresponding value of N Or, a question may be proposed as follows "By what must 96 be divided so as to give a quotient 5, and a remainder 11?" Here we have given $N=96$, $Q=5$, $R=11$, and therefore from the formula we obtain

$$96 = D \times 5 + 11,$$

whence $D=17$, the required divisor

119 In Geometry we have the following formulæ

(1) If a triangle, on a base b, has a height h its area (A) is given by the formula $A = \frac{1}{2}hb$

(2) If a circle has a radius r, the circumference (C) is given by the formula $C = 2\pi r$, and the area (A) by $A = \pi r^2$

Here π stands for a number which cannot be found exactly, approximately its value $= 3\,1416$, or $\frac{22}{7}$, roughly

(3) If a pyramid of height h stands on a base whose area is A, its volume (V) is given by the formula $V = \frac{1}{3}Ah$

In these cases any linear unit, inch, foot, being chosen, the superficial and solid units will be respectively the square and cubic inch, foot, , and in each of these formulæ any one of the quantities can be found by Arithmetic when the others are given

Example *The Great Pyramid of Egypt stands on a square base each side of which is 764 feet, and its height is 480 feet Find the number of cubic feet of stone used in its construction*

Here $V = \frac{1}{3}Ah$, where $A = 764^2$ and $h = 480$

Hence $$V = \tfrac{1}{3} \times (764)^2 \times 480$$
$$= 160 \times 764 \times 764$$
$$= 93391360 \text{ cubic feet.}$$

120 If a body, starting from rest, has a velocity v and passes over a space of s feet in t seconds, s is given by the formula

$$s=vt \qquad (1)$$

Here v is the number of feet passed over in 1 second

Again, if a body falls freely under the action of gravity, and describes s feet in t seconds,

$$s=\tfrac{1}{2}gt^2, \qquad (2)$$

where $g=32\ 2$ approximately

EXAMPLE 1 *If a train has a velocity of 75 feet a second, how long will it take to cross a viaduct which is 300 yards in length?*

Substituting the values of s and v (expressed in feet) in formula (1), we get

$$900=75t,$$

$$t=\frac{900}{75}=12$$

Therefore the time is 12 seconds

EXAMPLE 2 *A stone dropped from the Clifton Suspension Bridge takes 4 seconds before it reaches the water Find (to the nearest foot) the height of the bridge above the river*

From formula (2), $s=\frac{1}{2}\times 32\ 2\times(4)^2$

$=257\ 6$

Thus the required height is 258 feet

EXAMPLES IX c

1. From the formula for the area of a triangle in Art 119, find

   (i) the area, when the base is 24 ft, and the height 17 ft,

   (ii) the base, when the area is 72 sq ft and the height 9 ft,

   (iii) the height (in chains) when the area is 3 24 acres and the base 13 5 chains,

   (iv) the area, to the nearest square centimetre, when the base is 13 4 cm, and the height 5 8 cm

2. By means of formula (3) of Art 119, find

   (i) the volume of a pyramid of height 8 ft on a base whose area is 12 sq ft;

   (ii) the volume of a pyramid of height 9 ft on a square base each of whose sides is 2 ft,

   (iii) the height of a pyramid whose volume is 32 cu. ft and whose base has an area of 16 sq ft

3. By means of formulæ (2) of Art 119, find (i) the circumferences, (ii) the areas of two circles whose radii are $1\frac{3}{4}$ inches and 2 ft 4 in respectively Take $\pi = \frac{22}{7}$

4. The surface S of a sphere of radius r is given by the formula

$$S = 4\pi r^2$$

Find (i) the surface of a sphere whose radius is 2 1 in ;

(ii) the radius of a sphere whose surface is $17\frac{1}{9}$ sq ft

5 A ring is formed by two concentric circles of radii R and r respectively, if R be the radius of the greater circle, find the formula for the area (A) of the ring Use this formula to find

(i) the area of a ring when the radii are 3 5 cm and 2 8 cm respectively,

(ii) the radius of the outer circle when the area of the ring is 16 94 sq cm., and the radius of the inner circle is 3 54 cm

6 If a parallelogram on a base b has a height h, its area (A) is given by the formula

$$A = bh$$

Find the area of parallelograms in which

(i) the base = 3 5 in, and the height = 1 6 in ,

(ii) the base = 16 6 cm, and the height = 6 5 cm.

7 The area of a parallelogram is 4 2 sq in, and the base is 2 8 in. Find the height

8 From the formula $s = vt$ (Art 120), find

(i) how many miles a train will run in 27 min at 40 mi per hour,

(ii) how long a train will take to run 51 mi at 34 mi. per hour,

(iii) the velocity in miles per hour of a train which runs 6600 yards in 5 minutes

9. By means of the formula $s = \frac{1}{2}gt^2$ (Art 120), find

(i) the height of a flagstaff if a stone dropped from the top takes 3 seconds to reach the ground,

(ii) how long it will take a stone to drop from a balloon whose height above the ground is 402 ft 6 in

10. If a room is l feet long, b feet broad, and h feet high, find formulæ for (i) the area (A) of the floor, (ii) the perimeter (P), (iii) the area of the surface (S) of the four walls

11. From the formulæ in the last example, find A, P, and S in the case of rooms with the following dimensions

(i) length 18 ft, breadth 11 ft, height 9 ft,

(ii) length 20 ft 3 in, breadth 14 ft 8 in, height 12 ft

12. Find the height of a room when the length and breadth are 17 ft 9 in, 12 ft 3 in respectively, and the area of the walls is 630 sq ft

13. The area of a trapezium is equal to

$$\frac{1}{2}(\textit{sum of parallel sides}) \times (\textit{distance between them})$$

Express this in algebraical symbols, and apply the formula to find the area of a trapezium when the parallel sides are 6 ft 4 in and 7 ft 2 in and the distance between them is 4 ft

14 Use the formula of Art 115, Ex 1, to find a number which when divided by 19 gives a quotient 17 and remainder 5

15. By what number must 566 be divided so as to give a quotient 15 and remainder 11?

16. In a right angled triangle if a and b denote the lengths of the sides containing the right angle and c denotes the length of the hypotenuse, it is known that $c^2 = a^2 + b^2$

By substitution find which of the following sets of numbers can be taken to represent the sides of a right-angled triangle

(i) 7, 24, 25 (ii) 12, 35, 36 (iii) 1 6, 6 3, 6 5

17 The rectangle contained by two straight lines, one of which is divided into any number of parts, is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line

Prove this by taking algebraical symbols to represent the undivided line and the segments of the divided line

18. AB is a straight line divided into two parts at O Prove algebraically, as in the last example

(i) $AB^2 = AB \cdot AO + AB \cdot OB$

(ii) $AB \cdot AO = AO^2 + AO \cdot OB$

Express these two results in a verbal form as in Example 17.

19. If a and l stand for the first and last of a series containing n of the natural numbers 1, 2, 3, 4, 5, ..., taken consecutively, their sum (s) is given by the formula

$$s = \frac{n}{2}(a + l)$$

Use this formula to find the sum of

(i) all the natural numbers from 1 to 300,
(ii) " " 1 to 1000,
(iii) " " 301 to 1000

Check this last result by means of (i) and (ii)

20. A mechanic's wages are raised by £1 each year If he received £13 in his first year and £27 in his last, for how many years had he worked if the total of his wages amounted to £300?

[*Use the formula of* Ex 19]

21. With the notation of Example 16, find the value of

(i) c when $a=15$, $b=8$, (ii) a when $c=25$, $b=7$;

(iii) b when $c=41$, $a=9$, (iv) a when $c=1{\cdot}7$, $b=0{\cdot}8$

22. Find a formula which will give the simple interest (£I) on a principal (£P) for n years at r per cent.

Use this formula to find

(i) the interest on £435 for 4 years at 3 p.c. per annum,

(ii) at what rate per cent. the interest on £240 will amount to £21 in $2\frac{1}{2}$ years,

(iii) in what time the interest on £920 will amount to £207 at $4\frac{1}{2}$ p.c. per annum,

(iv) what principal will produce interest £10 1s. in 219 days at $2\frac{1}{2}$ p.c. per annum?

23. In the formula $F=\dfrac{mv^2}{gr}$, given $m=12{\cdot}075$, $r=3$, $g=32{\cdot}2$, $F=200$, find v.

24. In the formula $v^2-u^2=2as$, find the value of a when $v=50$, $u=10$, and $s=100$.

25. From the formula $s=\frac{n}{2}(a+l)$, find

(i) the value of s, when $n=20$, $a=14$, $l=964$,

(ii) the value of a, when $s=25{\cdot}2$, $n=12$, $l=3{\cdot}2$,

(iii) the value of n, when $s=46{\cdot}8$, $a={\cdot}6$, $l=7{\cdot}2$,

(iv) the value of l, when $s=-175{\cdot}5$, $a=13{\cdot}5$, $n=13$.

26. If $y=4+\frac{3}{10}x$, find the value of y when x has the values 0, 4, 8, 12, 16, 20.

There is a wall 20 ft. long, whose height at any point x ft. from one end is $4+\frac{3}{10}x$ feet. Draw the wall on a scale of 1 inch to 4 feet, marking on it the height at each end and at intervals of 4 ft.

CHAPTER X

Solution of Problems

121 To obtain the equation by which a problem may be solved we first represent the unknown quantity by a symbol x, and then state the conditions of the problem in symbolical language so as to obtain two expressions which are numerically equal We thus obtain an equation which may be solved by the methods already given in Chap VIII

Example 1 *Find two numbers whose sum is 36, and whose difference is 10*

Let x be the smaller number, then $x+10$ is the greater

Their sum is $x+(x+10)$, which is to be equal to 36

Hence, $\quad x+x+10=36,$

that is, $\quad 2x=26,$

$\quad x=13,$

and $\quad x+10=23,$

so that the numbers are 23 and 13

Verification

$23+13=36,$

$23-13=10$

The solution should always be tested to see whether it satisfies the conditions of the problem or not

Example 2 *Divide 54 into two parts so that four times the greater equals five times the less*

Let x be the greater part, then $54-x$ is the less

Four times the greater part is $4x$;
five times the less is $5(54-x)$

Hence the symbolical statement of the problem is

$4x=5(54-x),$

that is, $\quad 4x=270-5x,$

$9x=270,$

$x=30,$ the greater part,

and $\quad 54-x=24,$ the less

Verification

$30\times4=120,$

$24\times5=120,$

and $\quad 30+24=54$

Note The beginner's principal difficulty at this stage is in the *formation* of the equations The solution will usually present no difficulty, hence in the examples which follow we shall usually leave the solution and verification to be completed by the pupil

EXAMPLE 3 *If a certain number is increased by 5, one-half of the result is three-fifths of the excess of 61 over the number Find the number*

Let x represent the number

The sum of x and 5 is $x+5$ And the excess of 61 over x is $61-x$

Thus the symbolical statement of the problem is

$$\frac{1}{2}(x+5)=\frac{3}{5}(61-x)$$

Clearing of fractions, and solving this equation, we obtain $x=31$

EXAMPLES X a

1 One number exceeds another by 8, and their sum is 26, find them

2 The sum of two angles of a triangle is 48°, and their difference is 22°, find them What is the third angle?

3 Twice a certain number increased by 5 is equal to 23, find it

4. If a number is multiplied by 5, and then 4 is taken away, the result is 31, find the number

5 If 3 be taken from a number, and the result multiplied by 8, the product is 96, find the number

6 If 4 be added to a number, and the sum multiplied by 3, the result is 51, find the number

7 I thought of a number, doubled it, then added 3 The result multiplied by 4 came to 52 What was the number I thought of?

8 Find three consecutive numbers whose sum shall equal 45

9 One number is three times another, and four times the smaller added to five times the greater amounts to 133, find them

10 Find three consecutive numbers such that twice the greatest added to three times the least amounts to 34

11. Divide £40 between A and B so that twice A's share may equal three times B's share

12 Divide 60 into two parts so that three times the greater may exceed 100 by as much as eight times the less falls short of 200

13 Find three consecutive *even* numbers such that their sum is 78

14 Find three consecutive numbers such that three times the middle one shall be greater than the sum of the other two by 22

15 Divide 20 into two parts such that the square of the greater shall exceed the square of the less by 80

16 Find two sums of money differing by £10 whose difference is equal to one-half their sum

17 Find a number whose third part is less than 37 by as much as 29 exceeds its fifth part

18 Divide 99 sheep into two flocks so that four-fifths of one flock may be equal in number to two-thirds of the other flock

19 There are 28 sovereigns in a purse, how many more must be added so that the added sum may be one eighth of the whole contents?

20 Divide 92 into two parts so that one-third of one part may exceed one-seventh of the other part by 4

21. Divide 81 into two parts such that five-sixths of the smaller part shall exceed seven-fifteenths of the larger by 9

22 The difference between the squares of two consecutive numbers is 59 find the numbers

23 Divide 22 into two such parts that the square of the greater shall exceed the square of the less by 88

24 Find a number whose half added to 24 exceeds the sum of its third and fourth parts by 7

25 There are two consecutive numbers such that one-fifth of the greater exceeds one-seventh of the less by 3 find them

26 The third, sixth, and eighth parts of a number together make up 60 what is the number?

27 When the sixth part of a certain number is taken from the half of it, the result is 3 less than the sum of its fourth and eighth parts find the number

122 EXAMPLE 1 *Divide £47 between A, B, and C, so that A may have £10 more than B, and B £8 more than C*

Let x represent the *number* of pounds that C has, then B has $x+8$ pounds, and A has $x+8+10$ pounds

Hence $$x+(x+8)+(x+8+10)=47;$$
whence it will be found that $x=7$, so that C has £7, B £15, A £25

NOTE The symbol x represents a *number*, and such loose and inexact expressions as "*Let x equal what C has,*" or "*Let x equal C's money,*" must never be used

EXAMPLE 2 *A has £9, and B has 4 guineas, after B has received from A a certain sum, the latter has five sixths of what B has, how much did B receive?*

Let x represent the number of *shillings* B received from A

B would then have $(84+x)$ shillings,

and A " $(180-x)$ "

Hence $$180-x=\frac{5}{6}(84+x),$$

whence it will be found that $x=60$ Therefore B received 60 shillings, or £3.

NOTE It is important to express all the quantities in the same denomination shillings are here selected as being the most convenient

EXAMPLE 3 *A is three years older than B, eight years ago five-sixths of A's age exceeded three-fifths of B's age by 6 years find their present ages*

Let x represent B's age in years, then A's age is $(x+3)$ years

B's age 8 years ago was $(x-8)$ years, and

A's age 8 years ago was $(x+3-8)$ years, or $(x-5)$ years

Hence $$\frac{5}{6}(x-5)-\frac{3}{5}(x-8)=6,$$

whence $x=23$ Thus B's age is 23 years, and A's age is 26 years

EXAMPLE 4 *A person spent £28 4s in buying geese and ducks; if each goose cost 7s and each duck 3s, and if the total number of birds bought was 108, how many of each did he buy?*

Let x be the number of geese, then $108-x$ is the number of ducks

Since each goose costs 7 shillings, x geese cost $7x$ shillings

And since each duck costs 3 shillings, $108-x$ ducks cost $3(108-x)$ shillings

Therefore the amount spent is

$$7x+3(108-x) \text{ shillings}$$

But the question also states that the amount is £28 4s, that is, 564 shillings

Hence $\qquad 7x+3(108-x)=564,$

that is, $\qquad 7x+324-3x=564,$

or $\qquad 4x=240,$

$\qquad x=60$, the number of geese,

and $\qquad 108-x=48$, the number of ducks

NOTE Here again it should be observed (1) that we say "Let x be the *number* of geese," and (2) that all the quantities are expressed in the *same denomination*

EXAMPLES X b

1 Divide £67 between A, B, and C, so that A may have £15 more than B, and B £8 more than C

2 Divide £66 between A, B, and C, so that A's share may be half of B's, and C may have £4 less than B

3 If a sum of £85 is divided between A, B, and C, so that A has £10 less than B, and C has three times as much as A, find the share of each

4 Divide £188 between A, B, and C, so that A may have £37 less than B, and C's share may be £11 more than twice A's share

5 How must a sum of £156 be divided between three persons so that the first takes half, and of the other two one takes six-sevenths as much as the other?

6 Three trucks together contain a load of 100 tons The first holds 5 tons more than the second, and 3 tons more than the third What was the load of each truck?

7 Two men share £60 in such a way that one-fifth of one share is equal to one seventh of the other How is the money divided?

8. Divide 650 yards into two lengths so that one may be 20 yards longer than half of the other

9 A has 5 guineas, and B has £3 15s After B has paid A a certain sum the former finds that he has one-fifth as much money as A has; how much did A receive from B?

10 A and B have £12 between them, A receives £1 5s from B and finds that he has seven times as much money as B how much had each at first?

11. A has three times as much money as B, after giving B ten shillings he has only twice as much what had each at first?

12 Two boys together have £1 10s, if one had 6s less and the other 9s more, the former's money would be one-half that of the latter what has each of them?

13. A's age is twice B's, 4 years ago A was three times as old as B, find their present ages

14. B's age is one-third of A's, 10 years hence A will be 16 years older than B, find their ages

15 What is A's present age if he is now three times as old as B, and was four times as old 5 years ago?

16 B's present age is four times A's, 6 years ago B was ten times as old as A how old are they?

17. In 12 years a man will be three times as old as his son, the difference of their ages is 30 years how old are they?

18. A says to B, "I am 10 years your senior, in five years I shall be twice as old as you", find their ages

19. A roll of cloth was bought at 5s 6d a yard, and another roll, 25 yards longer, at 5s a yard, the two together cost £100 15s how many yards were there in each roll?

20 How many pounds of tea at 1s 6d and at 2s 6d a lb must be mixed to make a box of 200 lbs worth altogether £18?

21 Divide three guineas between A and B so that for every half-crown A receives B may receive a shilling

22 A hundredweight of tea worth £19 12s is made up of two sorts, part worth 4s a pound and the rest worth 2s a pound, how much was there of each sort?

23 B's age exceeds A's by 3 years, and two thirds of A's age is less than five sixths of B's by 10 years, what are their ages?

24 A is 9 years younger than B, and 6 years older than C, three-fourths of A's age, four-fifths of B's and one-half of C's together amount to 37 years find their ages

25 A man spent £1 18s in buying tea at 2s 2d per pound and coffee at 1s 4d per pound He bought 21 pounds altogether how many pounds of each did he buy?

26. A farmer has a certain number of oxen worth £18 each, and twice as many sheep worth £3 10s each, if their total value is £500, how many has he of each?

27 A purse contains £2 10s made up of pence, shillings, and half-crowns, the half-crowns number half as many again as the pence, but only one third of the number of shillings find the number of coins of each kind

28. Of two boys one was the taller by 5 in, the shorter has now grown 3 in., and the taller 2 in, and at present the difference of their heights is $\frac{1}{15}$ of the height of the taller boy What were their former heights?

123 It will sometimes be found easier not to put x equal to the quantity directly required, but to some other quantity involved in the question by this means the equation is often simplified

EXAMPLE 1 *A woman spends* 4s $4\frac{1}{2}$d *in buying eggs, and finds that* 9 *of them cost as much over one shilling as* 15 *cost under two shillings, how many eggs did she buy?*

Let x be *the price of an egg* in pence, then 9 eggs cost $9x$ pence, and 15 eggs cost $15x$ pence

Hence $$9x - 12 = 24 - 15x,$$
or $$24x = 36,$$
$$x = 1\tfrac{1}{2}$$

Thus the price of an egg is $1\frac{1}{2}d$, and the number of eggs
$$= 52\tfrac{1}{2} - 1\tfrac{1}{2} = 35$$

EXAMPLE 2 *At noon A starts to ride at 8 mi an hour, two hours later B starts after him on a bicycle at 12 mi an hour How far will A have ridden before he is overtaken by B? Find also at what times A and B will be 8 miles apart*

Let x represent the number of hours A has ridden before he is overtaken, then B has ridden for $x - 2$ hours

A rides $8x$ miles in x hours,
B $12(x-2)$ $x-2$ hours

And when B overtakes A he has covered the same distance as A,
$$12(x-2) = 8x,$$
whence $x = 6$

A has ridden for 6 hours, and has covered 48 miles

For the second part of the question, if x represents the required number of hours after noon, we have by similar reasoning
$$12(x-2) = 8x \pm 8,$$
where in the last term the upper or lower sign is to be taken according as B is 8 miles *ahead of* or *behind* A In the former case $x = 8$, and in the latter $x = 4$

Thus the required times are 4 p m and 8 p m

EXAMPLES X c

1. How many books can be bought for £5, if 17 cost as much over £2 as 7 of them cost under a sovereign?

2 If the price of 16 eggs is as much under half-a-crown as the price of 12 exceeds $5d$, how many can be obtained for $3s$ $9d$?

3 A gardener plants out 386 cabbages, some in rows of 15 and the remainder in rows of 17, there are 24 rows in all how many are planted in rows of 17?

4. Two boys have 252 marbles between them, one arranges them in heaps of 6 each, the other in heaps of 9 each, and there are 34 heaps in all how many marbles has each boy?

5. In 17 years a father will be twice as old as his son, whose age at the present time is one-third of his father's age How old is the father now?

6 A is 12 years older than B, 12 years ago he was twice as old as B then was How old is A now, and how many years ago is it since he was three times as old as B then was?

7. A person bought a number of oranges for 3*s* 9*d*, and finds that 12 of them cost as much over 5*d* as 16 of them cost under 2*s* 6*d*, how many oranges were bought?

8 By buying eggs at 15 for a shilling and selling them at a dozen for 15*d* a man gained 13*s* 6*d*, find the number of eggs

9. A man's age is three times the sum of the ages of his two sons, one of whom is twice as old as the other, in 12 years the sum of the sons' ages will be three-fourths of their father's age find their respective ages

10. A and B start at noon from two towns $37\frac{1}{2}$ miles apart, A's rate of walking being twice B's If they walk 5 hours before they meet, find their rates of walking

11. Two cyclists starting at the same time from two towns 48 miles apart meet in 2 hrs 24 min Find their rates of riding, given that one is two thirds of the other

12 A man can cycle from his house to a railway station and back in a certain time at 12 mi an hour If he rides out at 8 mi an hour, and returns by motor at 15 mi an hour he takes 15 minutes longer on the double journey Find the distance between his house and the station

13 A carriage, horse and harness are together worth £144, the carriage is worth four-fifths of the horse's value, and the harness three-fifths of the difference between the values of the horse and carriage what is the value of each?

14 A's age is equal to the sum of the ages of B and C Ten years ago A was twice as old as B Shew that ten years hence A will be twice as old as C [*Let* x *years represent B's age ten years ago*]

15 Two cyclists start from the same place to ride in the same direction A starts at noon at 8 mi an hour, and B starts at 1 30 p m at 10 mi an hour How far will A have ridden before he is overtaken by B? Find also at what times A and B will be 5 miles apart

16 Two men ride towards each other from two places 60 miles apart, one at 12 mi an hour, and the other at 9 mi an hour Find when they are first 18 miles apart How must your equation be altered so as to find the time when they are 18 miles apart after meeting?

17. If P and Q represent two towns 28 miles apart, and if A walks from P to Q at 4 mi an hour while B walks from Q to P at 3 mi an hour, both starting at 9 a m, when will they be 7 miles apart?

CHAPTER XI

GRAPHS

124 ONE quantity is often related to another in such a way that if a change is made in the value of one there is a corresponding change in the value of the other

For example, suppose we know the cost of a certain weight of tea, if we double the weight we double the cost, if we treble the weight we treble the cost, and so on In such a case the cost is said to be *directly proportional* to the weight

Similarly when a train is travelling at a uniform speed, the distance travelled is directly proportional to the time

125 Any expression involving x will have different values if different values are substituted for x Suppose we wish to find the values of the expression $2x+5$ when x has the series of values 3, 2, 1, 0, -1, -2, -3, then the following arrangement will be found convenient

Let y stand for the expression, that is, suppose $y=2x+5$, and arrange the values as in the following table

x	3	2	1	0	-1	-2	-3
$2x$	6	4	2	0	-2	-4	-6
$y=2x-5$	11	9	7	5	3	1	-1

Thus corresponding to the values 3, 2, 1 0, -1, -2, -3 for x we have the values 11, 9, 7, 5, 3, 1, -1, for y, or $2x+5$

Here there is no *direct* proportion between the values of x and y, but each value of y is dependent on the corresponding value of x

126 A quantity which may have a series of different values is called a variable In the above table x is a variable, and y (whose value depends on that of x) is also a variable The relation between two variables thus connected may often be conveniently shewn by means of diagrams which give the values of the variables at a glance.

127 Axes of Reference Coordinates. On a piece of squared paper select a pair of the thicker horizontal and vertical lines Let these be marked XOX′, YOY′ as in Fig 1 below Then the position of any point P with reference to these lines can be found when we know its distances from each of them Such lines are known as **axes of reference**, XOX′ being known as the **axis of x**, and YOY′ as the **axis of y** Their point of intersection O is called the **origin.**

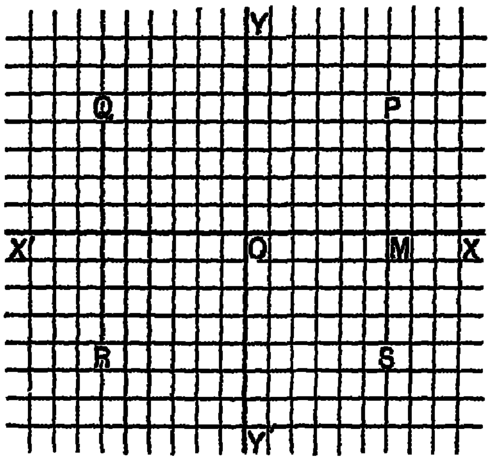


Fig 1

Consider the point P in the figure It will be seen that we can get to P by marking 6 divisions of the paper along OX, that is to the point M, and then taking 4 divisions vertically up from M Thus if the perpendicular distances of a point from the axes are known the position of the point is fixed The distances 6 and 4 are known as the **coordinates** of the point P OM is known as the **abscissa** of P, and PM is known as the **ordinate** of P

When symbols are used the abscissa is always denoted by x, and the ordinate by y A point whose coordinates are x and y is spoken of as "the point (x, y)," the abscissa of the point always being named first

This process of marking the position of a point by means of its coordinates is known as **plotting the point.**

In practice the most convenient paper is that ruled to tenths of an inch, and one or more of the divisions may be taken as the unit of length

128 The axes of reference divide the plane of the paper into four spaces XOY, YOX′, X′OY′, Y′OX, known respectively as the first, second, third, and fourth **quadrants.**

It is clear that in each quadrant there is a point whose distances from the axes are equal to those of P in the above figure, namely, 6 units and 4 units

The coordinates of these points are distinguished by the use of the *positive* and *negative* signs, according to the following system distances measured along the x-axis to the *right* of the origin are *positive*, those measured to the *left* of the origin are *negative* Distances measured vertically *upwards* from the x-axis (that is, in the first and second quadrants) are *positive*, those measured *downwards* from the x-axis (that is, in the third and fourth quadrants) are *negative*

Thus the coordinates of the points Q, R, S, in Fig 1 are

$$(-6, 4),\ (-6, -4),\ \text{and}\ (6, -4)\ \text{respectively}$$

The pupil may be reminded that this is a natural extension of the explanation of *opposite signs* given in Art 26 (ii)

EXAMPLE 1 *Plot the points*

(i) (6, 8), (ii) (−2, 2), (iii) (6, 0), (iv) (0, 0),

and find the distance between the first two, taking one-tenth of an inch as unit

(i) We first take 6 units to the *right* along OX, and then 8 units at right angles to OX and *above* it The resulting point P is in the first quadrant

(ii) Here we may briefly describe the process as follows Take 2 steps to the *left* then 2 *up*, the resulting point Q is in the second quadrant

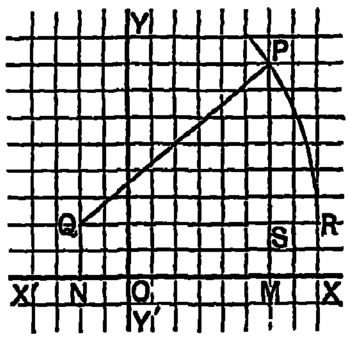


FIG 2

(iii) Take 6 steps to the *right*, then *no steps either up or down* from OX Thus the resulting point M is on the axis of x

(iv) The point (0, 0) obviously represents the origin O

To find the distance between Q and P, draw an arc of a circle with centre Q and radius QP Let this arc cut the horizontal line through Q at R Then QP=QR

But QR=10 units, each of which is one-tenth of an inch

Thus QP=1 inch

Otherwise By Geometry, $QP^2 = QS^2 + SP^2$

$$= 8^2 + 6^2 = 100,$$

$$\therefore QP = 10 \text{ units} = 1 \text{ inch.}$$

EXAMPLE 2 *A ship sails from harbour, first she sails 4 miles due West to a fort, thence 6 miles due South, then 6 miles due East, and then 11 miles due North Find to the nearest mile her final distance from the fort*

Here we may conveniently take the origin to denote the position of the harbour, and mark the axes WOE, NOS in order to shew the points of the compass Let each division of the paper represent one mile, then 4 steps to the left brings us to P which represents the fort. From this point the ship's course is shewn by the dotted lines, and the final position is T A circle described with centre P and radius PT cuts OE at V Then PT = PV, which is very nearly 8 divisions from P Thus to the nearest mile the distance between P and T is 8 miles

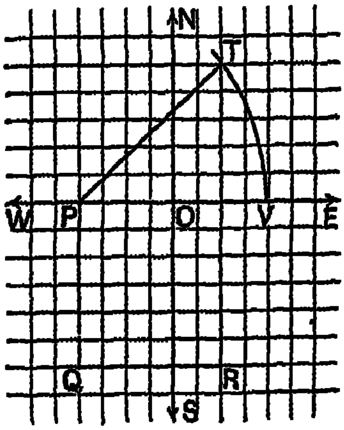


FIG 8

EXAMPLES XI. a.

[*The following examples are intended to be done mainly by actual measurement on squared paper, where possible, they should also be verified by calculation Unless otherwise stated one-tenth of an inch should be taken as unit*]

Plot the following pairs of points and draw the line which joins them:

1. (1, 3), (6, 2)	2 (5, 0), (4, 7)	3. (4, 0), (0, 4)
4 (−2, 1), (3, 4)	5. (6, 3), (−7, 0)	6 (−3, 0), (0, −5)
7. (0, 0), (−2, −4)	8. (−5, 5), (3, −3)	9. (3, 3), (−5, −5)

Taking 1 inch as unit plot the following pairs of points and draw the line which joins them

10. (3 5, 2 4), (0, −3 2)   11 (−1 6, 2 3), (4 0, −1 7)

12 Plot the points (5, 5), (−5, 5), (−5, −5), (5, −5) How many small squares are there in the figure formed by joining the points?

13. Plot the points (3, 4), (−3, 4), (−3, −2), (3, −2) What kind of figure is obtained by joining these points? How many units of area does it contain?

14. Plot the points (0, 0), (8, 0), (3, 6), and shew that they form a triangle containing 24 units of area

15 Draw the triangle whose vertices are (0, 0), (0, 12), (6, 7), and find its area Shew that the points (0, 0), (0, 12), (6, 0) determine a triangle of equal area Explain this result geometrically

16 Plot the points (2, 4), (−1, −2), and shew that they lie on a line passing through the origin Name the coordinates of other points on this line

17. Plot the following points, and shew experimentally that each set lie in one straight line

(i) (−4, −9), (2, 0), (4, 3),

(ii) (−6, −7), (0, 3), (3, 8)

18. Plot the points (2, 4), (2, 8), (−6, 8), (−6, 4) If each division of the paper represents one mile, how many square miles are there in the rectangle formed by joining these points?

19 Plot the following pairs of points, and in each case calculate the distance between them [See Art 128, Ex 1]

(i) (0, 5), (12, 0), (ii) (3, 9), (9, 1); (iii) (5, 6), (0, −6),

(iv) (5, −8), (−4, 4), (v) (16, 16), (6, −8), (vi) (15, 19), (3, 3)

Verify your calculation by measurement

20 Shew that the following points are all at the same distance from the origin.

(0, 10), (8, 6), (−6, 8), (−10, 0), (−8, −6), (6, −8)

21 How far will a man be from his starting point after walking North for 9 miles and then East for 12 miles?

22 A man walks 2 miles due West and then 3 miles due South How far will he have to walk in order to reach a place 2 miles due East of his starting point?

23 The course for a yacht race is marked off by 5 buoys as follows the second is 3 mi S of the first, the third 6 mi E of the second, the fourth 11 mi N of the third, and the last 12 mi W of the fourth How far in a straight line is the last buoy from the first?

24 Shew that the points (−3, 3), (7, 3), (5, 9) are the vertices of an isosceles triangle Calculate the lengths of the equal sides Verify by measurement

25 Find the perimeter of the triangle whose vertices are the points (1, 4), (6, 16), (15, 4)

26 Plot the two following series of points

(i) (3, 0), (3, 4), (3, 6), (3, −1), (3, −4),

(ii) (−3, 7), (0, 7), (2, 7), (4, 7), (7, 7),

and shew that they lie on two lines parallel respectively to the axis of y and the axis of x *What are the coordinates of the point at which they intersect?*

27 Draw the figure whose angular points are given by

(0, −3), (8, 3), (−4, 8), (−4, 3), (0, 0)

Find the lengths of its sides, taking the points in the above order

28. Plot the following series of points

(i) (2, 2), (−5, −5), (0, 0), (8, 8), (−1, −1);

(ii) (2, 0), (4, 0), (−3, 0), (8, 0), (−6, 0),

(iii) (5, 6), (0, 6), (−3, 6), (2, 6), (−7, 6),

(iv) (5, 0), (5, −1), (5, 3), (5, −8), (5, 10)

In each set state a common property possessed by all the points in that set

29 Find by trial a series of points with integral coordinates which satisfy the equation $3y=2x$, and shew experimentally that they all lie on a straight line through the origin

30 If $y=3x+9$, find the values of y when x has the values 0, 1, −2, −3, −4 Plot the points given by these pairs of values and shew experimentally that they lie on a straight line Where does it cut the axis of y?

31. If $y=\frac{2x+7}{3}$, find the values of y when x has the values 0, 1, 3, −2, −5, and shew that the points determined by these values lie on a straight line

32 Plot the points

(13, 0), (0, −13), (12, 5), (−12, 5), (−5, −12), (5, −12)

Find their locus (i) by measurement, (ii) by calculation

[*It will be convenient here to take three-tenths of an inch as the unit*]

Function Graph of a Function.

129 Any expression which involves a variable quantity x, and whose value depends on that of x, is called a **function of x.**

The words "function of x" are often briefly expressed by the symbol $f(x)$ If two quantities x and y are connected by a relation $y=f(x)$, by giving to x a series of numerical values for x, we can obtain a corresponding series of values for $f(x)$, that is for y If these are set off as abscissæ and ordinates respectively, we plot a succession of points We thus arrive at a line, straight or curved, which is known as the **graph** of the *function* $f(x)$, or the **graph** of the *equation* $y=f(x)$ Thus the graph of the *function* $2x+5$ is the same as the graph of the *equation* $y=2x+5$

130 Before going further the pupil should verify by trial each of the following statements

(i) The coordinates of the origin are (0, 0)

(ii) For every point on the axis of x the value of y is 0

Thus the graph of $y=0$ *is the axis of* x

(iii) For every point on the axis of y the value of x is 0

Thus the graph of $x=0$ *is the axis of* y

(iv) The graph of all points which have the same abscissa is a line parallel to the axis of y

Thus on page 99, Ex 26, (i) gives a line parallel to the axis of y, and this line is the graph of $x=3$

(v) The graph of all points which have the same ordinate is a line parallel to the axis of x

Thus on page 99, Ex 26, (ii) gives a line parallel to the axis of x, and this line is the graph of $y=7$

(vi) The distance of any point P(x, y) from the origin is given by $OP^2=x^2+y^2$

131 We now return to the expression $2x+5$ discussed in Art 125 Using the same values of x as before, and putting y to represent the value of the expression, we have the following table of values

x	3	2	1	0	−1	−2	−3
$y=2x+5$	11	9	7	5	3	1	−1

If we now plot the points given by each pair of values we mark L, M, N, P, Q, R, S in the adjoining figure

It will be seen that they all lie on a *straight line* This line may be produced in either direction, and is called the **graph** of the expression $2x+5$

Since y is always equal to $2x+5$, the variations of this expression are seen at a glance by noting the values of the ordinates of the different points

The advantage of this graphical method of illustration is that we can read off from the graph the value of y (that is, of the expression $2x+5$) for *any* value of x

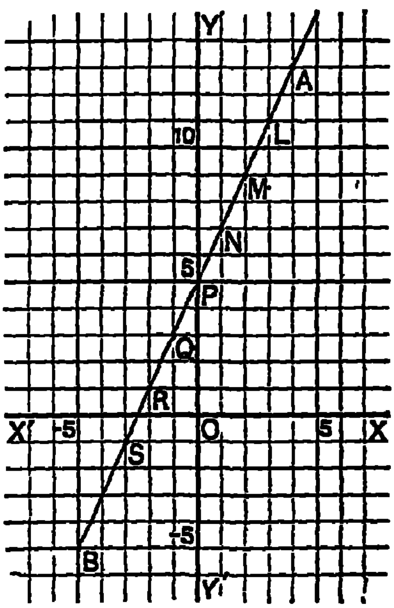


FIG 4

Thus, from the graph,

when $x=4$, y (or $2x+5$)$=13$, at the point A;

and when $x=-5$, $y=-5$, at the point B

132 The following examples deserve particular attention

EXAMPLE 1 *Plot the graph of* y=x

When $x=0$, $y=0$, *thus the origin is one point on the graph*

Also, when $x=1, 2, 3, \quad -1, -2, -3, \quad ,$

$y=1, 2, 3, \quad -1, -2, -3,$

Thus the graph passes through O, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by the straight line POP′ in Fig 5 below

The pupil may here plot the graphs of

$$y=-x, \quad y=2x, \quad y=3x,$$

shewing that each equation represents a line through the origin

EXAMPLE 2 *Plot the graph of* y=x+3

Arrange the values of x and y as follows

x	3	2	1	0	−1	−2	−3	
y	6	5	4	3	2	1	0	

By joining these points we obtain a line MN parallel to that in Example 1

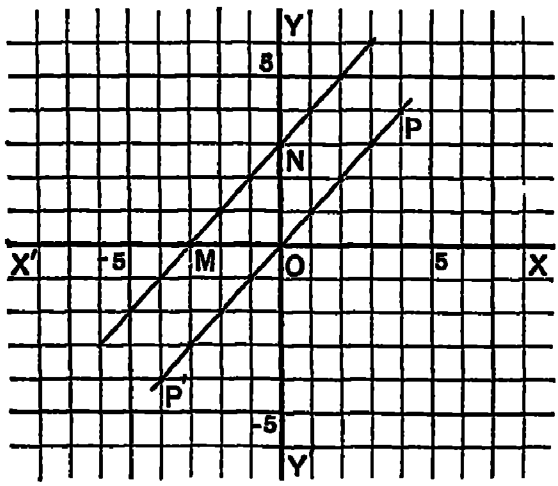


FIG 5

NOTE By observing that in Example 2 each ordinate is 3 units greater than the corresponding ordinate in Example 1, the graph of $y=x+3$ may be obtained from that of $y=x$ by simply producing each ordinate 3 units in the positive direction

In like manner the equations

$$y=x+5, \quad y=x-5$$

represent two parallel lines on opposite sides of $y=x$ and equidistant from it, as the pupil may easily verify for himself

EXAMPLE 3 *Plot the graphs represented by the equations*

$$\text{(i) } 3y=2x, \quad \text{(ii) } 3y-2x=4, \quad \text{(iii) } 3y+5=2x$$

First put the equations in the equivalent forms

$$\text{(i) } y=\frac{2x}{3}; \quad \text{(ii) } y=\frac{2x}{3}+\frac{4}{3}, \quad \text{(iii) } y=\frac{2x}{3}-\frac{5}{3},$$

and in each case find values of y corresponding to

$$x=-3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3$$

For example, in (ii) we have the following values of y

$$y=-\frac{2}{3}, \ 0, \ \frac{2}{3}, \ \frac{4}{3}, \ 2, \ \frac{8}{3}, \ \frac{10}{3}$$

To avoid fractions it will be found convenient to take *three* divisions of the paper as our unit

The graphs will be found to be as in Fig 6

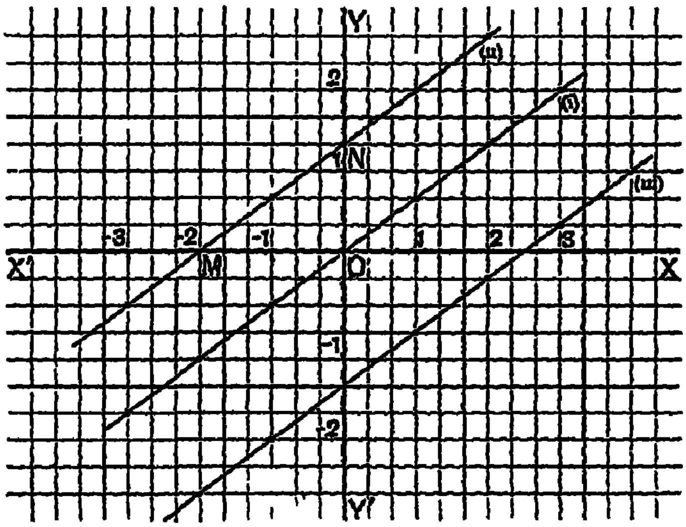


FIG 6

Each graph should be worked out in full by the pupil.

EXAMPLES XI. b

By finding five points on each, plot the graphs of the following equations, shewing each set of three on the same diagram

1 (i) $y=4x$, (ii) $y=4x+2$, (iii) $y=4x-5$

2 (i) $y=-5x$, (ii) $y=-5x+6$, (iii) $y=-5x-10$

3 (i) $y+x=0$, (ii) $y+x=8$, (iii) $y+4=x$

4 (i) $2y-3x=0$, (ii) $2y=3x+2$, (iii) $3y+2x=0$

133 The points where a graph cuts the axes can always be found by putting $y=0$, $x=0$ successively in the equation Thus in $3y-2x=4$, equation (ii) on the last page,

when $y=0$, $x=-2=\mathrm{OM}$ in the figure,

when $x=0$, $y=\frac{4}{3}=\mathrm{ON}$ " "

The distances OM, ON are known as the *intercepts on the axes*

134 The pupil who has worked intelligently through the foregoing examples should now be prepared for the following inferences

(i) For all numerical values of a the equation $y=ax$ represents a straight line *through the origin*

If a is positive, x and y have the same sign, and the line lies in the first and third quadrants, if a is negative, x and y have opposite signs, and the line lies in the second and fourth quadrant In either case a is called the slope or gradient of the line

(ii) For all numerical values of a and b the equation $y=ax+b$ represents a line parallel to $y=ax$, and cutting off an intercept b from the axis of y

The graph of $y=ax+b$ is fixed in position as long as a and b retain the same values

If a alone is altered, the line has a different direction but still cuts the axis of y at the same distance (b) from the origin

If b alone is altered, the line is still parallel to $y=ax$, but cuts the axis of y at a different distance from the origin, further or nearer according as b is greater or less

Since the values a and b fix the position of the line we are considering in any one piece of work, they are called the **constants** of the equation

NOTE The *slope* of $y=ax+b$ is the same as that of $y=ax$

(iii) From the way in which the plotted points are determined from an equation, it follows that the graph passes through all points whose coordinates satisfy the equation, and through no other points

135 Since every equation involving x and y only in the first degree can be reduced to one of the forms $y=ax$, $y=ax+b$, it follows that *every simple equation connecting two variables represents a straight line* For this reason an expression of the form $ax+b$ is said to be a **linear function** of x, and an equation such as $y=ax+b$, or $ax+by+c=0$, is said to be a **linear equation.**

136 Since a *straight* line can always be drawn when *any* two points on it are known, in drawing a *linear* graph only two points need be plotted The points where the line meets the axes will always suffice, though they are not always the best to select.

EXAMPLE *Draw the graph of* $4x-3y=13$

When $y=0$, $x=\frac{13}{4}$ (intercept on the x-axis),

and when $x=0$, $y=-\frac{13}{3}$ (intercept on the y-axis)

As both of these values involve fractions of the unit, it would be difficult to draw the line accurately In such a case it is better to find by trial *integral* values of x and y which satisfy the equation

Thus when $x=1$, $y=-3$, and when $y=1$, $x=4$

The graph can now be drawn by joining the points $(1, -3)$, $(4, 1)$

EXAMPLES XI b (*Continued*)

By finding the intercepts on the axes, or by joining any two convenient points, plot on the same diagram the graphs of

5 (i) $2x+3y=6$, (ii) $2x+3y=12$, (iii) $3x-2y=18$

6 (i) $x+2y=0$, (ii) $x+2y=5$, (iii) $y-2x=3$

7. (i) $x-2=0$, (ii) $y-3=0$, (iii) $3x=2y$

8 (i) $x+8=0$, (ii) $4y+3x=0$, (iii) $y-6=0$

9 Draw in one figure the graphs represented by

$$y=5-3x, \quad 3y=x+5,$$

and find by measurement the coordinates of the point where they meet.

10 Draw on the sames axes the graphs corresponding to

$$3x+2y=5, \quad 2x+y=4, \quad 4x=2-5y,$$

and shew that they have a common point Find its coordinates

11. Plot six points all having the ordinate equal to 5 What is the equation of the line which passes through these points?

12 Plot the graph of the function $\frac{5x+17}{2}$, and from the graph read off the value of the function when $x=5$, and $x=8$

13 Draw on the same axes the graphs of

$$x=4, \quad x=7, \quad y=3, \quad y=10,$$

and find the number of units of area enclosed by them

14 Taking one tenth of an inch as unit, find the area included between the graphs of $x=-4$, $x=11$, $y=3$, $y=-3$

15 Find the area included between the graphs of

$$x-2=0, \quad y-1=0, \quad 2x+5y=19 \qquad [\textit{Half-inch unit}]$$

16. Draw graphs to shew the variations of the functions $1{\cdot}2x-3$, $3{\cdot}5-3{\cdot}8x$ between the values 0, 1, 2, 3, 4 of x Hence find the value of x which satisfies the equation $1{\cdot}2x-3=3{\cdot}5-3{\cdot}8x$

137 **Measurement on Different Scales.** In the foregoing examples we have measured abscissæ and ordinates on the same scale for the sake of simplicity, but there is no necessity for so doing, and it will often be convenient to measure the variables on different scales so as to get a better diagram

For example, in drawing the graph of $y=6x+3$,

when x has the values 0, 1, 2, 3, 4,

the corresponding values of y are 3, 9, 15, 21, 27

Thus some of the ordinates are much larger than the corresponding abscissæ, and rapidly increase as x increases

If these points are plotted with x and y measured on the same scale it will be found that with a small unit (such as one-tenth of an inch) the graph is inconveniently placed with regard to the axes If a larger unit is used the graph requires a diagram of inconvenient size

[The pupil should prove this for himself by trial]

The inconvenience may be avoided by measuring the values of y on a considerably smaller scale than those of x For example, let us take one inch as unit for x, and one-tenth of an inch as unit for y, then the graph of $y=6x+3$ will be found to be as in Fig 7

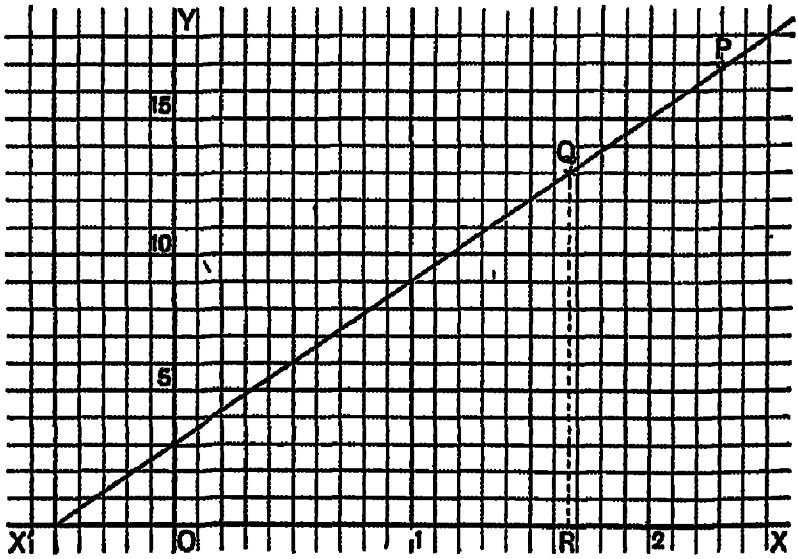


FIG 7

NOTE Speaking generally, whenever one variable increases much more rapidly than the other, a small unit should be chosen for the rapidly increasing variable and a large one for the other

138 When a graph has been accurately drawn from plotted points, it can be used to *read off* (without calculation) corresponding values of the variables at intermediate points The process is known as **interpolation** Or if one coordinate of a point on the graph is known the other can be found approximately by measurement

EXAMPLE *From the graph of the expression* $6x+3$ *find its value when* $x=2\ 3$ *Also find the value of* x *which will make the expression equal to* 13

Put $y=6x+3$, then the graph is that given in Fig 7 Now we see that $x=2\ 3$ at the point P, and here $y=17$, nearly

Again $y=13$ at the point Q, and $x=1\ 66$ very nearly In reading off this last result we observe that OR is greater than 1 6 and less than 1·7, and we mentally divide the tenth in which R falls into *ten* equal parts (*i e* into *hundredths of the unit*) and judge as nearly as possible how many of these hundredths are to be added to 1 6

EXAMPLES XI. c

[*In some of the following Examples the scales are specified, in others the pupil is left to select suitable units for himself When two or more equations are involved in the same piece of work, their graphs must all be drawn on the same scale In every case the units employed should be marked on the axes*]

1 By finding the intercepts on the axes draw the graph of $4x+5y=14$ [*Take* 1 0″ *as unit on both axes*]

2 Draw the graphs of

$$\text{(i) } 15x+20y=6, \qquad \text{(ii) } 12x+21y=14$$

[*In* (i) *take* 1 *inch as unit, and in* (ii) *take six-tenths of an inch as unit In each case explain why the unit is convenient*]

3. Taking the x-unit as 1 0″, and the y-unit as 0 1″, draw the graph of the function $\dfrac{36-5x}{3}$ From the graph find the value of the function when $x=1\ 8$, also find for what value of x the function becomes equal to 8

4. From the graph of the function $11x+6$ find its value when $x=1\ 28$ Also find the approximate value of x which will make the function equal to 26

5. On one diagram draw the graphs of

$$y=5x+11, \qquad 10x-2y=15$$

What is the slope of these graphs? Find the length of the y-axis intercepted between them

6 With the same units as in Ex 3 draw the graphs of

$$x+0\ 35y=2, \qquad 10x-6y=1,$$

and find the coordinates of the point at which they intersect

7. Draw the graph of the function $7x+5$, and read off its value when $x=1\ 5$ Also find the approximate value of x which will make the function equal to 22

Some Applications of Graphs.

139 A graph accurately drawn on a suitable scale may often be used as a 'ready reckoner'

It is particularly important that the pupil should draw his diagrams on a sufficiently large scale, and that he should be careful in the choice of units These should always be clearly marked on the axes

In some of the examples which follow the diagrams are limited by the size of the page In such cases the pupil is recommended to draw his own graphs on a considerably larger scale

EXAMPLE 1 *Given that 5 kilograms are roughly equal to 11 pounds, shew graphically how to express any number of kilograms in pounds Express $7\frac{1}{2}$ pounds in kilograms, and 7 kilograms in pounds*

Since 11 pounds = 5 kilograms,

$$x \text{ pounds} = \frac{5}{11}x \text{ kilograms}$$

Hence if y kilograms are equivalent to x pounds, we have the equation $y=\frac{5}{11}x$ connecting the variables x and y

This is a straight line *through the origin*, and when $x=11$, $y=5$

On the horizontal axis let the scale be 1 inch to 5 lbs, so that each pound is represented by 0·2″, and on the vertical axis let the scale be 1 inch to 10 Kg, so that each kilogram is represented by 0 1″

The required graph is obtained at once by joining the origin to the point P whose coordinates are 11 and 5

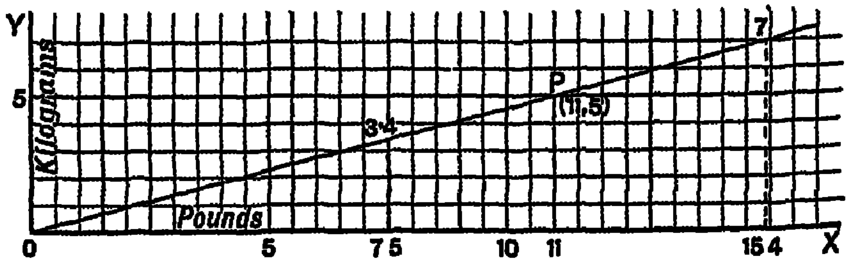


FIG 8

Now, by measurement, when $x=7$ 5, $y=3$ 4

Thus $7\frac{1}{2}$ lbs = 3 4 Kg

And when $y=7$, $x=15$ 4

Thus 7 Kg = 15 4 lbs

EXAMPLE 2 *The salary of a clerk is increased each year by a fixed sum After 6 years' service his salary is raised to £128, and after 15 years to £200 Draw a graph from which his salary may be read off for any year, and determine from it* (i) *his initial salary*, (ii) *the salary he should receive for his 21st year*

Let £y represent the salary *after* x years, £a, £b the annual increase and the initial salary respectively.

Then we have the relation $y=ax+b$, which represents a straight line.

When $x=6$, $y=128$, and when $x=15$, $y=200$. The line can be drawn by joining these points.

The graph presents no difficulty and is left as an exercise for the pupil. A scale of 10 years to the inch horizontally, and £80 to the inch vertically will be found convenient. Thus each vertical division of the paper will represent £8.

To find the initial salary, we have only to find the intercept on the y-axis, where $x=0$. This gives £80.

The salary for the 21st year (that is *after* 20 years) is given by the ordinate corresponding to an abscissa 20, and will be found to be £240.

EXAMPLES XI. d.

1. Given that 35 yards are approximately equal to 32 metres, draw the graph shewing the equivalent of any number of yards when expressed in metres.

Shew that 22.2 yards = 20.3 metres, approximately.

[*Take* 1 *inch to the yard along the axis of* x, *and* 1 *inch to the metre along the axis of* y. *Join the origin to the point whose coordinates are* (3.5″, 3.2″), *and read off the ordinate corresponding to an abscissa* 2.22″.]

2. Given that 5.5 centimetres are approximately equal to 2.15 inches, draw a graph to convert any number of inches into centimetres, or centimetres into inches. Express 1 inch in centimetres, and 4 centimetres in inches.

[*By taking* 1 *inch as unit on each axis the ruled lines will mark tenths of the unit. We can thus read* accurately *to one place of decimals. The second place can be judged by the eye as explained in the example of* Art. 138.]

3. Given that 20 litres = 4.4 gallons, draw a graph to convert litres to gallons, or gallons to litres.

Express (i) $2\frac{1}{2}$ gallons in litres; (ii) 20.9 litres in gallons.

[*Take* 1 *gallon to the inch on the axis of* x, *and* 10 *litres to the inch on the axis of* y.]

4. If 24 men can reap a field of 29 acres in a given time, find roughly by means of a graph the number of acres which could be reaped in the same time by 15, 33, and 42 men respectively.

5. Draw a graph to serve as a ready reckoner for wages at £15 a year. Read off to the nearest penny the wages for 1 week, 20 days, and 51 days.

How long had a servant worked who received £2. 11*s*. as wages?

[*Take* 0.1″ *to represent* 1 *day on the* x-*axis and the same unit to represent* 1 *shilling on the* y-*axis. Then since the wages for* 73 *days amount to* £3, *the graph is at once obtained by joining the origin to the point* (73, 60).]

6 If 1 cwt of coffee costs £9 12*s*, draw a graph to give the price of any number of pounds Read off the price (to the nearest penny) of 13 lbs, 21 lbs, 23 lbs

[*Suppose* x *lbs cost* y *shillings, then from the data*

$$\frac{y}{192}=\frac{x}{112}, \text{ whence } y=\frac{12}{7}x$$

Draw the graph of this equation, and read off the values of y *when* x=13, 21, 23]

7. If the wages for a day's work of 8 hours are 4*s* 6*d*, draw a graph to shew the wages for any fraction of a day, and find (to the nearest penny) what ought to be paid to men who work $2\frac{1}{2}$, $3\frac{1}{2}$, $6\frac{1}{2}$ hours respectively How many hours' work might be expected for 2*s* 10*d* ?

[*Take* 1 *inch to represent* 1 *hour, and one-tenth of an inch to represent* 1 *penny*]

8. Draw a graph to shew the connection between the retail and cost prices of certain articles, on the supposition that they are retailed so as to make a profit of 20 per cent From the graph find the cost price of articles which were sold for 9*d*, 2*s*, 3*s* 6*d* Also find, to the nearest penny, the retail price of articles which cost 10*d*, 1*s* 9*d*, 3*s*

9. The highest marks gained in an examination were 136, and these are to be raised so that the maximum is 200 Shew how this may be done by means of a graph, and read off, to the nearest integer, the final marks of candidates who scored 71 and 49 respectively

10. For a certain book it costs a publisher £100 to prepare the type and 2*s* to print each copy Find an expression for the total cost in pounds of x copies Make a diagram on a scale of 1 inch to 1000 copies and 1 inch to £100 to shew the total cost of any number of copies up to 5000 Read off the cost of 2500 copies, and the number of copies costing £525

11. By measuring time along OX (1 inch for 1 hour), and distance along OY (1 inch for 10 miles) shew that a line may be drawn from O through the points (1, 8), (2, 16), (3, 24), to indicate distance travelled towards Y in a specified time at 8 miles an hour

A starts from London at noon at 8 miles an hour, two hours later *B* starts, riding at 12 miles an hour Find graphically at what time and at what distance from London *B* overtakes *A* At what times will *A* and *B* be 8 miles apart? If *C* rides after *B*, starting at 3 p m at 15 miles an hour, find from the graphs

(i) the distances between *A*, *B*, and *C* at 5 p m,

(ii) the time when *C* is 8 miles behind *B*

12 At noon *A* starts to walk at 6 miles an hour, and at 1 30 p m *B* follows on horseback at 8 miles an hour When will *B* overtake *A*? Also find

(i) when *A* is 5 miles ahead of *B*,

(ii) when *A* is 3 miles behind *B*

140 We have seen that the variations of a linear function $f(x)$ can always be shewn graphically by first selecting values of x and y which satisfy the equation $y=f(x)$ and then drawing a line through the plotted points The method is quite general, and it may be applied *when the function is not linear* In such a case it will be found that the resulting graph will take the form of some *curve* differing in shape according to the *form* of the equation which connects the variables Moreover, whenever two variable quantities depend on each other so that a change in one produces a corresponding change in the other, we can draw a graph to shew their variations without knowing any algebraical relation between them, *provided that we are furnished with a sufficient number of corresponding values accurately determined*

EXAMPLE 1 *Draw a graph to shew the variations of the function* x^2-2x *between the values* -2 *and* 4 *of* x

Put $y=x^2-2x$, and use the following table of values, taking 0 5″ as unit for x, and 0 1″ as unit for y

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	-2	-4	-6	-8
y	8	3	0	-1	0	3	8

If the points we have now determined are plotted and connected by a continuous line drawn with a free hand, we shall obtain the curve shewn in Fig 9

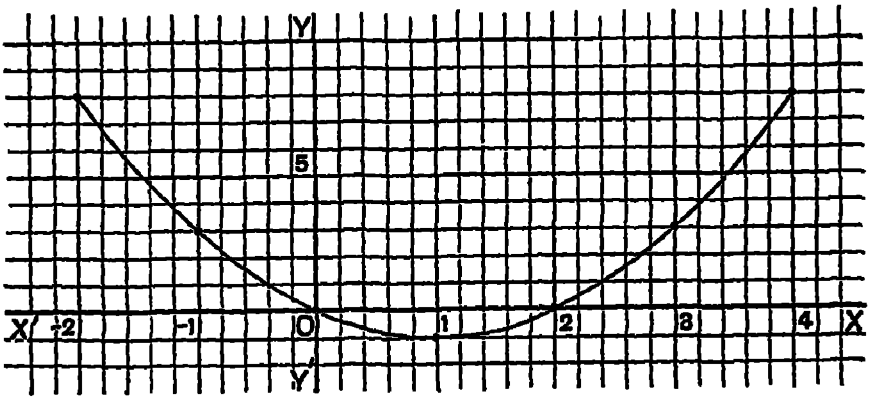


FIG 9

It may be observed that by taking other values of x and y which satisfy the equation $y=x^2-2x$ the curve may be extended in either direction

The least (or minimum) value of the function is shewn by the least value of y on the graph This is at the lowest point Thus the minimum value of the function is -1

EXAMPLE 2 *Draw a graph to shew the relation between* x *and* y *from the following corresponding values of* x *and* y

$$x = -8, \quad -5, \quad -2, \quad 1, \quad 5, \quad 10;$$
$$y = 2, \quad 5, \quad 6\ 8, \quad 8, \quad 8\ 8, \quad 8\ 4$$

From the graph read off, as accurately as possible, the values of y *corresponding to* x=3 *and* x=-3

Taking 0 1″ as unit on each axis the curve through the given points will be as in Fig 10

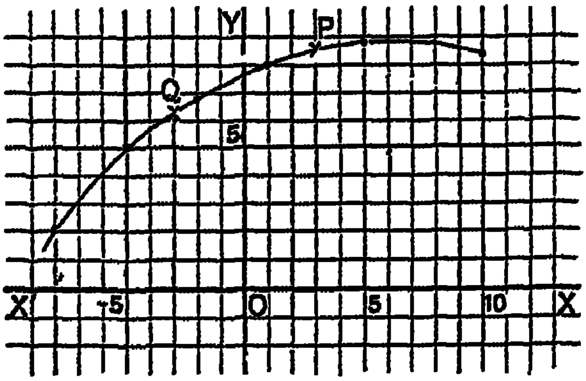


FIG 10

When $x=3$, $y=8\ 6$, at the point P
When $x=-3$, $y=6\ 3$, at the point Q

Here as we have no equation connecting x and y we cannot plot any points except those whose coordinates are given, but we can interpolate *intermediate* values

EXAMPLES XI. e

[*The selection of suitable units for x and y is very important here* *See* Art 137, Note]

Plot the points given by the tables in Examples 1-7, and in each case draw a graph passing through all the points

1 $\qquad x= -1, \quad 0, \quad 2, \quad 2\ 5, \quad 4, \quad 5\ 5,$
$\qquad y= 15\ 5, \quad 12\ 5, \quad 6\ 5, \quad 5, \quad 0\ 5, \quad -4$

From the graph find as accurately as possible the coordinates of the point where it cuts the axis of x

2 $\qquad x= -5, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3,$
$\qquad y= -36, \quad -25, \quad -14, \quad -3, \quad 8, \quad 19, \quad 30$

Find approximately the value of y when $x=2\ 6$, and the value of x when $y=4{\cdot}7$ [*Use a large unit for x*]

3. $\qquad x= -5, \quad -4, \quad -1, \quad 0, \quad 1, \quad 3, \quad 5,$
$\qquad y= 28, \quad 18, \quad 0 \quad -2, \quad -2, \quad 4, \quad 18$

Find the value of x when $y=10$, and the value of y when $x=-3\ 5$.

4. $x = 0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12;$
$y = 0,\ 0{\cdot}5,\ 2,\ 4{\cdot}5,\ 8,\ 12{\cdot}5,\ 18$

Find, to the nearest integer, the value of y when $x=11$

5. $x = 2,\ 1{\cdot}5,\ 1,\ 0{\cdot}5,\ 0,\ -0{\cdot}5,\ -1,\ -1{\cdot}5,\ -2,\ -2{\cdot}5;$
$y = -21,\ -12,\ -5,\ 0,\ 3,\ 4,\ 3,\ 0,\ -5,\ -12.$

[*Take* 0·4″ *as unit for* x, *and* 0·1″ *as unit for* y]

6. $x = 25,\ 10,\ 5,\ 3\tfrac{1}{3},\ 2\tfrac{1}{2},\ 1\tfrac{2}{3},\ 1,$
$y = 1,\ 2\tfrac{1}{2},\ 5,\ 7\tfrac{1}{2},\ 10,\ 15,\ 25$

Also draw with the same pair of axes the graph corresponding to the above values of x and y, each taken negatively

7. $x = -2,\ -1{\cdot}5,\ -1,\ -0{\cdot}5,\ 0,\ 0{\cdot}5,\ 1,\ 1{\cdot}5,\ 2;$
$y = -8,\ -3{\cdot}375,\ -1,\ -0{\cdot}125,\ 0,\ 0{\cdot}125,\ 1,\ 3{\cdot}375,\ 8$

[*Take* 1·0″ *as unit for* x, *and* 0·2″ *as unit for* y]

8 Make a table giving the value of the expression x^2-3x when x has the values $-1, 0, 1, 2, 3, 4$ taking 1 inch as the unit Then draw a curve that will shew how x^2-3x varies while x varies from -1 to 4

In the equations below, numbered 9 to 21, choose values for one of the variables and find corresponding values of the other Tabulate these, and in each case draw a graph through the points given by them

9 $y=x^2$ 10 $4y=x^2$ 11 $16x=y^2$

12 $y=8x^2$ [*Take the* x-*unit ten times as great as the* y-*unit*]

13 $y=x(x+1)$ 14 $y=4x-x^2$ 15 $x=(y-1)^2$

16. $y=x^2-5x+3$ 17 $8y=x^3$ 18 $xy=16$

19 $y+1=(x-2)^2$ 20. $y=x^3-8x$ 21 $(x+1)(y+2)=60$

Graphs of Statistics

141 There are some cases in which we have to deal with a limited number of corresponding values of two variables obtained by *observation* or *experiment* In such cases the data may involve inaccuracies, and consequently the positions of the plotted points cannot be absolutely relied on Moreover, as there is no mathematical law connecting the variables, we cannot correct irregularities in the graph by selecting other points whose coordinates satisfy a given equation One method of procedure is to join successive points by *straight* lines The graph will then be represented by an irregular broken line, sometimes with abrupt changes of direction as we pass from point to point In cases where no great accuracy of detail is required this simple method is often used to illustrate statistical results A familiar instance is a Weather Chart giving the height of the barometer at equal intervals of time.

The chief disadvantage of the above method is that, although it gives a general idea of the total change that has taken place between the plotted points, it furnishes no accurate information with regard to intermediate points

EXAMPLE *The readings of a thermometer taken at intervals of 2 hours beginning at* 10 a m *were* 62 5°, 64°, 69 6°, 69°, 66 5°, 65 7°

Draw a chart to shew the changes of temperature

Let the hours be measured on the horizontal axis, taking 5 divisions to represent each interval of 2 hours, beginning at 10 a m On the vertical axis let each division represent 1° of temperature, beginning at 60°

After plotting the points furnished by the data of the question, and joining them by straight lines we obtain the broken line PQRSTV shewn in Fig 11

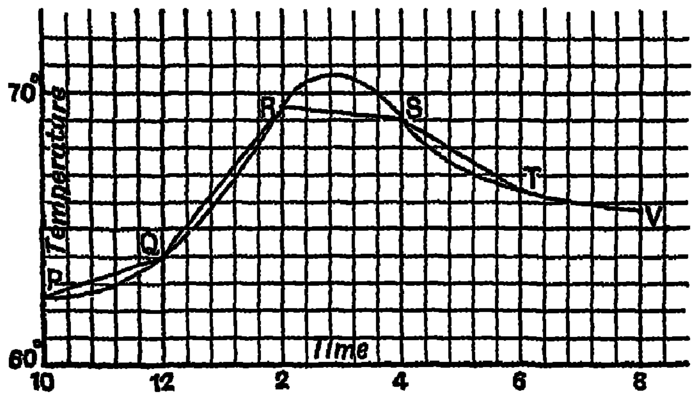


FIG 11

But it is contrary to experience to suppose that the abrupt changes of direction at Q and R accurately represent the change of temperature at noon and 2 p m respectively Moreover, it is probable that the maximum temperature occurred at some time between 2 and 4, and not at the time represented at R, the highest of the plotted points Now if the chart had been obtained by means of a self-registering instrument, the graph (representing change from instant to instant instead of at long intervals) would probably have been somewhat like the continuous waving curve drawn through the points previously registered From this it would appear that the maximum temperature occurred shortly before 3 p m , and that TV (which represents a very gradual change) is *the only* portion of the broken line which records with any degree of accuracy the variation in temperature during two consecutive hours

142 Although in the last example we were able to indicate the *form* of the curved line which from the nature of the case *seemed most probable*, it is evident that it is possible to draw any number of curves through a limited number of plotted points In such a case the best plan is to draw a curve to lie *as evenly as possible* among the plotted points, passing through some perhaps, and with the rest fairly distributed on either side of the curve As an aid to drawing an even continuous curve (usually called a *smooth* curve), a thin

piece of wood or other flexible material may be bent into the requisite shape, and held in position while the line is drawn When the plotted points lie approximately on a straight line, the simplest plan is to use a piece of tracing paper on which a straight line has been drawn When this has been placed in the right position the extremities can be marked on the squared paper, and by joining these points the approximate graph is obtained

EXAMPLE 1 *The following table gives statistics of the population of a certain country, where* P *is the number of millions at the beginning of each of the years specified*

Year	1830	1835	1840	1845	1850	1855	1860
P	20	22	24 5	28	31	36	41

Let t *be the time in years from* 1830 *Plot the values of* P *vertically and those of* t *horizontally and shew the relation between* P *and* t *by a simple curve passing fairly evenly among the plotted points Find what the population was at the beginning of the years* 1847 *and* 1858

Take one-tenth of an inch as unit in each case, also it will be convenient if we begin measuring abscissæ at 1830, and ordinates at 20

The graph is given in Fig 12, it will be seen that it passes exactly through three of the points and lies evenly among the others

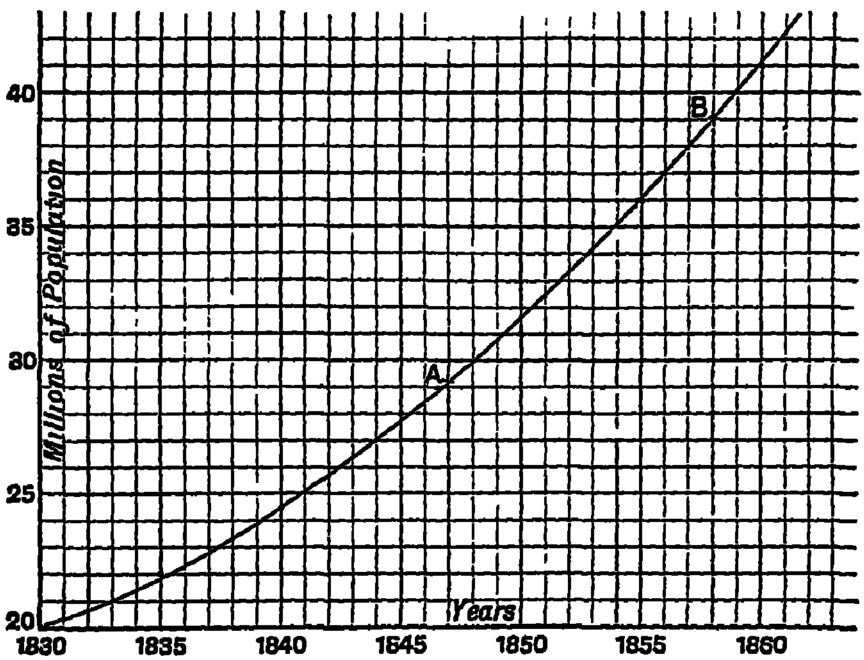


FIG 12

The population in 1847 and 1858, at the points A and B respectively, will be found to be $29\frac{1}{4}$ millions and 39 millions

EXAMPLE 2 *The following table gives corresponding values of* x *and* y

x	3	6 5	12	14	21	28 6	31 5
y	4	4 8	6 7	7	8 5	11	11 5

Supposing these values to involve errors of observation, draw the graph approximately Find the value of x *when* y=11 5 *and the value of* y *when* x=10

The points are plotted in Fig 13 where one-tenth of an inch has been taken as unit on each axis

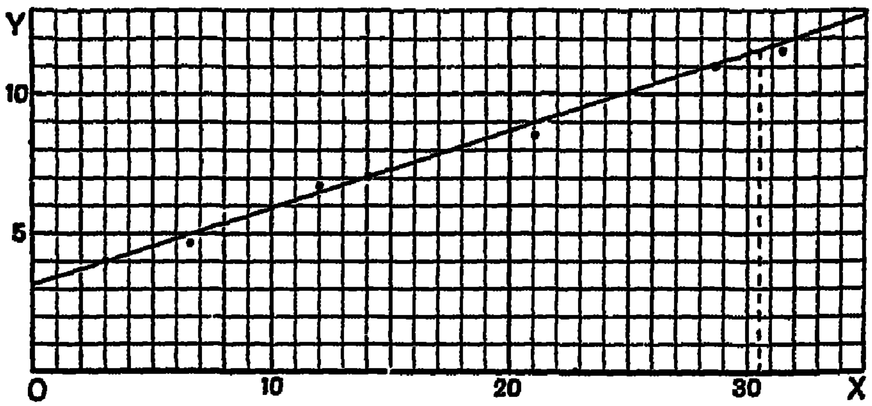


FIG 13

It is seen that the points lie nearly on a straight line By trial it is found that the line through the first and fourth points lies most evenly among the remaining points When this line has been drawn we can read off the required results

When $y=11\ 5$, we have $x=30\ 5$,
$x=10$, $y=5\ 9$

EXAMPLES XI f

[*In Examples* 1–6 *plotted points may be joined by straight lines In other cases the graph is to be a straight line or smooth curve lying evenly among the plotted points*]

1 In successive weeks a boy's place in his Form were as follows

6, 8, 5, 3, 4, 7, 2, 1, 1, 1

Shew these results by means of a graph.

2. The minimum temperatures for the first 10 days in January 1908 were as follows

31 5, 26·2, 24 1, 24 5, 18 0, 17 9, 44 4, 32 3, 29 9, 23 8

Draw a chart to shew these variations

3 Draw a graph illustrating the following scores in a series of 12 innings at cricket

12, 20, 17, 9, 0, 42, 35, 16, 0, 25, 70, 55

4. The following details give the height of the barometer on certain days in Sept 1908

5th, 6th, 8th, 9th, 11th, 12th, 14th, 15th,

30 1, 30 0, 29 5, 29 5, 29 6, 29 8, 30 0, 29 8

Draw a graph to shew these variations, and explain why it cannot be used to read the height of the barometer on the 7th, 10th, and 13th days

5 The highest and lowest prices of Consols for the years 1899 to 1908 were as follows

Year	'99	'00	'01	'02	'03	'04	'05	'06	'07	'08
Highest	$111\frac{1}{2}$	$103\frac{1}{4}$	$98\frac{1}{8}$	98	$93\frac{7}{8}$	$91\frac{7}{16}$	92	$91\frac{1}{4}$	$87\frac{9}{16}$	$88\frac{9}{16}$
Lowest	$97\frac{3}{4}$	$96\frac{3}{4}$	91	$92\frac{1}{8}$	$86\frac{7}{8}$	85	$87\frac{1}{2}$	$85\frac{9}{16}$	$80\frac{3}{4}$	$83\frac{1}{4}$

Make a chart to shew these variations graphically on the same diagram.

[*Take* 1 0" *to* £10 *vertically, beginning at* 80, *and* 0 5" *to* 1 *year horizontally*]

6 The following table gives the census returns of the population (in millions) of Scotland and Ireland in the years specified

Year	1851	1861	1871	1881	1891	1901
Scotland	2 9	3 1	3 4	3 7	4 0	4 5
Ireland	6 6	5 8	5 4	5 2	4 7	4 5

Shew these variations graphically on the same diagram

7 Draw a graph to shew the connection between circumferences and corresponding diameters of different circles from the following data

Diameter 2, 5, 7, 8, 10

Circumference 6 3, 15 7, 22, 25 1, 31 8

Find approximately the circumference of a circle 9 5 inches in diameter, and the diameter of a circle which has a circumference of 18 9 inches

8 The following details taken from "Compound Interest Tables" give approximately the amount of £1 at 4 p c for different periods

No of years 1, 6, 11, 16, 21, 26, 31, 36

Amount 1 04, 1 27, 1 54, 1 87, 2·28, 2 77, 3 37, 4 10

Illustrate these data by means of a curve, and determine graphically (i) the amount of £100 in 15 years, (ii) in how many years £100 will amount to £247

9 The following table gives the population (P) in millions at the end of each of the years specified

Year	1840	1845	1850	1860	1870	1875	1880	1890
P	10	12 1	13 5	19 0	24 2	28 2	31 0	39 4

From a graph read off the population at the beginning of the years 1858 and 1885

10. The following data shew the connection between the areas of equilateral triangles and their bases, in corresponding square and linear units

Area	0 43,	1 73,	3 90,	6 93,	10 82,	15 59
Base	1,	2,	3,	4,	5,	6

Illustrate these results graphically, and determine the area of an equilateral triangle on a base of 2 4 ft

11. In catering for a ball-supper the following scale of charges was given

Number of guests	150,	200,	250,	300,	350,	400,
Charge per head	4*s*,	3*s* 3*d*,	2*s* 9*d*,	2*s* 6*d*,	2*s* 2*d*,	2*s*

Represent these data by a graph, and estimate to the nearest penny, the charge per head for 175, 225, and 375 guests

12. Corresponding values of x and y, obtained by experiment, are given in the following table

x	1	3 1	6	9 5	12 5	16	19	23
y	2	2 8	4 2	5 3	6 6	8 3	9	10 8

Draw the graph which passes most evenly through the points, find the value of y when $x=15$, and the value of x when $y=3$ 6

13 In an Insurance Company the premium (£P) to insure £100 at different ages is given approximately by the following table

Age	20	23	27	30	32	35	40	45
P	1 7	1 8	2 0	2 2	2 3	2 5	2 9	3 5

Illustrate these statistics graphically and estimate to the nearest shilling the premiums for persons aged 25 and 38 Shew also that between the ages 20 and 28 the premiums are very nearly proportional to the ages

14. In a certain machine P is the force in pounds required to raise a weight of W pounds The following corresponding values of P and W were obtained experimentally

P	2 8	3 7	4 8	5 5	6 5	7 3	8	9 5	10 4	11 75
W	20	25	31 7	35 6	45	52 4	57 5	65	71	82 5

Draw the graph connecting P and W, and read off the value of P when W=60 Also find the weight which could be raised by a force of 7 lbs

15 The following details give the population (in millions) of two countries A and B at the beginning of each of the years specified

Year	1850	1860	1870	1885	1890	1900
A	20 4	21 9	25	33 7	38	50
B	23 8	24 7	26 5	32·2	34 9	40 5

Plot the graphs on the same diagram In what year were the populations approximately equal? Find also in what year the population of A exceeded that of B by about 6 5 millions

16 An elastic cord was loaded with weights, and a measurement of its length was taken for each load. Plot a graph to shew the relation between the length of the cord and the loads from the following data

Load in pounds	4 5,	7 5,	10 5,	15,	18,	19 5
Length in feet	10 6,	11 4,	12 2,	13 4,	14 2,	14 6

Find the unstretched length of the cord also determine the weight it will support when its length is 13 feet

17. A manufacturer wishes to stock a certain article in many sizes; at present he has five sizes made at the prices given below.

Length in inches	22,	29,	34,	39,	44
Price	*12s 6d*,	*16s*,	*21s*,	*27s*,	*33s 6d*

Draw a graph to shew suitable prices for intermediate sizes, and find what the prices should be when the lengths are 27 in and 33 in

18. At a given temperature p lbs per square inch represents the pressure of a gas which occupies a volume of v cubic inches Draw a curve connecting p and v from the following table of corresponding values.

p	36	30	25 7	22 5	20	18	16 4	15
v	5	6	7	8	9	10	11	12

Find the volume when the pressure is 29 lbs per square inch.

MISCELLANEOUS EXAMPLES III.

Exercises for Revision.

A.

1. Simplify $(x+2y)^2-(x-2y)^2-(2y)^2$

2. How many miles can a person walk in 75 minutes if he walks x miles in y hours?

3. Add together

$$\tfrac{1}{2}a^3-2a^2b-\tfrac{3}{2}b^3,\quad \tfrac{3}{2}a^2b-\tfrac{3}{4}ab^2+2b^3,\quad -2a^3+ab^2+\tfrac{1}{2}b^3$$

4. What expression must be added to $(2a-3b)x+(4b-3c)y$ to make $(3a+2b)x-(4b+3c)y$?

5. Solve the equations

$$\text{(i)}\ \frac{3x}{4}-\frac{5x}{8}+6=\frac{x}{2},\qquad \text{(ii)}\ 6(x-2)-\tfrac{1}{2}(5-x)=26-7x$$

6. If $\mathsf{P}=(x-1)^2$, $\mathsf{Q}=(x+1)^2$, and $\mathsf{R}=x^2-1$, find the value of $\mathsf{PQ}-\mathsf{R}^2$

7. Divide £1000 between two persons so that one may have £10 more than half what the other has

B

8. Simplify $12(x+y)-[2x-\{3y-2(5x+y)\}]$

9. From the formula $a^2-b^2=(a+b)(a-b)$ find by how much the square of 69843 exceeds the square of 30157

10. Of what dimensions are the first and the last terms of the following expression?

$$\tfrac{1}{3}abc+\tfrac{1}{4}b^2c-\tfrac{1}{2}ad^3-\tfrac{3}{4}ab^3c+ac^2d^3$$

If $a=1$, $b=-2$, $c=3$, $d=0$, find the numerical value of the expression

11. Draw the graph of $3x+4y=7$, by finding its intercepts on the axes [*Take* 1 2" *as unit on each axis*]

12. A man bought $4a$ sheep for $5p$ shillings each, and $5b$ sheep for $4q$ shillings each. How many pounds did he spend?

If he sold the $4a$ sheep for $6p$ shillings each, and the $5b$ sheep for $5q$ shillings each, how many pounds did he gain?

13. Solve the equations

$$\text{(i)}\ 5x-\frac{2x-3}{5}=\frac{17x}{6}+16\tfrac{1}{2};$$

$$\text{(ii)}\ \frac{3}{5}(x-7)-\frac{2}{3}\left(\frac{x}{2}-8\right)=4+\frac{1}{15}(2x+1)$$

14. How may a sum of £10 be paid in sovereigns and half-crowns, so that the number of half-crowns is double the number of sovereigns?

C

15 Subtract the sum of the squares of $2x+3y$ and $2x-3y$ from $(3x+4y)(3x-4y)$ What does the result reduce to when $x=6y$?

16. What value of x will make $\frac{1}{2}(x-1)+\frac{2}{3}(x+2)$ equal to $\frac{9}{4}(x-3)$?

17. How many seconds will it take to travel b yards at the rate of a miles an hour? How many yards will be passed over in b minutes?

18 What expression must be subtracted from $10a^2-11b^2+12c^2$ to leave $20a^2-11b^2-12c^2$, and what expression must be added to your answer to produce $a^2+b^2+c^2$?

19 A train travelling 40 miles an hour takes two hours less in going a certain distance than a train travelling 24 miles an hour What is the distance?

20 If $F=\dfrac{mv^2}{gr}$, find F, to the nearest unit, when

$$m=25, \quad r=12, \quad v=60, \quad g=32\cdot 2$$

21. Plot the graphs of the functions $2x+9$, $\frac{1}{3}(7-4x)$

For what value of x will they be equal?

D

22 Explain the meanings of 93, 9·3, and 9×3, if x stands for 9, and y for 3, write down algebraically the numbers given above

23. Rearrange the following expression in ascending powers of x, and use brackets to shew the coefficients of the different powers

$$36x^2-7ax-ax^3-6b-2bx^2+bx^3-1-7ax^2$$

Find also the value of the expression when $a=2$, $b=0$, $x=-1$

24. Distinguish between the meanings of

(i) $(a+b)(a-b)$, (ii) $(a+b)a-b$, (iii) $a+b(a-b)$

Find the sum of the three expressions

25 By means of squared paper shew that

(i) $(x+3)(x+5)=x^2+8x+15$, (ii) $a(a+b)=a^2+ab$

26. Solve the equations

(i) $\dfrac{3x}{2}-\dfrac{4-x}{3}=2\frac{1}{3}-3(x-2)$, (ii) $\dfrac{x}{6}-\dfrac{1}{3}\left(x-\dfrac{1}{2}\right)-\dfrac{1}{3}\left(\dfrac{2}{5}-\dfrac{x}{3}\right)=0$

27 If ducks' eggs cost $4d$ a dozen more than hens' eggs, determine the price of each per dozen, when 7 ducks' eggs and 19 hens' eggs can be bought for two shillings

28. An artisan's wages are raised by the same sum yearly after two years they rise to £19, after eight years to £28 Plot the graph for reading off his wages for any year, and find (i) the annual rise, (ii) the starting wages, (iii) after how many years his wages will be £38 10s.

CHAPTER XII

SIMULTANEOUS EQUATIONS

143 CONSIDER the equation $y-2x=5$, which contains the two unknown quantities x and y

Here, since $y=2x+5$, it is clear that for every value we choose to give to x, there will be one corresponding value of y Thus we can find as many pairs of values as we please which satisfy the given equation

Thus, when x has the values 3, 2, 0, -1, -2,
we get for y the values 11, 9, 5, 3, 1

If, however, the values of x and y which satisfy the equation

$$y-2x=5 \tag{1}$$

also satisfy another equation of the same kind, such as

$$3x+y=15, \tag{2}$$

we shall find that there is only *one solution*

For from (1), we have $\quad y=2x+5,$

and from (2), $\quad y=15-3x,$

and since the values of y in the two equations are to be the same, we must have

$$2x+5=15-3x,$$

whence $\quad x=2$

If we substitute this value of x in *either* of the given equations, we obtain $y=9$

Thus, $x=2$, $y=9$ is the only solution possible if the two equations are to be satisfied by the *same* pair of values of x and y

144 Since every equation, involving x and y in the first degree only, can be represented graphically by a straight line, the conclusions arrived at in the preceding article may be explained as follows

The graph of each of the given equations passes through an indefinite number of points, the coordinates of which satisfy that equation *taken by itself* But two straight lines can only intersect at one point, the coordinates of this point give the values of x and y which satisfy the two equations *taken together*

NOTE The graph of $y=2x+5$ is given in Art 131; the graph of $3x+y=15$ can be drawn by joining the points (5, 0), (0, 15) It will be found that the graphs intersect at the point M in Fig 4, p 101

145 When two or more equations are satisfied by the same values of the unknowns, they are called **simultaneous equations**

Since such equations are true for the same values of the unknowns, *any* equation formed by combining them will also be true for those values of the unknowns which satisfy the original equations In combining the given equations, our first object is to obtain a new equation which involves only one of the unknowns

The process by which we get rid of an unknown quantity is called **elimination**, and it can be effected in different ways according to the nature of the equations given for solution

146 First method, by equalising coefficients

EXAMPLE 1 *Solve*
$$3x+7y=27, \qquad (1)$$
$$5x+2y=16 \qquad (2)$$

To eliminate x we multiply (1) by 5 and (2) by 3, so as to make the coefficients of x in both equations equal This gives
$$15x+35y=135,$$
$$15x+\ 6y=48,$$
subtracting,
$$29y=87,$$
$$y=3$$

To find x, substitute this value of y in *either* of the given equations

Thus, from (1),
$$3x+21=27,$$
$$x=2$$

Therefore the complete solution is $x=2$, $y=3$

[*Verification* $3x+7y=3\times2+7\times3=\ 6+21=27,$
$5x+2y=5\times2+2\times3=10+\ 6=16$]

NOTE When one of the unknowns has been found, it is immaterial which of the equations we use to complete the solution

EXAMPLE 2 *Solve*
$$11x+8y=31, \qquad (1)$$
$$13x-6y=83 \qquad (2)$$

Here it will be more convenient to eliminate y Since 24 is the L C M of 8 and 6, we shall make the coefficients of y numerically equal by multiplying (1) by 3, and (2) by 4

Thus we obtain
$$33x+24y=93,$$
$$52x-24y=332,$$
adding,
$$85x=425,$$
$$x=5$$

Substituting in (1),
$$55+8y=31,$$
whence
$$y=-3$$

Thus the solution is $x=5$, $y=-3$

We *add* when the coefficients of one unknown are equal and *unlike* in sign, and *subtract* when the coefficients are equal and *like* in sign

147 Second Method. Elimination by Substitution.

EXAMPLE *Solve*

$$2x=5y+1, \quad (1)$$
$$24-7x=3y \quad (2)$$

Here we can eliminate x by substituting in (2) its value obtained from (1) Thus

$$2x=5y+1, \quad (1)$$
$$x=\tfrac{1}{2}(5y+1)$$

Substituting this value of x in (2), we have

$$24-\tfrac{7}{2}(5y+1)=3y,$$
$$48-35y-7=6y,$$
$$41=41y,$$

and from (1),
$$\left.\begin{array}{l} y=1, \\ x=3 \end{array}\right\}$$

Or thus From (1), $x=\frac{1}{2}(5y+1)$, and from (2), $x=\frac{1}{7}(24-3y)$.

By equating these values of x, we obtain

$\frac{1}{2}(5y+1)=\frac{1}{7}(24-3y)$, whence $y=1$, as before

This method is sometimes called elimination by comparison.

EXAMPLES XII. a.

Solve the following equations by the *First Method*, and verify the solutions

1. $x+y=12,$ $x-y=6$
2. $x-y=5,$ $x+y=19$
3. $x+y=16,$ $x-y=0$
4. $3x+2y=13,$ $3x-2y=5$
5. $x+2y=4,$ $2x+3y=7$
6. $2x+y=23,$ $4x-y=19$
7. $x+3y=38,$ $3x-y=24$
8. $7x-5y=45,$ $2x+3y=4$
9. $7x+3y=10,$ $35x-6y=1$
10. $11y-6x=36,$ $7y+24x=9$
11. $5y-3x=85,$ $12y+5x=21$
12. $7x+6y=71,$ $5x-8y=-23$

Solve the following equations as in Art 147

13. $x=3y-2,$ $9y=4x-7$
14. $3x-2y=6,$ $6y-5x=30$
15. $3x=7+y,$ $5x=9y+41$
16. $4x+3y=0,$ $5y+53=11x$
17. $3x+2y=118,$ $x+5y=191$
18. $13+2y=9x,$ $3y=7x$
19. $2x-9y=0,$ $7x+27=18y$
20. $12x+9=y,$ $18y-5x=56\frac{1}{2}$
21. $7x=10y+4,$ $12x=1-18y$

Solve the following equations

22. $9x-11y=15,$
$\quad 7x-13y=25$

23. $21x-10y=109,$
$\quad 13x-7y=61$

24. $23x-11y=1,$
$\quad 15x+7y=29$

[*In Examples* 25–27 *first form two new equations by adding the two equations, and by subtracting one from the other*]

25. $7x+5y=1,$
$\quad 5x+7y=11$

26. $14x+9y=19,$
$\quad 9x+14y=4$

27. $13x+11y=70,$
$\quad 11x+13y=74$

[*In Example* 28, *observe that the L C M of* 39 *and* 52 *is* $13\times3\times4$]

28. $39x+7y=107,$
$\quad 52x+11y=131$

29. $11x-45y=211,$
$\quad 24x+75y=114$

30. $95y=49+23x,$
$\quad 76y=102-13x$

31. If $x+5y=18$, and $3x+2y=41$, find the value of $x-8y$, and of $15y-x$

32. If the equation $y=ax+b$ is satisfied by $x=4$, $y=8$, and also by $x=12$, $y=20$, find the values of a and b

148 Sometimes it will be necessary to simplify the equations before applying any one of the methods of solution

EXAMPLE *Solve*

$$5(x+2y)-(3x+11y)=14, \qquad (1)$$
$$7x-9y-3(x-4y)=38 \qquad (2)$$

From (1),
$$5x+10y-3x-11y=14,$$
$$2x-y=14 \qquad (3)$$

From (2),
$$7x-9y-3x+12y=38,$$
$$4x+3y=38$$

From (3),
$$6x-3y=42$$

By addition, $10x=80$, whence $x=8$ From (3), we obtain $y=2$

149 When the value of one of the unknowns has been found, instead of substituting this value to find the value of the second unknown, it will sometimes prove easier to employ the process of elimination

EXAMPLE *Solve*

$$3x-\frac{y-5}{7}=\frac{4x-3}{2}, \qquad (1)$$

$$\frac{3y+4}{5}-\frac{1}{3}(2x-5)=y \qquad (2)$$

Clearing of fractions, we have

from (1),
$$42x-2y+10=28x-21,$$
$$14x-2y=-31, \qquad (3)$$

and from (2),
$$9y+12-10x+25=15y,$$
$$10x+6y=37 \qquad (4)$$

Eliminating y from (3) and (4), we find that $x=-1\frac{1}{13}$

Eliminating x from (3) and (4), we find that $y=7\frac{25}{26}$

150 Definition If the product of two quantities is equal to unity, each quantity is said to be the reciprocal of the other.

Thus the following pairs are *reciprocals*:

$$3 \text{ and } \frac{1}{3}, \quad x \text{ and } \frac{1}{x}; \quad \frac{a}{b} \text{ and } \frac{b}{a}$$

Simultaneous equations may often be conveniently solved by taking the reciprocals of x and y, that is $\frac{1}{x}$ and $\frac{1}{y}$, as the unknowns.

Example *Solve* (1) $\frac{8}{x}-\frac{9}{y}=1$, (2) $\frac{10}{x}+\frac{6}{y}=7$

Multiplying (1) by 2, and (2) by 3, we have

$$\frac{16}{x}-\frac{18}{y}=2,$$

$$\frac{30}{x}+\frac{18}{y}=21$$

By addition, $\frac{46}{x}=23$, whence $x=2$ From (1) we obtain $y=3$

EXAMPLES XII b.

Solve the following equations

1. $3x-y=23$, $\frac{x}{3}+\frac{y}{4}=4$

2. $2x-3y=24$, $\frac{5x}{3}-\frac{y}{2}=12$

3. $3x-y=3$, $\frac{5x}{8}+\frac{7y}{18}=6$

4. $12x+7y=2\frac{1}{2}$, $8y-9x=18$

5. $\frac{4y-2}{3}=\frac{5x}{2}$, $18x-20y=3$

6. $\frac{x-3}{5}=\frac{y-7}{2}$, $11x=13y$

7. $\frac{11x-5y}{11}=\frac{3x+y}{16}$, $8x-5y=1$

8. $\frac{1}{5}(x-2)=\frac{1}{4}(1-y)$, $26x+3y+4=0$

9. $4(x-2y)-(5x+3y)=30$, $3(3x+7y)-2(x+9y)=12$

10. $\frac{3x+1}{7}-\frac{2x-y}{2}=\frac{2y-x}{8}$, $\frac{4x-2}{3}-\frac{4y-5x}{2}=\frac{x+y}{5}$

11. $\frac{2y-25}{3}-\frac{6-x}{7}=\frac{2(y-7)}{5}$, $\frac{29-x}{8}-\frac{3y-1}{10}=\frac{4-x}{3}$

12. $\frac{x}{3}+\frac{y}{4}=3x-7y-37=0$

13. $\frac{x+3}{5}=\frac{8-y}{4}=\frac{3(x+y)}{8}$.

14. $3{\cdot}75x-1{\cdot}5y=27$, $7x+6y=68$

15. $2x+0{\cdot}4y=1{\cdot}2$, $3{\cdot}4x-0{\cdot}02y=0{\cdot}01$

Solve the following equations as in Art 150

16. $\frac{9}{x}-\frac{4}{y}=8,\quad \frac{13}{x}+\frac{7}{y}=101$

17. $\frac{15}{x}-\frac{1}{y}=4\frac{1}{2},\quad \frac{9}{x}+\frac{2}{y}=4$

18. $\frac{2}{x}+\frac{5}{y}=\frac{5}{6},\quad \frac{3}{x}+\frac{4}{y}=\frac{9}{10}$

19. $2y-x=4xy,\quad \frac{4}{y}-\frac{3}{x}=9$

20. $\frac{2}{x}+3y=15,\quad \frac{5}{x}-4y=3$

21. $\frac{1}{x}+2y=1\frac{1}{4},\quad \frac{2}{x}+y=1$

22. $\frac{8}{x}-\frac{9}{y}=7,\quad 6\left(\frac{1}{x}+\frac{1}{y}\right)=1$

23. $\frac{2}{x}-\frac{3}{2y}=\frac{41}{35},\quad \frac{2\frac{1}{2}}{2x}+\frac{3\frac{1}{2}}{y}=-\frac{73}{70}$

Simultaneous Equations Treated Graphically

151 The graph of any simple equation involving x and y is a straight line, and any pair of values which satisfy the equation will give the coordinates of some point on this line The number of such points is unlimited If, however, x and y are connected by *two* simultaneous equations their linear graphs intersect in *one point only* The coordinates of this point are the values of x and y which satisfy the equations

EXAMPLE *Solve graphically the equations*

$$(1)\ 3y-x=6,\qquad (2)\ 3x+5y=38$$

In (1) the intercepts on the axis are -6, 2 Thus the line is found by joining $P(-6, 0)$ and $P'(0\ 2)$

In (2) when $x=1$, $y=7$, and when $y=1$, $x=11$

Thus the line is found by joining $Q(1, 7)$ and $Q'(11, 1)$

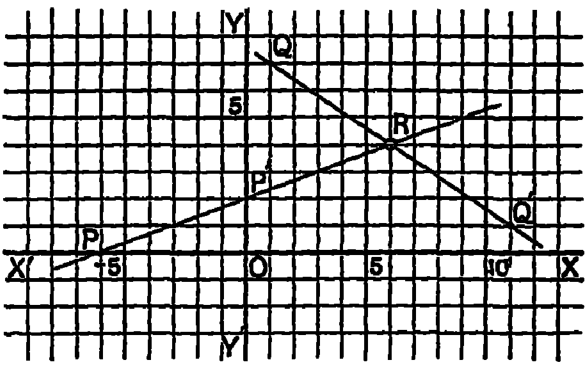


FIG 14

It is seen from the diagram that these lines intersect at the point R, whose coordinates are 6, 4 Thus the solution of the given equations is

$$x=6,\quad y=4$$

The pupil should verify this result by solving the equations algebraically by any of the methods given in this chapter

152 Since an unlimited number of values can be found to satisfy an equation of the first degree in x and y, such an equation is said to be **indeterminate** Graphically we see that one such equation determines a straight line, but that two are required to determine a point

153 Two simultaneous equations lead to no finite solution if they are *inconsistent with each other* For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$ The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection

154 Again, two simultaneous equations must be *independent* The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by multiplying throughout by 4 Thus *any pair of values* which will satisfy one equation will satisfy the other Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points

EXAMPLES XII. c.

Solve the following equations graphically

1 $y=2x-1,$ $x+y=5$	2. $x=2y-3,$ $y=3x-6$	3. $2x=3y,$ $x-y=2$
4. $2x-5y=16,$ $4x+y=10$	5. $4x+3y=2,$ $x-y=4$	6 $2y-5x=20,$ $4x+3y=7$

7. Shew that the straight lines represented by the equations

$$2x=3y+14, \quad 3x+y=10, \quad x+2y=0,$$

meet in a point, and find its coordinates

8 Solve the following pairs of equations correctly to one-tenth of the unit

(i) $4x+5y=28,$ $x+y=6{\cdot}1,$ (ii) $5x-3y=8{\cdot}8,$ $7x-5y=10{\cdot}4$

[*Take one inch as unit, and in* (ii) *estimate the intercepts on the axes to one-hundredth of the unit*]

9. With one inch as unit draw the graphs of the equations

$$3{\cdot}4x+5y=17, \quad x-y=0{\cdot}8, \quad y-0{\cdot}5x=0{\cdot}45,$$

and shew that they all pass through one point Read off the coordinates of this point

10 Draw the triangle whose sides are given by the equations

$$3y-x=9, \quad x+7y=11, \quad 3x+y=13,$$

and find the coordinates of its vertices

11 Explain by a diagram why it is not possible to find the coordinates of the point of intersection of the lines

$$\frac{x}{18}-\frac{y}{16}=2\cdot 5, \quad \frac{2x}{3}-\frac{3y}{4}=12$$

***155** When a linear graph has been drawn through certain plotted points it is often convenient to be able to find its equation

EXAMPLE *Shew that the points* (3, -4), (9, 4), (12, 8) *lie on a straight line, and find its equation*

The first part of this example may be solved graphically by drawing the line which joins any two of the points, and shewing experimentally that it passes through the remaining point

The following method has the advantage of giving the equation of the linear graph which passes through the given points

Since the equation of any straight line is of the first degree in x and y, we may assume $y=ax+b$ as the equation of the line If it passes through the first two of the given points, their coordinates must satisfy the equation

Substituting $x=3$, $y=-4$, we have $-4=3a+b$, (1)

again, substituting $x=9$, $y=4$ we have $4=9a+b$ (2)

From (1) and (2), we obtain $a=\frac{4}{3}$, $b=-8$

Hence, $\quad y=\frac{4}{3}x-8 \quad$ or $\quad 4x-3y=24$,

is the equation of the line joining the points (3, -4), (9, 4) On trial we find that $x=12$, $y=8$ satisfy this equation, so that the line also passes through the point (12, 8)

*EXAMPLES XII c (*Continued*)

12 If the equation $y=ax+b$ represents a line through the points (2, 5), (-3, 4), find the values of a and b

13 Shew that the points (4, 5), (11, 11), (-3, -1) lie on a straight line, and find its equation

14 Prove that the points (2, 4) (-3, 8), (12, -4) lie on a straight line which cuts the axis of x at a distance of 7 units from the origin

15 Find the equation of the line which joins the points (0, 3 1) and (3, 2 5)

16 Find the values of a and b so that the line represented by $y=ax+b$ may pass through the intersection of the lines $4x=3y-16$, $4x=5y-24$, and also through the point (-2, 2)

17 Shew that the points (2, -2), (-4, 7), (6, 22) lie on a graph whose equation is of the form $y=ax^2+b$, and find the values of a and b

*156 In Art 141 we have explained how a graph may be drawn to lie evenly among a number of plotted points, provided that corresponding values of two connected variables are known from observation or experiment When the graph is linear it can be produced to any extent within the limits of the paper, and so any value of one of the variables being determined, the corresponding value of the other can be read off When large values are in question this method is inconvenient, in such a case it is best to make use of the equation of the graph, which can be found as in the preceding article

Example *Corresponding values of* x *and* y, *some of which are slightly inaccurate, are given in the following table*

x	1	*4	*6.8	8	9.5	*12	14.4
y	4	8	12.2	13	15.3	20	24.8

Draw the most probable graph and find its equation Also find the value of y *corresponding to* $x=80$

Let 1 inch be taken to represent 5 units along OX, and 20 units along OY

After carefully plotting the given points we see that a straight line can be drawn passing through the points marked with an asterisk and lying evenly among the others This is the required graph

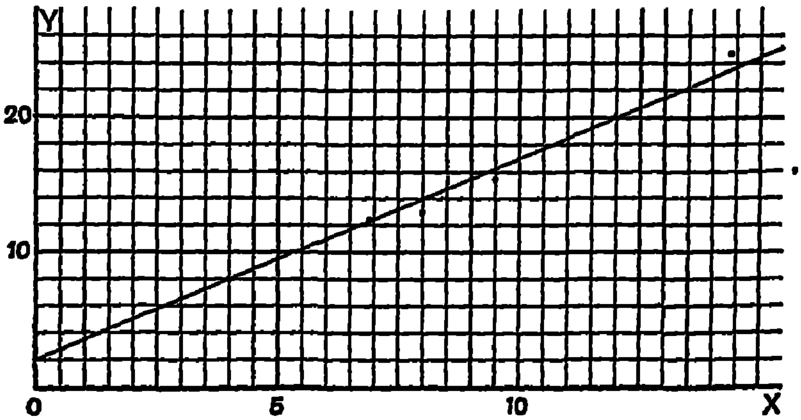


Fig 15

Assuming $y=ax+b$ for the equation of the linear graph, we can find the values of a and b by substituting the coordinates of two points through which the line passes

Thus, putting $x=4$, $y=8$, we obtain $8=4a+b$,

again, when $x=12$, $y=20$, we have $20=12a+b$

By solving these equations we obtain $a=1.5$, $b=2$

Hence the equation of the graph is $y=1.5x+2$, and the coordinates of any number of points on the line may now be found by trial.

Thus when $x=80$, $y=122$

*EXAMPLES. XII. d.

1 Plot on squared paper the following measured values of x and y, and determine the most probable equation between x and y

$$x=3,\ 5,\ 8{\cdot}3,\ 11,\ 13,\ 15{\cdot}5,\ 18{\cdot}6,\ 23,\ 28,$$
$$y=2,\ 2{\cdot}2,\ 3{\cdot}4,\ 3{\cdot}8,\ 4,\ 4{\cdot}6,\ 5{\cdot}4,\ 6{\cdot}2,\ 7{\cdot}25$$

2 Corresponding experimental values of x and y are given as follows

$$x=1,\ 3{\cdot}1,\ 6,\ 9{\cdot}5,\ 12{\cdot}5,\ 16,\ 19,\ 23;$$
$$y=2,\ 2{\cdot}8,\ 4{\cdot}2,\ 5{\cdot}3,\ 6{\cdot}6,\ 8{\cdot}3,\ 9,\ 10{\cdot}8$$

Draw the most probable graph and find its equation Find the correct value of y when $x=19$, and the correct value of x when $y=2{\cdot}8$

3 For a given temperature C degrees on a Centigrade are equal to F degrees on a Fahrenheit thermometer The following are corresponding values of C and F

$$C=-10,\ -5,\ 0,\ 5,\ 10,\ 15,\ 25,\ 40,$$
$$F=14,\ 23,\ 32,\ 41,\ 50,\ 59,\ 77,\ 104.$$

Draw a graph to shew the Fahrenheit reading corresponding to a given Centigrade temperature, and find the Fahrenheit readings corresponding to 12·5° C and 31° C

Find also the algebraical relation connecting F and C

4 The keeper of a hotel finds that when he has G guests a day his total daily profit is P pounds If the numbers below are averages from many days' accounts, find a simple algebraical relation between P and G

$$G=21,\ 27,\ 29,\ 32,\ 35,$$
$$P=-1{\cdot}8,\ 2,\ 3{\cdot}2,\ 4{\cdot}5,\ 6{\cdot}6$$

For what number of guests would he just have no profit?

5 In a certain machine P is the force in pounds required to raise a weight of W pounds The following corresponding values of P and W were obtained experimentally

$$P=3{\cdot}08,\ 3{\cdot}9,\ 6{\cdot}8,\ 8{\cdot}8,\ 9{\cdot}2\ \ 11,\ 13{\cdot}3,$$
$$W=21,\ 36{\cdot}25,\ 66{\cdot}2,\ 87{\cdot}5,\ 103{\cdot}75,\ 120,\ 152{\cdot}5$$

Draw the graph connecting P and W, and read off the value of P when W=70 Also find the linear equation connecting P and W, find the force necessary to raise a weight of 310 lbs, and also the weight which could be raised by a force of 180·6 lbs

6 The following values of x and y, some of which are slightly inaccurate, are connected by an equation of the form $y=ax^2+b$

$$x=1,\ 1{\cdot}6,\ 3,\ 3{\cdot}7,\ 4,\ 5,\ 5{\cdot}7,\ 6,\ 6{\cdot}3,\ 7,$$
$$y=3{\cdot}25,\ 4,\ 5,\ 6{\cdot}5,\ 7{\cdot}4,\ 9{\cdot}25,\ 10{\cdot}5,\ 11{\cdot}6,\ 14,\ 15{\cdot}25$$

By plotting these values draw the graph, and find the most probable values of a and b

Find the true value of x when $y=4$, and the true value of y when $x=6$

Simultaneous Equations with Three Unknowns.

157. If we have *two* simple equations with *three* unknowns, one of these can be eliminated as already explained, the result will be a simple equation in two unknowns, and therefore *indeterminate* [Art 152] If, however, three unknowns are connected by three consistent and independent equations, we may eliminate one unknown from any pair of the given equations, and then *the same unknown* from a *different pair* The two resulting equations, containing two unknowns, can be solved by the rules already given The third unknown can then be found by substituting the values so found in any of the original equations.

EXAMPLE 1 *Solve the equations*

(1) $7x+5y-7z=-8$, (2) $4x+2y-3z=0$, (3) $5x-4y+4z=35$

Choose y as the unknown to be eliminated.

Multiplying (2) by 5, $20x+10y-15z=0$;
multiplying (1) by 2, $14x+10y-14z=-16$;
subtracting, $6x-z=16$. (4)

Again, multiplying (2) by 2, $8x+4y-6z=0$;
from (3), $5x-4y+4z=35$,
adding, $13x-2z=35$
Multiplying (4) by 2, $12x-2z=32$

The last two equations give $x=3$, $z=2$ Substituting these values in (2), we obtain $y=-3$

EXAMPLE 2. *Solve the equations*

$$\frac{y+z}{4}=\frac{z-x}{3}=\frac{x+y}{2}, \quad x+y+z=27$$

Here we must first form *three* equations in x, y, z

From $\dfrac{y+z}{4}=\dfrac{x+y}{2}$, we obtain $2x+y-z=0$. (1)

From $\dfrac{z+x}{3}=\dfrac{x+y}{2}$, we obtain $x+3y-2z=0$. (2)

Also $x+y+z=27$ (3)

From (1) and (3), by addition, $3x+2y=27$

Multiply (3) by 2 and add to (2), thus $3x+5y=54$.

The last two equations give $y=9$, $x=3$ Hence from (3), $z=15$

NOTE. A system of equations containing four or more unknowns can be solved in a similar way if the number of equations is the same as the number of the unknowns For example, with four such equations we first eliminate one unknown from *three different* pairs of equations We thus obtain three new equations containing the three remaining unknowns, and the solution can be completed as already explained

EXAMPLES XII e

Solve the equations

1. $x+y+z=7, \quad 2x+3y-z=0, \quad 3x+4y+2z=17$

2 $x+y-z=8, \quad 4x-y+3z=26, \quad 2x+y-4z=8$

3. $2x+y+z=8, \quad 5x-3y+2z=3, \quad 7x+y+3z=20$

4 $3x+y-z=3, \quad 2x-y+3z=20, \quad 7x+y+z=23$

5 $5x-4y+z=3, \quad 3x+y-2z=31, \quad x+4y+z=15$

6 $4x-5y+6z=3, \quad 8x-7y-3z=9, \quad 7x-8y+9z=6$

7. $5z-3x=4(1+y), \quad 2(x+2z)=8+3y, \quad 2y+3z=14-x$

•8. $x+z=2y, \quad 9x+3z=8y, \quad 2x+3y+5z=36$

•9 $x-\frac{y}{5}=6, \quad y-\frac{z}{7}=8, \quad z-\frac{x}{2}=10$

10. $\frac{1}{4}x+\frac{1}{6}y+\frac{1}{3}z=8, \quad \frac{1}{2}x-\frac{1}{9}y+\frac{1}{6}z=5, \quad \frac{1}{3}x+\frac{1}{2}y-z=7.$

•11 $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12, \quad \frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8, \quad \frac{z}{3}+\frac{x}{2}=10$

12 $y+z-x=z+x-3y=\frac{1}{2}(x+y-2z)=1$

13 $x+z-1=\frac{1}{2}(x+4z-8)=\frac{1}{3}(x+9z-27)=y$

14 $x+y=6, \quad y+z=10, \quad z+x=20$

15. $x+y-z=2{\cdot}3, \quad y+z-x=6{\cdot}7, \quad z+x-y=8{\cdot}5$

16 Find x, y, z, w from the following equations

$$x+y+z+w=5, \quad 2x+y-z-2w=3,$$
$$x-y+2z+w=3, \quad x-3y+z+w=9$$

17 When $x=7$, $y=-2$ the expression $ax+by$ is equal to 22, and when $x=3$, $y=1$, the expression is equal to 15, find a and b

18 Find a linear expression of the form $ax+b$ such that its value is 5 when $x=2$, and 17 when $x=5$

19 Shew that the equations

$$3x-4y=1, \quad 8y-7=6x$$

are inconsistent Illustrate graphically

20. Shew that the three equations

$$5x-3y-z=6, \quad 13x-7y+3z=14, \quad 7x-4y=8$$

are not independent

21. Find the values of a and b in order that the equations

$$4x+7y=5, \quad 9x+8y=19, \quad ax+2by=11, \quad ax-by=17$$

may be consistent

22 What peculiarity is there about the following three equations?

$$x+2y-3z=9, \quad -x+6y+11z=7, \quad x+26y+21z=53$$

If in the first equation 8 is written in the place of 9, how is the nature of the equations altered?

CHAPTER XIII

Problems leading to Simultaneous Equations

158 In the last chapter we have seen that there must always be as many equations as there are unknown quantities whose values are to be found Consequently, in problems which may be solved by using simultaneous equations, there must be as many independent conditions, each of which can be stated in the form of a separate equation, as there are quantities to be found

Example 1 *Three fourths of the sum of two numbers is 57, and if the greater is subtracted from three times the less the remainder is 40, find the numbers*

Let x be the greater number, and y the less

From the first condition, $\frac{3}{4}(x+y)=57$,

or $\quad 3x+3y=228 \qquad (1)$

From the second, $\quad 3y-x=40 \qquad (2)$

Subtracting, we obtain $\quad 4x=188$,

$x=47$,

and from (2), $\quad y=29$

Thus the numbers are 47 and 29

Example 2 *If unity is subtracted from the numerator of a certain fraction, and 2 added to the denominator, it reduces to $\frac{2}{3}$, if 3 is added to the numerator, and the denominator is multiplied by 2, it reduces to $\frac{3}{5}$ What is the fraction?*

Let x be the numerator of the fraction, and y the denominator, then the fraction is $\frac{x}{y}$

From the first supposition, $\quad \frac{x-1}{y+2}=\frac{2}{3}, \qquad (1)$

from the second, $\quad \frac{x+3}{2y}=\frac{3}{5} \qquad (2)$

To clear of fractions multiply (1) by $3\times(y+2)$, and (2) by $5\times 2y$,

thus from (1), $\quad 3(x-1)=2(y+2)$,

or $\quad 3x-2y=7 \qquad (3)$

And from (2), $\quad 5x+15=6y \qquad (4)$

Equations (3) and (4) give $x=9$, $y=10$

Therefore the fraction is $\frac{9}{10}$.

EXAMPLE 3 *Five cows and nine sheep are worth £102, while 6 cows and 7 sheep are worth £111, find the value of a cow and a sheep respectively*

Suppose a cow to cost x pounds,
and a sheep „ y „

Then from the question we have

$$5x+9y=102, \qquad (1)$$

$$6x+7y=111 \qquad (2)$$

From these equations $x=15$, $y=3$

Thus the cost of a cow is £15, and the cost of a sheep £3

EXAMPLES XIII a

1. Find two numbers whose sum is 25, and whose difference is 7

2 Find two numbers whose sum is 61, and whose difference is 15

3 One fourth of the sum of two angles is 13°, and one-sixth of their difference is 3°, find them

4 One-seventh of the sum of two numbers is 6, and four times their difference is 64, find them

5 Find two numbers which are such that twice the greater exceeds three times the less by 10, and such that one fifth of the greater is less than 20 by one third of the less

6 Find two numbers such that one-third of the greater exceeds one-half of the less by 1, and one-fifth of the greater added to one sixth of the less equals one half of the less

7 The difference of two numbers is five sixths of their sum, and the greater exceeds 10 times the less by 3, find the numbers

8 One man said to another "If you give me half your money I shall have £5" The other replied "I shall have £5 if you give me a third of your money" How much had each?

9 Find a fraction which becomes equal to $\frac{1}{2}$ if 1 is subtracted from both numerator and denominator, and equal to $\frac{2}{3}$ if 1 is added to both numerator and denominator

10 Find a fraction such that if its numerator is diminished by unity, it reduces to $\frac{1}{2}$, and becomes equal to 2 when the numerator is increased by 5 and the denominator diminished by 6

11. If 4 is taken from the denominator of a certain fraction it reduces to 1, if the numerator is multiplied by 3, and the denominator increased by 5 it reduces to $\frac{3}{2}$ What is the fraction?

12 Add 1 to the numerator and denominator of a certain fraction and it reduces to $\frac{4}{5}$, subtract 5 from each, and it reduces to $\frac{1}{2}$ required the fraction

13 A horse and a cow are together worth £42, while 4 horses and 7 cows cost £213 find the price of each animal

14 If either 9 tables and 7 chairs, or 10 tables and 2 chairs, can be bought for £78, what is the price of each?

15 A man sells 15 animals, consisting of horses and sheep, for £275 If the price of a horse is £45 and of a sheep £5, how many of each did he sell?

16. If 10 lbs of tea and 8 lbs of coffee cost £1 16*s* 6*d*, while 6 lbs of tea and 5 lbs of coffee cost £1 2*s* 3*d*, find the price per pound of tea and coffee

17 The wages of 24 men and 16 boys amount to £5 16*s* per day, half that number of men, with 21 more boys, would earn the same money What are the daily wages of each man and boy?

18. I buy two pieces of cloth for £25 6*s*, one piece being 16*s* and the other 18*s* per yard I sell them at a profit of 2*s* per yard, and gain on the whole £3 How long was each piece?

19 If 15 lbs of tea and 17 lbs of coffee together cost £3 5*s* 6*d*, and 25 lbs of tea and 13 lbs of coffee together cost £4 6*s* 2*d*, find the price of each per pound

159 The following examples contain some special features, and should be carefully studied by the pupil before he works through the miscellaneous problems of the next Exercise

EXAMPLE 1 *I spend 3s in buying apples at 4 a penny and oranges at 3 a penny, and then dispose of three-fourths of my apples and half of my oranges for 2s, which was a penny more than they cost me, how many of each did I buy?*

Let x be the number of apples and y the number of oranges

Then x apples cost $\frac{x}{4}$ pence,

and y oranges cost $\frac{y}{3}$ pence,

hence $$\frac{x}{4}+\frac{y}{3}=36,$$

or $$3x+4y=432 \qquad (1)$$

Again, $\frac{3}{4}x$ apples cost $\frac{3}{4}\cdot\frac{x}{4}$, or $\frac{3x}{16}$ pence,

and $\frac{1}{2}y$ oranges cost $\frac{1}{2}\cdot\frac{y}{3}$, or $\frac{y}{6}$ pence

But, by the question, these together cost 1*s* 11*d*,

hence $$\frac{3x}{16}+\frac{y}{6}=23,$$

or $$9x+8y=1104 \qquad (2)$$

By combining equations (1) and (2) we obtain $x=80$, $y=48$

Thus there were 80 apples and 48 oranges

EXAMPLE 2 *A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed, find the number* [See Art 115, Ex 3]

Let x be the digit in the tens' place, y the digit in the units' place, then the number will be represented by $10x+y$, and the number formed by reversing the digits will be represented by $10y+x$

The sum of the digits is $x+y$

Hence we have the two equations

$$10x+y=3(x+y) \qquad (1)$$

and

$$10x+y+45=10y+x \qquad (2)$$

From (1), $7x=2y$,

from (2), $y-x=5$

From these equations we obtain $x=2$, $y=7$

Thus the number is 27

EXAMPLE 3 *Two persons, 27 miles apart, setting out at the same time, are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions, find their rates of walking*

Suppose the faster walker goes x miles per hour,

and ,, slower ,, ,, y ,, ,,

When they walk in the *same* direction the faster walker gains on the other $(x-y)$ miles per hour, and in 9 hours he will gain $9(x-y)$ miles

Therefore $9(x-y)=27$,

or

$$x-y=3 \qquad (1)$$

When they walk in *opposite* directions they lessen the distance between them by $(x+y)$ miles per hour, and in three hours this decrease is $3(x+y)$ miles

Therefore $3(x+y)=27$,

or

$$x+y=9 \qquad (2)$$

From (1) and (2), we find $x=6$, $y=3$

Thus the rates of walking are 6 and 3 miles per hour respectively

EXAMPLES XIII b

1 I spend 3*s* 4*d* in buying eggs at 2 for 1*d*, and apples at 3 for 1*d* If I were to sell them all alike at the rate of 20 for 1*s*, I should gain 1*s* 2*d* How many of each did I buy?

2 By purchasing pen holders at 8*d* a score, and lead pencils at 9*d* a dozen at a total outlay of 5*s* 5*d*, and selling them all at a uniform price of 11 for 8*d*, I gain 1*s* 3*d* How many of each did I buy?

3 A man sold one sort of oranges at 5 for 2*d*, and another sort at 16 for 1*s* If he had sold them all at a half-penny each, he would have received 2*d* less, if he had sold all at 3 for 2*d*, he would have received 1*s* 6*d* more How many of each sort did he sell?

4 A number of two digits is such that if 9 be added to it the digits will be reversed, if the sum of the digits is 7, find the number

5 A number of two digits is equal to eight times the sum of its digits, if 45 be subtracted from the number, the digits will be reversed find the number

6 A number of two digits has its digits reversed if 18 is taken from it, the sum of the digits is 12, find the number

7. A certain number of two digits is two-ninths of what it would be if the digits were reversed If the number is increased by the sum of its digits the result is 27, find the number

8. A number of two digits exceeds four times the sum of its digits by 3, if the number is increased by 18, the result is the same as if the number formed by reversing the digits were diminished by 18 Find the number

9 On Monday a hawker disposes of the whole of his stock of boot-laces at 6 for $2\frac{1}{2}d$, and four-ninths of his stock of buttons at 6 for $1\frac{1}{2}d$, the proceeds amounting to *3s 6d*, and on Tuesday he sells the remainder of the buttons at the same price, for *1s 3d* less than he received for the boot-laces with how many of each did he start out on Monday?

10 A boy has *6s* with which he is to buy two kinds of note books He finds that if he asks for 11 of the smaller size and 13 of the larger he will require *2d* more, if he asks for 13 of the smaller and 11 of the larger he will have *2d* over Find the price of each kind

11 A certain sum of money is divided among A, B, and C B's share is sixpence more than half the sum of the shares of A and C A's share is four shillings less than half the sum of the shares of B and C If the shares of A and B together amount to *33s*, find how much each receives

12. At a certain election there were two rival candidates, and their supporters were conveyed to the polling-booths in carriages capable of accommodating 8 and 12 voters respectively If the voters, 740 in all, just filled 75 carriages, find by what majority the election was won

13. Two numbers are formed by the same two digits, and if the smaller number is divided by the greater the quotient is $\frac{4}{7}$, and if the smaller is subtracted from the greater the remainder is 27 Find the numbers

14. A certain number consists of two digits If 5 is added to the number, and the result divided by the units' digit, the quotient is 6, and if 10 is subtracted from the number, and the remainder divided by the sum of the digits, the quotient is 3 What is the number?

15 Of a number consisting of two digits the units' digit is the greater If the number is increased by 3, and the result divided by the difference of the digits, the quotient is 30, and if it is diminished by 48, the remainder is equal to three fourths of the sum of the digits Find the number

16. At a flower show, at which 1250 attended, outsiders were charged 1*s*, villagers 6*d*, and school children 1*d*, and the total receipts were £35 There were three times as many villagers as outsiders, how many outsiders came?

17 A bag contained shillings and half crowns amounting to £5 Half of the shillings were taken out and replaced by half crowns If the value of the contents was then £6 17*s* 6*d*, how many shillings did the bag contain at first?

18 By selling 6 horses and buying 8 cows a dealer increases his cash by £70 He then, at the same prices, buys 7 horses and sells 12 cows, and thereby decreases his cash by £43 Find the price of each horse and cow

19 At 9 a m a man starts from A and walks continuously at the rate of $3\frac{1}{2}$ miles per hour to meet his friend, who starts at the same hour from B, 56 miles away If the latter walks at the rate of 4 miles per hour, but stops an hour by the way, when and where do they meet?

20 *A*, *B* and *C* travel from the same place at the rates of 5, 6, and 8 miles an hour respectively, if *B* starts 2 hours after *A*, how long after *B* must *C* start in order that they may overtake *A* at the same instant?

21 A boat goes up stream 30 miles and then down stream 44 miles in 10 hours, and it also goes up-stream 40 miles and down stream 55 miles in 13 hours, find the rate of the stream and of the boat

22. Find the distance between two towns when by increasing the speed 7 miles per hour a train can perform the journey in 1 hour less, and by reducing the speed 5 miles per hour can perform the journey in 1 hour more

23 A train travelled a certain distance at a uniform rate Had the speed been 9 miles an hour more, the journey would have occupied 3 hours less, and had the speed been 6 miles an hour less the time taken would have been 3 hours more Find the distance

24. If 2 rabbits and 4 pheasants cost 17*s* 6*d*, 3 pheasants and two chickens cost 17*s* 3*d*, one chicken and three rabbits cost 6*s* 9*d*, find the price of each

25 There is a number whose three digits, from left to right, are in descending order of magnitude and differ from each other in succession by the same amount If the number is divided by the sum of its digits the quotient is 48 and if from the number 198 is subtracted, the digits of the difference are the same as in the original number, but in reverse order find the number

26 A train running from A to B meets with an accident 50 miles from A, after which it travels with three-fifths of its original velocity and arrives 3 hours late at B, if the accident had occurred 50 miles further on, it would have been only 2 hours late Find the distance from A to B, and the original velocity of the train

CHAPTER XIV.

RESOLUTION INTO FACTORS (*A First Course*)

160 DEFINITION When an algebraical expression is the product of two or more expressions each of these latter quantities is called a **factor** of it, and the determination of these quantities is called the **resolution** of the expression into its factors

161 *When each of the terms of an expression is divisible by a common factor*, the expression may be simplified by dividing each term separately by this factor, and enclosing the quotient within brackets, the common factor being placed outside as a coefficient

EXAMPLE 1 The terms of the expression $3a^2-6ab$ have a common factor $3a$,

$$3a^2-6ab=3a(a-2b)$$

EXAMPLE 2 The terms of the expression $m^2n^2-3m^2n^3$ have a common factor m^2n^2,

$$m^2n^2-3m^2n^3=m^2n^2(1-3n)$$

EXAMPLE 3 $5a^2bx^3-15abx^2-10bx^2=5bx^2(a^2x-3a-2)$

NOTE The pupil should always verify his results by multiplying the factors together mentally If these have been correctly chosen, the product should be the original expression

EXAMPLES XIV a

Resolve into factors

1 a^2+ab	2 a^3-a^2x	3 $2a^2-2a$	4. b^2-b^3
5 $cd-c^2$	6 c^3-c^2d	7. $5a^2-10a$	8. $3a-9a^2$
9. $3x^2-6xy$	10 $2p^2q+p^3$	11 y^2-xy^2	12. y^5-y^4

13 $4a^2-16a^2b$	14. $15d+45d^2$	15. $18c^3-9cd^2$
16 $16m-64m^2n$	17. $13x^2y^2+39y^4$	18. $9x^2y^2-3x^2z^2$
19. $81x-54$	20 $10p^3+25p^4q$	21. $51x^2y^2-17$
22. $4x^3+x^2-x$	23 $2a^3-4a^2-2a$	24 $3x^3-6x^2+9x$
25 $x^3-x^2y+xy^2$	26 $12xy^2+9x^2y+3x^3$	27. $2c^2d^3-6c^2d^2+2c^3d$
28 $2a^5-6a^4b+2a^3b^2$	29 $3x^4y-6x^3y^2+9x^2y^3$	30. $7a^3-7a^2b+14ab^2$

162 An expression may have a *compound* factor common to all its terms

Thus the expression

$$a(x-y)+3(x-y)=(x-y) \text{ taken } a \text{ times}$$
$$\textit{plus } (x-y) \text{ taken 3 times}$$
$$=(x-y) \text{ taken } (a+3) \text{ times}$$
$$=(x-y)(a+3)$$

163 An expression may be resolved into factors *if the terms can be arranged in groups which have a compound factor common.*

EXAMPLE 1 *Resolve into factors* $x^2-ax+bx-ab$

Noticing that the first two terms contain a common factor x, and the last two terms a common factor b, we enclose the first two terms in one bracket, and the last two in another Thus

$$\begin{aligned} x^2-ax+bx-ab &= (x^2-ax)+(bx-ab) \\ &= x(x-a)+b(x-a) \\ &= (x-a) \text{ taken } x \text{ times } \textit{plus } (x-a) \text{ taken } b \text{ times} \\ &= (x-a) \text{ taken } (x+b) \text{ times} \\ &= (x-a)(x+b) \end{aligned}$$

EXAMPLE 2 *Resolve into factors* $2x^2y+2cy-5x^2-5c$

$$\begin{aligned} 2x^2y+2cy-5x^2-5c &= (2x^2y+2cy)-(5x^2+5c) \\ &= 2y(x^2+c)-5(x^2+c) \\ &= (x^2+c)(2y-5) \end{aligned}$$

EXAMPLE 3 *Find the factors of* $12b^2-3bx^2-4b+x^2$

$$\begin{aligned} 12b^2-3bx^2-4b+x^2 &= (12b^2-3bx^2)-(4b-x^2) \\ &= 3b(4b-x^2)-(4b-x^2) \\ &= (4b-x^2)(3b-1) \end{aligned}$$

EXAMPLES XIV. b.

Resolve into factors

1. $(m+n)y+(m+n)z$
2. $(c+d)x-(c+d)y$
3. $2a(y^2+z^2)-b(y^2+z^2)$
4. $c^2(x-2y)-2(x-2y)$
5. $5(x-y)-(x-y)n$
6. $ab(l+m)+y(l+m)$
7. $a^2+ab+ac+bc$
8. $a^2-ac+ab-bc$
9. $a^2c^2+acd+abc+bd$
10. $a^2+3a+ac+3c$
11. $2x+cx+2c+c^2$
12. $x^2-ax+5x-5a$
13. $5a+ab+5b+b^2$
14. $ab-by-ay+y^2$
15. $ax-bx-az+bz$
16. $pr+qr-ps-qs$
17. $mx-my-nx+ny$
18. $mx-ma+nx-na$
19. $2ax+ay+2bx+by$
20. $6ac-2cy-3a+y$
21. $6x^2+3xy-2ax-ay$
22. $mx-2my-nx+2ny$
23. $ax^2+bx^2+2a+2b$
24. $x^2-3x-xy+3y$
25. $2x^4-x^3+4x-2$
26. x^4+x^3+2x+2
27. y^3-y^2+y-1
28. $axy+bcxy-az-bcz$
29. $f^2x^2+g^2x^2-ag^2-af^2$
30. $2ax^2+3axy-2bxy-3by^2$
31. $ax-bx+by+cy-cx-ay$
32. $a^2x+abx+ac+aby+b^2y+bc$

Factors of Trinomial Expressions

164 Before proceeding to the next case of resolution into factors we again draw the pupil's attention to the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial product

Thus $$(x+5)(x+3)=x^2+8x+15, \qquad (1)$$
$$(x-5)(x-3)=x^2-8x+15 \qquad (2)$$

By considering the way in which these trinomial products are formed we can learn, by a converse process, how to obtain their respective factors

By examining the factors on the left-hand side of each of the above results, we notice that

(i) The *first* term of each *factor* is x, that is, the square root of the first term of the trinomial

(ii) The *second* terms of the *factors* are such that their product gives the *third* term of the *trinomial* expressions on the right-hand side

Thus in (1) we see that $+15$ is the product of $+5$ and $+3$, and in (2) we see that $+15$ is the product of -5 and -3 Also it is to be observed that the numerical quantities 5, 3 must be either *both positive* or *both negative* in order to give the product $+15$

(iii) The *second* terms of the *factors* are such that their sum (taken with their proper signs) gives the coefficient of the *second* term in the *trinomials*

Thus $$5+3=8, \quad -5-3=-8$$

The application of these principles is illustrated in the following examples

EXAMPLE 1 *Resolve into factors* $x^2+11x+24$

The second terms of the factors must be such that their product is $+24$, and their sum $+11$ It is clear that they must be $+8$ and $+3$

$$x^2+11x+24=(x+8)(x+3)$$

EXAMPLE 2 *Resolve into factors* $x^2-10x+24$

The second terms of the factors must be such that their product is $+24$, and their sum -10 Hence they must *both* be *negative*, and it is easy to see that they must be -6 and -4

$$\therefore x^2-10x+24=(x-6)(x-4)$$

EXAMPLE 3 $$a^2-14a+49=(a-7)(a-7)$$
$$=(a-7)^2$$

EXAMPLE 4 $$x^4+10x^2+25=(x^2+5)(x^2+5)$$
$$=(x^2+5)^2$$

EXAMPLE 5 *Resolve into factors* $x^2-11ax+10a^2$

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$ Hence they must be $-10a$ and $-a$

$$x^2-11ax+10a^2=(x-10a)(x-a)$$

EXAMPLES XIV. c

Resolve into factors

1.	x^2+3x+2	2.	x^2+5x+6	3.	x^2+4x+3
4.	x^2-3x+2	5.	x^2-5x+6	6.	x^2-4x+3
7.	y^2+5y+4	8.	y^2+6y+8	9.	$y^2+7y+12$
10.	$y^2-9y+20$	11.	y^2-8y+7	12.	$y^2-7y+10$
13.	$z^2+8z+15$	14.	$z^2-7z+10$	15.	$z^2+9z+18$
16.	$z^2-16z+15$	17.	$z^2+13z+42$	18.	$z^2+8z+16$
19.	a^2-9a+8	20.	$a^2+10a+21$	21.	$a^2+10a+24$
22.	$a^2+9ab+14b^2$	23.	$a^2-8a+12$	24.	$a^2+11ab+24b^2$
25.	b^2-6b+9	26.	$b^2-14b+13$	27.	$b^2+11b+28$
28.	$b^2-10bc+9c^2$.	29.	$b^2+9bc+8c^2$	30.	$b^2+12b+11$
31.	$x^2+16xy+63y^2$	32.	$x^2+10xy+25y^2$	33.	$x^2-14xy+24y^2$
34.	$a^2b^2-4ab-4$	35.	$a^2b^2+10ab+16$	36.	$a^2b^2+12ab+35$
37.	n^4-18n^2+65	38.	n^4-25n^2+136	39.	n^6-10n^3+25
40.	$p^2-18pq+17q^2$	41.	p^4+26p^2+69	42.	$p^2q^2-15pq+44$.

165 Next consider the following cases

$$(x+5)(x-3)=x^2+2x-15, \quad \ldots \quad (1)$$

$$(x-5)(x+3)=x^2-2x-15 \quad (2)$$

By examining the factors on the left-hand side of each of the above results, we notice that

(i) The *second* terms of the factors must have different signs in order to give the negative product which forms the *third* term of the trinomial

Thus $(+5)\times(-3)$ and $(-5)\times(+3)$ both give the product -15

(ii) The *second* terms of the factors are such that their *algebraical sum* gives the coefficient of the *second* term of the trinomial

Thus $+5-3=+2$, and $-5+3=-2$

EXAMPLE 1 *Resolve into factors* $x^2+2x-35$

The second terms of the factors must be such that their product is -35, and their *algebraical sum* $+2$ Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum

The required terms are therefore $+7$ and -5

$$x^2+2x-35=(x+7)(x-5)$$

EXAMPLE 2 *Resolve into factors* $x^2-3x-54$

The second terms of the factors must be such that their product is -54, and their *algebraical sum* -3 Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum

The required terms are therefore -9 and $+6$

$$x^2-3x-54=(x-9)(x+6)$$

EXAMPLES XIV. d.

Resolve into factors

1. a^2-a-2	2. a^2-2a-3	3. a^2-a-6
4. a^2+a-2	5. a^2+2a-3	6. a^2+a-6
7. b^2-4b-5	8. $b^2+2b-15$	9. $b^2-4b-12$
10. b^2+3b-4	11. $b^2-3b-10$	12. b^2-b-12
13. $c^2-cd-20d^2$	14. $c^2-4c-12$	15. c^2+c-20
16. c^2+c-56	17. $c^2-4cd-21d^2$.	18. $c^2+3c-40$
19. $x^2+9x-36$	20. $x^2-5xy-24y^2$.	21. $x^2-4x-45$
22. $x^2-5xy-36y^2$	23. $x^2-2x-24$	24. $x^2+4xy-5y^2$
25. $y^2+y-110$	26. $y^2+2y-63$	27. $y^2-11y-60$
28. $y^2+yz-156z^2$	29. y^4-2y^2-35	30. y^4+17y^2-60
31. $z^2-12z-85$	32. $z^2-9z-90$	33. z^4+7z^2-78
34. z^2+z-72	35. z^4+3z^2-54	36. $z^2+22z-75$
37. $x^2-2xy-8y^2$	38. $x^2+5xy-24y^2$	39. $x^2-4xy-77y^2$
40. $x^2-11xy-26y^2$	41. $x^2+11xy-102y^2$	42. $x^2+6xy-91y^2$
43. $a^2b^2+2ab-15$	44. $a^2b^2-ab-56$	45. $a^2b^2+3ab-54$
46. $2+m-m^2$	47. $14-5x-x^2$	48. $98-7y-y^2$

166. The following Exercise contains miscellaneous examples of trinomials to be separated into factors.

EXAMPLES XIV. e.

Resolve into factors

1. x^2-3x+2	2. $a^2+7ab+10b^2$	3. b^2+b-12
4. $y^2-4y-21$	5. $c^2+12c+11$	6. x^2-4x-5
7. $n^2+12n+20$	8. $y^2+9y-10$	9. $p^2-2pq-24q^2$
10. $y^2+y-110$	11. $z^2-9z-90$	12. $l^2-14l+48$
13. $a^2+18ab+81b^2$	14. $b^2-24bc-81c^2$	15. $c^2+30c+81$
16. $x^2-14x+49$	17. $y^2+10yz+21z^2$	18. $z^2+2z-63$
19. $n^2+11n+24$	20. $p^2-5pq-24q^2$	21. $l^2+9l-36$
22. $a^2b^2-4ab+4$	23. $a^2b^2+10ab+16$	24. $b^2-4bc-45c^2$
25. $m^2+3m-88$	26. $n^2-12n-45$	27. $p^2+10p-39$
28. $x^2y^2-xy-72$	29. z^2-z-20	30. $x^2+xy-56y^2$
31. $a^2-11ab-26b^2$	32. $a^2b^2-ab-56$	33. y^4+y^2-156
34. z^4-7z^2-78	35. y^4-2y^2-35	36. $x^2+6xy-91y^2$
37. $63+2y-y^2$	38. $52-9x-x^2$	39. $132+23a^2+a^4$

The Difference of Two Squares.

167 By multiplying $a+b$ by $a-b$ we obtain the identity

$$(a+b)(a-b)=a^2-b^2,$$

a result which in Art 85 was expressed as follows

The product of the sum and the difference of any two quantities is equal to the difference of their squares

Conversely, *the difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities*

Thus any expression which is the difference of two squares may at once be resolved into factors

EXAMPLE 1 *Resolve into factors* $25x^2-16y^2$

$$25x^2 \quad 16y^2=(5x)^2-(4y)^2$$

Therefore the first factor is the sum of $5x$ and $4y$,

and the second factor is the difference of $5x$ and $4y$,

$$25x^2-16y^2=(5x+4y)(5x-4y)$$

The intermediate steps may usually be omitted

EXAMPLE 2 $\qquad 1-49c^6=(1+7c^3)(1-7c^3)$

The difference of the squares of two numerical quantities may readily be found by the formula $a^2-b^2=(a+b)(a-b)$

EXAMPLE 3 $\qquad (37)^2-(32)^2=(37+32)(37-32)$
$\qquad\qquad =69\times5=345$

EXAMPLE 4 $\qquad (329)^2-(171)^2=(329+171)(329-171)$
$\qquad\qquad =500\times158=79000$

EXAMPLES XIV f

Resolve into factors

1 x^2-1	2. x^2-4	3 x^2-9	4 x^2-25
5. x^2-16	6 $9a^2-b^2$	7 $36-c^2$	8 d^2-49
9. y^2-64	10. $100-z^2$	11 p^2q^2-1	12 c^2d^2-4
13. $9-x^2y^2$	14. $16-a^6$	15 $25-4y^2$	16 $81-25p^2$.
17. $100m^2-49$	18. z^4-121	19 $9a^4-25b^4$	20 x^2y^6-16
21. $4x^2y^2-a^2b^2$	22. $144-a^2x^4$	23 $16a^2x^2-49$	24 k^2-169
25. $a^2b^2c^4-64$	26 $l^2-81m^2n^2$	27. $25m^2-64n^2$	28 a^8-4b^4
29 x^4a^2-49	30. $9x^4-25y^4$	31 $16x^2-y^2z^2$	32 $1-25b^6$
33 p^4q^2-121	34 $49z^2-81$	35 $25b^4-81c^2$	36. x^4y^4-64.

Find by factors the value of

37 $(29)^2-(21)^2$	38 $(51)^2-(49)^2$	39 $(101)^2-(99)^2$
40 $(81)^2-(19)^2$	41 $(1001)^2-1$	42 $(66)^2-(34)^2$
43 $(75)^2-(25)^2$	44 $(102)^2-(98)^2$	45 $(875)^2-(125)^2$.

The Sum or Difference of Two Cubes.

168 If we divide a^3+b^3 by $a+b$ the quotient is a^2-ab+b^2, and if we divide a^3-b^3 by $a-b$ the quotient is a^2+ab+b^2

We have therefore the following identities

$$\begin{cases} a^3+b^3=(a+b)(a^2-ab+b^2), \\ a^3-b^3=(a-b)(a^2+ab+b^2) \end{cases}$$

These important results may be quoted verbally as follows

1 *The* **sum** *of the cubes of any two quantities is equal to the product of two expressions, one of which is the* **sum** *of the two quantities, and the other the sum of their squares* **diminished** *by their product*

2 *The* **difference** *of the cubes of any two quantities is equal to the product of two expressions, one of which is the* **difference** *of the two quantities, and the other the sum of their squares* **increased** *by their product*

EXAMPLE 1
$$\begin{aligned} 8a^3+27b^3 &= (2a)^3+(3b)^3 \\ &= (2a+3b)\{(2a)^2-(2a)(3b)+(3b)^2\} \\ &= (2a+3b)(4a^2-6ab+9b^2) \end{aligned}$$

EXAMPLE 2
$$\begin{aligned} 64x^3-1 &= (4x)^3-(1)^3 \\ &= (4x-1)(16x^2+4x+1) \end{aligned}$$

We may usually omit the intermediate steps and write down the factors at once

EXAMPLES
$$8x^9+729=(2x^3+9)(4x^6-18x^3+81)$$
$$a^6-27x^3=(a^2-3x)(a^4+3a^2x+9x^2)$$

EXAMPLES XIV. g

Resolve into factors

1 a^3-1	2 x^3+1	3. $1+m^3$	4 $1-n^3$
5 $8-b^3$	6 c^3+27	7. d^3+64	8 $1+8p^3$
9. $27y^3-1$	10 $x^3y^3+z^3$	11 a^3b^3-8	12 m^3+27n^3
13. $64-p^3q^3$	14 $125p^3-8$	15 $x^3+1000y^3$	16 $343-y^3$
17. b^3+729	18 x^3+125y^3	19 $216-a^3b^3$	20. n^3+64m^3
21 $125-z^3$	22 $512a^3+b^3$	23 $8c^3-343$	24 $x^3y^3z^3-27$.
25. x^6+64y^3	26 $125a^6+1$	27. $729p^3-8q^3$	28. $8+1000a^6$
29 $64x^6-125y^3$	30. $c^3d^6e^9-1$	31. p^6+8q^3	32 $1-27a^3b^3$
33 z^3+216		34 $343a^3-125b^3$	
35. $64p^6q^6+1$		36. $729x^3y^3-512z^3$	

169 We shall now give an exercise containing miscellaneous examples illustrating all the rules and processes explained in this chapter

In some of the examples which follow it will be found that a simple factor is common to every term Such a factor must always be removed as a first step

EXAMPLES (i) $3x^2y - 21xy^2 + 30y^3 = 3y(x^2 - 7xy + 10y^2)$
$$= 3y(x-5y)(x-2y)$$

(ii) $a^5x^4 - 16a = a(a^4x^4 - 16)$
$$= a(a^2x^2+4)(a^2x^2-4)$$

Now $a^2x^2 - 4 = (ax+2)(ax-2)$,
$$a^5x^4 - 16a = a(a^2x^2+4)(ax+2)(ax-2)$$

EXAMPLES XIV. b.

Resolve into two or more factors

1. $m^3n^2 - 3m^2n^3$	2. $10x^3 + 25x^4y$	3 $y^2 - 2y - 15$
4 $(a+b)p + (a+b)q$	5 $x^2 - xz + xy - yz$	6 $4(a-b) - c(a-b)$
7 $a^4 + a^3 + 2a + 2$	8 $x^5 - 25x$	9 $b^4c^4 - 1$
10. $z^4 - 81$	11. $m^4 - 15m^2 - 100$	12 $a^2b^2 - ab - 110$
13. $p^2 - 14p + 49$	14 $p^2q^2 + 8pq + 16$	15 $z^3 - z^2 - 6z$
16 $a^3 + a^2 - 42a$	17 $25 - 81a^2$	18 $a^4b^4 - 9$
19 $27 + l^3$	20 $1 - 64m^3$	21 $k^4 - 25l^2$
22 $p^3q^3 - 1$	23 $8z^3 + 1$	24 $1 - 64a^2$
25. $2m^4 - m^3 + 4m - 2$	26 $a^4 - 3a^3 - a^3b + 3a^2b$	
27 $p^2 - pq - 20q^2$	28 $l^3 - l^2 - 42l$	29 $a^2b^2c^2 - 81d^2$
30 $x^2 + 21x + 108$	31. $a^2 + 6a - 91$	32 $x^2 - 20xy + 96y^2$
33 $a^2b^2 + 14ab - 51$	34 $c^3 + c^2 - 156c$	35 $m^2n - 6mn^2 + 9n^3$
36 $(a+b)(a-3b) + (a+b)$	37 $(x^2 - y^2) + (x + y)$	

Write down the value of the following products

38 $(3a^2+5)(3a^2-5)$ 39 $(a+2)(a-2)(a^2+4)$ 40 $(1+7x^3)(1-7x^3)$

Write down the value of the following quotients

41 $\dfrac{a^3b^3+27}{ab+3}$ 42 $\dfrac{8x^3-1000}{2x-10}$ 43. $\dfrac{1-64p^3}{1+4p+16p^2}$ 44 $\dfrac{729+y^6}{9+y^2}$

45. $\dfrac{x^2+2x-99}{x-9}$ 46 $\dfrac{a^2+20ab+96b^2}{a+8b}$ 47. $\dfrac{c^2d^2-3cd-180}{cd-15}$

$*_*^*$ *Examples illustrating the application of easy factors will be found in* Examples XVIII b, XIX. b, XX. b

CHAPTER XV.

HARDER CASES OF MULTIPLICATION AND DIVISION

170 EASY cases of Multiplication and Division of algebraical expressions have been dealt with in previous chapters The principles already explained will now be applied to examples of greater difficulty

EXAMPLE 1 *Find the product of* $3x^2-2x-5$ *and* $2x-5$

$$\begin{array}{llll}
3x^2 & -\ 2x & -5 & \\
 & \ \ 2x & -\ 5 & \\
\hline
6x^3 & -\ 4x^2 & -10x & \\
 & -15x^2 & +10x & +25 \\
\hline
6x^3 & -19x^2 & & +25
\end{array}$$

Each term of the first expression is multiplied by $2x$, the first term of the second expression, then each term of the first expression is multiplied by -5, like terms are placed in the same columns and the results added.

[*Check* Put $x=1$ in each expression, and in the product

$$\left.\begin{array}{r} 3x^2-2x-5=3-2-5=-4 \\ 2x-5=2-5\quad=-3 \end{array}\right\} \quad (-4)\times(-3)=12$$

Also $\qquad 6x^3-19x^2+25=6-19+25=12$]

If the expressions are not arranged according to powers, ascending or descending, of some common letter, a rearrangement will be found convenient

EXAMPLE 2 *Find the product of* $2a^2+4b^2-3ab$ *and* $3ab-5a^2+4b^2$

$$\begin{array}{lllll}
2a^2 & -\ 3ab & +4b^2 & & \\
-\ 5a^2 & +\ 3ab & +4b^2 & & \\
\hline
-10a^4 & +15a^3b & -20a^2b^2 & & \\
 & \ \ 6a^3b & -\ 9a^2b^2 & +12ab^3 & \\
 & & \ \ 8a^2b^2 & -12ab^3 & +16b^4 \\
\hline
-10a^4 & +21a^3b & -21a^2b^2 & & +16b^4
\end{array}$$

The rearrangement is not *necessary*, but convenient, because it makes the collection of like terms more easy

This may be checked as before by substituting any simple values of a and b, suitably chosen [See Note to Ex. 2, Art 65]

EXAMPLE 3 *Multiply* $2xz-z^2+2x^2-3yz+xy$ *by* $x-y+2z$

$$\begin{array}{llllllllll}
2x^2 & +\ xy & +2xz & -3yz & -z^2 & & & & & \\
x & -\ y & +2z & & & & & & & \\
\hline
2x^3 & +\ x^2y & +2x^2z & -3xyz & -\ xz^2 & & & & & \\
 & -2x^2y & & -2xyz & & -xy^2 & +3y^2z & +\ yz^2 & & \\
 & & 4x^2z & +2xyz & +4xz^2 & & & -6yz^2 & -2z^3 & \\
\hline
2x^3 & -\ x^2y & +6x^2z & -3xyz & +3xz^2 & -xy^2 & +3y^2z & -5yz^2 & -2z^3 &
\end{array}$$

EXAMPLES XV. a

Multiply together the following pairs of expressions and check the results in Examples 1–12

1. $x^2-x+1,\ 2x-1$
2. $a^2-2a+1,\ 3a+2$
3. $2x^2+3x-5,\ 3x-2$
4. $4z^2+3z+5,\ 3z-5$
5. $c^2-3c-6,\ c-3$
6. $3b^2-b-4,\ 4-3b$
7. $5x^3-3x^2+8,\ 3x-4$
8. $6d^2-d-7,\ 1-2d$
9. $x^2-6x+7,\ x-1$
10. $a^2b-ab^2-b^3,\ a+b$
11. $3y^2+5y-1,\ 4y-3$
12. $ax^2+bx-c,\ ax-c$
13. $a^2-ab+b^2,\ a+b$
14. $a^2+ab+b^2,\ a-b$
15. $x^2-6x+9,\ x-3$
16. $c^2+2cd+d^2,\ -c-2d$
17. $1-9x^2+20x^3,\ 1-3x$
18. $a^4+a^2b^2+b^4,\ a^2-b^2$
19. $m^2-2+n,\ m^2-n+2$
20. $x^2-3x+1,\ x^2-3x+1$

Find the product of

21. $a-b+c,\ a+b-c$
22. $2x-y+3z,\ 2x+y-3z$
23. $1-3d+d^2,\ 1+3d-d^2$
24. $2x^2-3ax+a^2,\ 3ax-2a^2-x^2$.
25. $y^3-5y+6,\ y-2+3y^2$
26. $a-b+c-d,\ a-b-c+d$
27. $x^2-xy-x+y^2-y+1,\ x+y+1$
28. $a^2+b^2+c^2-bc-ca-ab,\ a+b+c$
29. $x^3-y^3+3xy^2-3x^2y,\ x^3+3xy^2+3x^2y+y^3$
30. $2a^3b^3-c^3+2abc^2-3a^2b^2c,\ a^3b^2-a^2bc-ac^2$

The Method of Detached Coefficients

171 When two compound expressions contain powers of one letter only, the labour of multiplication may be lessened by using **detached coefficients**, that is, by writing down the coefficients only, multiplying them together in the ordinary way, and then inserting the successive powers of the letter at the end of the operation In using this method the expressions must be arranged according to ascending or descending powers of the common letter, and zero coefficients must be used to represent terms corresponding to missing powers of that letter

EXAMPLE *Multiply* $2x^3-4x^2-5$ *by* $3x^2+4x-2$

$$\begin{array}{r}
2-\ 4+\ 0-\ 5\\
\underline{3+\ 4-\ 2}\\
6-12+\ 0-15\\
8-16+\ 0-20\\
\underline{-\ 4+\ 8-\ 0+10}\\
6-\ 4-20-\ 7-20+10
\end{array}$$

Here there is no term containing x in the multiplicand, and we insert a zero coefficient to represent the missing power In the product the highest power of x is clearly x^5, and the others follow in descending order

Thus the product is

$$6x^5-4x^4-20x^3-7x^2-20x+10$$

172 The method of detached coefficients may also be used to multiply two compound expressions which are *homogeneous* and contain powers of *two* letters [Arts 43–45 should here be revised]

From the rule for *distributing a product* (Art 65) it follows that *the product of any two homogeneous expressions is itself a homogeneous expression, the degree of which is the sum of the degrees of the two factors which form the product*

For example, if each term of the first factor is of the fourth degree, and each term of the second factor of the second degree, all the partial products will be of the sixth degree Hence the complete product will be homogeneous and of the sixth degree

EXAMPLE 1 *Multiply* $3a^4+2a^3b+4ab^3+2b^4$ *by* $2a^2-b^2$

$$\begin{array}{r}
3+2+0+4+2 \\
\underline{2+0-1} \\
6+4+0+8+4 \\
\underline{-3-2-0-4-2} \\
6+4-3+6+4-4-2
\end{array}$$

The two expressions are written in descending powers of a and ascending powers of b We write a zero coefficient to represent the term containing a^2b^2 which is absent in the first expression Similarly, the term containing ab is represented by a zero coefficient in the second expression

It is easily seen how the powers of a and b arise in the successive terms, and the complete product is

$$6a^6+4a^5b-3a^4b^2+6a^3b^3+4a^2b^4-4ab^5-2b^6$$

NOTE The second line of multiplication is not written down as all the terms are zero

EXAMPLE 2 *Expand* $(2-x+3x^3-x^4)(1-2x^2+x^3+2x^5)$ *as far as the term involving* x^3

In distributing the product we may omit any term of higher dimensions than x^3 in each factor

$$\begin{array}{r|}
2-1+0+3 \\
\underline{1+0-2+1} \\
2-1+0+3 \\
-4+2 \\
\underline{2} \\
2-1-4+7
\end{array}$$

Here we omit terms which would fall to the right of the vertical line, as these would involve x^4 and higher powers of x

Thus the required result is

$$2-x-4x^2+7x^3$$

EXAMPLES XV. b

Distribute the following products, using detached coefficients

1 $(2a^4-4a^2-1)(2a^4+4a^2+1)$ 2. $(3x^3-x^2+2)(x^2-5)$

3 $(1+3a+3a^2+a^3)(1-2a+a^2)$ 4 $(3p^2-pq+q^2)(p^2+2pq-q^2)$

5 $(x-3+2x^2)(2-x^2-5x)$ 6 $(x^4+x^2y^2+y^4)(x^2-y^2)$

7 $(x^5+x^4+x^2+2x+1)(x^3+x-2)$

8 $(6y^2+y^4+1-4y^3-4y)(1+y^2-2y)$ in ascending powers

9. $(x^4+2x^3y+3x^2y^2+4xy^3+5y^4)(x^2-2xy+y^2)$

Expand the following products

10. $(1-x+2x^2+x^3)(1+x-2x^3+x^4)$ as far as x^2

11. $(2+x^2+x^3-x^4)(3-2x-x^3+x^4)$ as far as x^3

12. $(y-3y^2+y^3)(y+2y^3-y^5)$ as far as y^4

13. $(2-x+3x^2-2x^4)(1-2x+x^3)(1+3x+x^2)$ as far as x^2

14. Find the first four terms of

(i) $(1-2x+3x^2+x^3)^2$, (ii) $(1+a+a^2+a^3+a^4)^3$

Division.

173 The process of Art 73 will now be applied to harder cases of division of compound expressions

EXAMPLE 1 *Divide* $6x^4-x^3+4x^2+5x-6$ *by* $3x^2+x-2$

$$
\begin{array}{r|l}
3x^2+x-2 \,) & 6x^4-\ x^3+4x^2+5x-6\ (\,2x^2-x+3 \\
 & \underline{6x^4+2x^3-4x^2} \\
 & \quad -3x^3+8x^2+5x \\
 & \quad \underline{-3x^3-\ x^2+2x} \\
 & \qquad\qquad 9x^2+3x-6 \\
 & \qquad\qquad \underline{9x^2+3x-6}
\end{array}
$$

EXAMPLE 2 *Divide* $4x^3-5x^2+6x^5-18-x^4-3x$ *by* $3+2x^2-x$

First arrange each of the expressions in descending powers of x

$$
\begin{array}{r|l}
2x^2-x+3\,) & 6x^5-\ x^4+4x^3-\ 5x^2-3x-18\ (\,3x^3+x^2-2x-5 \\
 & \underline{6x^5-3x^4+9x^3} \\
 & \quad 2x^4-5x^3-\ 5x^2 \\
 & \quad \underline{2x^4-\ x^3+\ 3x^2} \\
 & \qquad -4x^3-\ 8x^2-3x \\
 & \qquad \underline{-4x^3+\ 2x^2-6x} \\
 & \qquad\qquad -10x^2+3x-18 \\
 & \qquad\qquad \underline{-10x^2+5x-15} \\
 & \qquad\qquad\qquad -2x-\ 3
\end{array}
$$

Now the division cannot be carried on any further without introducing fractional terms in the quotient, thus the quotient is $3x^3+x^2-2x-5$, and there is a remainder $-2x-3$

In all cases where the divisor and dividend are arranged in *descending powers of some common letter*, if the divisor is not exactly contained in the dividend, the work should be carried on until the highest power in the remainder is lower than that in the divisor

Some further remarks on inexact division will be found in a later chapter

174 The method of detached coefficients may be used in Division

(i) When the two compound expressions contain powers of one letter only

(ii) When the two compound expressions are homogeneous and contain powers of two letters only

EXAMPLE *Divide* $2a^5+6a^4+9a^2-17a+6$ *by* $2a^3+4a-3$

Here the missing powers, a^3 in the dividend and a^2 in the divisor, must be represented by zero coefficients

$$\begin{array}{r}
2+0+4-3\,)\,2+6+0+\ 9-17+6\,(\,1+3-2 \\
\underline{2+0+4-\ 3} \\
6-4+12-17 \\
\underline{6+0+12-\ 9} \\
-4+\ 0-\ 8+6 \\
\underline{-4+\ 0-\ 8+6}
\end{array}$$

Since the first power of a in the quotient is obviously a^2, the complete quotient is a^2+3a-2

EXAMPLES XV. c

(Most of the following Examples may be worked by Detached Coefficients.)

Divide

1. $2a^3-7a^2-a+2$ by a^2-3a-2
2. $8a^3+10a^2-7a-6$ by $4a^2-a-2$
3. $6b^3-11b^2+6b-1$ by $2b^2-3b+1$
4. $6x^3-25x^2+28x-49$ by $3x^2-2x+7$
5. $6y^3+11y^2-39y-65$ by $3y^2+13y+13$
6. $21c^3-5c^2-3c-2$ by $7c^2+3c+1$
7. $12d^3-19d^2-2d+8$ by $4d^2-d-2$
8. $21x^3-26x^2-27x+20$ by $7x^2+3x-4$
9. $8y^3-8y^2+4y-1$ by $4y^2-2y+1$
10. $5b^3-7b^2c+17bc^2-6c^3$ by $b^2-bc+3c^2$
11. $6x^3-17x^2-16x+7$ by $3x^2+2x-1$
12. $m^4-4m^3-18m^2-11m+2$ by m^2-7m+1
13. $10x^3-19x^2y+9xy^2-y^3$ by $5x^2-7xy+y^2$
14. $12x^4+x^3-8x^2+7x-2$ by $3x^2-2x+1$
15. $3a^5+3a^4+2a^3+1$ by $3a^3-a+1$
16. $4a^5+19a^3x^2+2a^2x^3-5ax^4+10x^5$ by a^2+5x^2
17. $30b+b^4-9-25b^2$ by $3-5b+b^2$
18. $18k^7+21k^4-24k^3+21k^2+6k-7$ by $3k^4+3k^2-1$.
19. $35c^4-3+c^3+11c+10c^2$ by $3c-1+7c^2$
20. $33p^2-13p^3+15p^4-9p+10$ by $5p^2-p+2$
21. $2a^4+4a^2+7a+1-a^3$ by a^2-a+3.

Divide

22 $c^5-2c^4-4c^3+19c^2$ by c^3-7c+5 23 $8x^6+y^6$ by y^2+2x^2

24. $81x^4-1$ by $3x-1$ 25 $3x^5-5x^3+2$ by x^2-2x+1

26 c^8+64 by c^4-4c^2+8 27 x^9-y^9 by x^3-y^3

28 $30y+9-71y^3+28y^4-35y^2$ by $4y^2-13y+6$

29 $6m-5m^3+12m^4+20-33m^2$ by $4m^2+m-5$

30 $4x^5-29x-36+8x^2-7x^3+6x^4$ by x^3-2x^2+3x-4

31. $3a^2+8ab+4b^2+10ac+8bc+3c^2$ by $3a+2b+c$

32 $25a^6-44a^4+4a^2-9$ by $5a^3-2a^2-4a-3$

Important Cases in Division.

175 The following example deserves special notice

EXAMPLE. *Divide* $a^3+b^3+c^3-3abc$ *by* $a+b+c$

$$
\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2 \\
\qquad\quad \underline{a^3+a^2b+a^2c} \\
\qquad\qquad -a^2b-a^2c-3abc \\
\qquad\qquad \underline{-a^2b-ab^2-abc} \\
\qquad\qquad\qquad -a^2c+ab^2-2abc \\
\qquad\qquad\qquad \underline{-a^2c\qquad -abc-ac^2} \\
\qquad\qquad\qquad\qquad ab^2-abc-ac^2+b^3 \\
\qquad\qquad\qquad\qquad \underline{ab^2\qquad\qquad +b^3+b^2c} \\
\qquad\qquad\qquad\qquad -abc+ac^2-b^2c \\
\qquad\qquad\qquad\qquad \underline{-abc\qquad -b^2c-bc^2} \\
\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}
$$

Here the work is arranged in *descending* powers of a, and the other letters are taken in alphabetical order; thus, in the first remainder a^2b precedes a^2c, and a^2c precedes $3abc$ A similar arrangement is preserved throughout the work

It is equally important to remember the result of this example in its converse form that is

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$$

EXAMPLES XV d.

Divide

1 $1-a^3+3a^4+a^9$ by $1-a+a^3$ 2 $x^3+3xy+y^3-1$ by $x+y-1$

3 $a^3-b^3-c^3-3abc$ by $a-b-c$ 4 $a^3+b^3+8c^3-6abc$ by $a+b+2c$

5 $x^3-27y^3+8z^3+18xyz$ by $x-3y+2z$

6 $a^6x^6-30a^3x^3+8a^3x^3+125$ by $5+2ax+a^2x^2$

7. $27y^3+18xy+8-x^3$ by $x-3y-2$

8 $8x^3-y^3+z^3+6xyz$ by $y-z-2x$

176 The following examples in division may be easily verified, they are of great importance, and should be carefully noticed

$$\text{I}\quad \begin{cases} \dfrac{x^2-y^2}{x-y}=x+y, \\ \dfrac{x^3-y^3}{x-y}=x^2+xy+y^2, \\ \dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3, \end{cases}$$

and so on, the divisor being $x-y$, the terms in the quotient *all positive*, and the index in the dividend *either odd or even*.

$$\text{II}\quad \begin{cases} \dfrac{x^3+y^3}{x+y}=x^2-xy+y^2, \\ \dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4, \\ \dfrac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6, \end{cases}$$

and so on, the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always odd*

$$\text{III}\quad \begin{cases} \dfrac{x^2-y^2}{x+y}=x-y, \\ \dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3, \\ \dfrac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5, \end{cases}$$

and so on, the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always even*

IV The expressions x^2+y^2, x^4+y^4, x^6+y^6, (where the index is *even*, and the terms *both positive*) are *never* divisible by $x+y$ or $x-y$

All these different cases may be more concisely stated as follows

(1) x^n-y^n is divisible by $x-y$ if n be *any* whole number

(2) x^n+y^n is divisible by $x+y$ if n be any *odd* whole number

(3) x^n-y^n is divisible by $x+y$ if n be any *even* whole number

(4) x^n+y^n is *never* divisible by $x+y$ or $x-y$, when n is an *even* whole number

EXAMPLES XV. e.

Without division write down the quotients in the following cases

1. $\dfrac{p^3+q^3}{p+q}$	2. $\dfrac{x^3+8}{x+2}$	3. $\dfrac{a^4-b^4}{a-b}$	4. $\dfrac{x^4-y^4}{x+y}$
5. $\dfrac{27-x^3}{3-x}$	6. $\dfrac{16-d^4}{2+d}$	7. $\dfrac{x^5+y^5}{x+y}$	8. $\dfrac{a^5-1}{a-1}$
9. $\dfrac{x^6-y^6}{x-y}$	10. $\dfrac{c^6-d^6}{c+d}$	11. $\dfrac{a^7+1}{a+1}$	12. $\dfrac{32-z^5}{2-z}$.
13. $\dfrac{c^4-d^4}{c^2-d^2}$	14. $\dfrac{x^6-1}{x-1}$	15. $\dfrac{a^6+64}{a^2+4}$	16. $\dfrac{8a^6+1}{2a^2+1}$

Write down the products in the following cases

17. $(a+b)(a^2-ab+b^2)$	18. $(c-d)(c^2+cd+d^2)$
19. $(1-x)(1+x+x^2+x^3)$	20. $(a+1)(a^3-a^2+a-1)$
21. $(x^2-y^2)(x^4+x^2y^2+y^4)$	22. $(2x+3y)(4x^2-6xy+9y^2)$
23. $(x+1)(x^4-x^3+x^2-x+1)$	24. $(x-1)(x^5+x^4+x^3+x^2+x+1)$
25. $(x+1)(x^5-x^4+x^3-x^2+x-1)$	26. $(x^2-5)(x^4+5x^2+25)$

Functional Notation Remainder Theorem

177 In Art 129 any expression involving x has been defined as a **function of x** At this stage we are only concerned with **rational integral functions.** A function is said to be *rational* when no term contains a square or other root, and it is said to be *integral with respect to x* when the powers of x are all positive integers

Thus lx^2+mx+n, px^3+qx^2+rx+s are rational integral functions of x of two and three dimensions respectively

Such functions are often briefly denoted by symbols such as $f(x)$ and $F(x)$

Thus in any example involving the functions $x^2-7x+10$ and $5x^3+6$ we might shorten the work by saying

'let $f(x)\equiv x^2-7x+10$, and $F(x)\equiv 5x^3+6$,'

and throughout that example $f(x)$ and $F(x)$ would be considered as short equivalents of these particular functions

If in the course of work any definite value is given to the variable x, that value must appear in the functional symbols

Thus $f(2)\equiv 2^2-7\cdot 2+10$, and $F(3)\equiv 5\cdot 3^3+6$, and, more generally, $f(a)$ stands for the value of the function $f(x)$ when x has the value a

178 When dividend and divisor are functions of x in descending powers, each successive remainder in the process of division is of lower dimensions than the preceding one Hence the division can always be carried on until the remainder is of lower dimensions than the divisor

179 The following example in division is very important

$$
\begin{array}{l}
x-a\,)\,px^3+qx^2+rx+s\,(\,px^2+(pa+q)x+pa^2+qa+r \\
\underline{px^3-pax^2} \\
\qquad (pa+q)x^2+rx \\
\qquad \underline{(pa+q)x^2-(pa^2+qa)x} \\
\qquad\qquad (pa^2+qa+r)x+s \\
\qquad\qquad \underline{(pa^2+qa+r)x-(pa^3+qa^2+ra)} \\
\qquad\qquad\qquad pa^3+qa^2+ra+s
\end{array}
$$

Here the division has been carried on until the remainder does not contain x, and its value is the result obtained by replacing x by a in the dividend This is a particular case of an important proposition known as the **Remainder Theorem**

If any rational integral function f(x) *is divided by* x − a *until the remainder does not contain* x, *the remainder is* f(a)

Again, the remainder is zero when the given expression is exactly divisible by $x-a$, hence

If a rational integral function of x *becomes equal to* 0 *when* a *is written for* x *it contains* x − a *as a factor*

Or in symbols $f(x)$ is divisible by $x-a$ when $f(a)=0$

Note also that $f(x)$ is divisible by $x+a$ when $f(-a)=0$

EXAMPLE 1 *Find the remainder when* x^4-2x^3+x-7 *is divided* (i) *by* $x-2$, (ii) *by* $x+3$

(i) Here $f(x)\equiv x^4-2x^3+x-7$,

$$f(2)=2^4-2\cdot 2^3+2-7=-5$$

the remainder is -5

(ii) Since $x+3=x-(-3)$, we must write -3 for x in $f(x)$

Thus $$f(-3)=(-3)^4-2\cdot(-3)^3-3-7$$
$$=81+54-3-7=125,$$

the remainder is 125

EXAMPLE 2 *Without division shew that* $x+7$ *is a factor of* $x^3-39x+70$ *What other factors has this expression?*

If $f(x)\equiv x^3-39x+70$,

$$f(-7)=-343+273+70=0$$

Since the remainder is zero, $x+7$ is a factor of $x^3-39x+70$

Again, since $70=7\times 10=7\times 2\times 5$, we may apply the Remainder Theorem to test divisibility by the pairs of factors $x-2$ and $x-5$, or $x+2$ and $x+5$

On trial, $f(2)=0$ and $f(5)=0$, thus $x-2$ and $x-5$ are factors of $f(x)$.

Hence finally $x^3-39x+70=(x+7)(x-2)(x-5)$

EXAMPLE 3 *Find what numerical values must be given to* a *and* b *in order that the expression* $2x^3+ax^2-13x+b$ *may be divisible by* $(x-3)(x+2)$

If $f(x)$ stands for the expression, we must have $f(3)=0$ and $f(-2)=0$

Now $\quad f(3)=54+9a-39+b \qquad f(-2)=-16+4a+26+b$

$\qquad =9a+b+15, \qquad\qquad =4a+b+10,$

a and b satisfy the equations $9a+b+15=0,$

$$4a+b+10=0$$

By subtraction, $5a+5=0$, whence $a=-1$

Hence, by substitution, $\qquad b=-6$

EXAMPLES XV. f.

1 If $f(x)\equiv x^3-3x+2$, find the values of $f(2)$, $f(5)$, $f(1)$

2 Find the values of $f(2)$, $f(-2)$, $f(3)$ when $f(x)\equiv x^3-3x^2-4x+12$. What do you infer from the results?

3. If $f(n)\equiv n^2+n$, find the value of $f(n+1)-f(n)$

Find the remainder (if any) which results from dividing

4. x^3+2x^2-x+6 by $x-3$ $\qquad$ 5 $x^5-x^4+x^3+2x+5$ by $x+1$.

6. $x^3+9x^2+26x+24$ by $x+4$ $\qquad$ 7 $x^3-8x^2-31x-20$ by $x-11$.

Without actual division shew that

8 $x-1$ is a factor of $x^{29}-13x+12$

9 $x+3$,, ,, $x^4+29x+6$

10 $x-a$,, ,, $x^3-4ax^2+4a^2x-a^3$

11. $x+2c$,, ,, $x^3+7cx^2+11c^2x+2c^3$

12 Prove that $x^5-7x^3-12x+18$ is divisible by x^2+2x-3

13 If $x^3-2ax+15$ is divisible by $x+5$, find the value of a

14 Determine the values of p and q in order that the expression $px^3+qx^2-58x-15$ may be divisible by $x^2+2x-15$

15 Apply the Remainder Theorem to shew that the factors of $x^3-37x-84$ are $x+3$, $x+4$, and $x-7$

16 If the expressions x^3+2x^2+3x+a and x^3+x^2+9 leave the same remainder when divided by $x+2$, find the value of a

17 By means of the Remainder Theorem find the factors of

(i) x^3-2x^2-5x+6, $\qquad$ (ii) $x^3-19x+30$;

(iii) $x^3+x^2-10x+8$, $\qquad$ (iv) x^4-2x^3-6x-9;

(v) $2x^3+13x^2-36$, $\qquad$ (vi) $2x^3-3x^2-12x+20$

18. Find what numerical values must be given to a and b in order that the expression $2x^3-(a-b)x^2-(4b-1)x+4a$ may be divisible by

CHAPTER XVI

INVOLUTION AND EVOLUTION

Involution

180 DEFINITION **Involution** is the general name for multiplying an expression by itself so as to find its second, third, fourth, or any other power

Involution may always be effected by actual multiplication Here, however, we shall deal with some cases in which the results may be written down at once

By definition $(a^2)^3=a^2\ a^2\ a^2=a^{2+2+2}=a^{2\times 3}=a^6$,

$$(-a^2)^3=(-a^2)(-a^2)(-a^2)=-a^{2+2+2}=-a^{2\times 3}=-a^6,$$

$$(-x^3)^2=(-x^3)(-x^3)=x^{3+3}=x^{3\times 2}=x^6,$$

$$(-x^5)^3=(-x^5)(-x^5)(-x^5)=-x^{5+5+5}=-x^{5\times 3}=-x^{15},$$

$$(4m^4)^2=(4)^2(m^4)^2=16m^{4\times 2}=16m^8$$

Hence we obtain a rule for raising a simple expression to any required power

Raise each of the LITERAL *factors of the expression to the required power by* MULTIPLYING *its index by the index of that power If there is a numerical coefficient, raise it to the required power by Arithmetic, and prefix the result, with its proper sign, to the literal expression already obtained*

181 The following general principles, which are evident from the Rule of Signs, should here again be noted

(1) The SQUARE *of every expression, whether such expression is positive or negative, is* POSITIVE

(2) *No* EVEN *power of* ANY *expression can be* NEGATIVE

(3) *Any* ODD *power of an expression will have the* SAME SIGN *as the expression itself*

EXAMPLE 1 $\quad (-2x^2)^5=(-2)^5x^{2\times 5}=-32x^{10}$

EXAMPLE 2 $\quad (-3ab^3)^6=(-3)^6a^6b^{3\times 6}=729a^6b^{18}$

EXAMPLE 3 $\quad \left(\dfrac{2ab^3}{3x^2y}\right)^4=\dfrac{16a^4b^{12}}{81x^8y^4}$

In Ex 3 it will be seen that the numerator and denominator are operated upon separately

EXAMPLES XVI. a.

Write down the square of each of the following expressions

1. $2a^3$	2. $3xy^3$	3. b^2c^3	4. $4ab^2$
5. $7c^3d^5$	6. $6a^3b^6$	7. $5a^2b^5c$	8. $-3ab^3c^5$
9. $-9p^4q^6$	10. $-4ab^8$	11. $-a^3b^3c^5d^4$	12. $-8m^3n^6$
13. $-\dfrac{3xy}{4}$	14. $\dfrac{2m^2n^3}{3p^5q^2}$	15. $-\dfrac{1}{3x^3}$	16. $-\dfrac{1}{4y^3}$
17. $-\dfrac{7k^3l^5}{8pq^4}$	18. $-\dfrac{1}{9a^5b^4c^4}$	19. $\dfrac{4x^4y^3z^5}{7}$	20. $-\dfrac{11}{10b^5c^6}$

Write down the cube of each of the following expressions

21. x^3y^2	22. $2x^2$	23. $3y^3$	24. $6x^2z^3$
25. $-3x^9$	26. $-2x^2y^3z$	27. $-4p^4q^2$	28. $-5c^3d^6$
29. $\dfrac{1}{3a^3b}$	30. $-\dfrac{2k^3l^3}{p^5q}$	31. $-\dfrac{3x^4y^2}{7}$	32. $-\dfrac{6}{5a^5b^2}$

Write down the value of each of the following expressions

33. $(x^2y)^3$	34. $(-xy^3)^4$	35. $(-2m^3n^2)^5$	36. $(-3p^2q^3)^3$
37. $\left(\dfrac{1}{4a^5}\right)^3$	38. $\left(-\dfrac{2a^3x}{p^2q^5}\right)^5$	39. $\left(-\dfrac{1}{a^2b^4c^5}\right)^8$	40. $\left(\dfrac{a^4b^5c}{x^3y^8z^2}\right)^9$

182. In Art. 85 the following rules were stated:

(1) *The square of the* SUM *of two quantities is equal to the sum of their squares* INCREASED *by twice their product.*

(2) *The square of the* DIFFERENCE *of two quantities is equal to the sum of their squares* DIMINISHED *by twice their product.*

Thus
$$(x+2y)^2 = x^2 + 2 \cdot x \cdot 2y + (2y)^2$$
$$= x^2 + 4xy + 4y^2;$$
$$(2a^3-3b^2)^2 = (2a^3)^2 - 2 \cdot 2a^3 \cdot 3b^2 + (3b^2)^2$$
$$= 4a^6 - 12a^3b^2 + 9b^4.$$

We may now obtain a rule for writing down the square of an expression which consists of more than two terms.

Thus since $(a+b)^2 = a^2 + 2ab + b^2$,
$$(x+y+z)^2 = \{(x+y)+z\}^2 = (x+y)^2 + 2(x+y)z + z^2$$
$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$$
$$= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$$

In the same way we may prove
$$(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc;$$
$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

In each of the above instances we observe that the square consists of

(1) the sum of the squares of the several terms of the expression, together with

(2) twice the sum of the products of those terms taken two and two with their proper signs; that is, each product has the sign + or − *according as the two factors composing it have like or unlike signs*

NOTE The *square terms* are always positive

The same laws hold whatever be the number of terms in the expression to be squared Hence the following general Rule

The square of any multinomial is equal to the sum of the squares of the several terms, together with the algebraical sum of twice the product of each term into each of the terms which follow it

Ex 1 $\quad (x-2y-3z)^2 = x^2+4y^2+9z^2-2\cdot x\cdot 2y-2\cdot x\cdot 3z+2\cdot 2y\cdot 3z$

$$= x^2+4y^2+9z^2-4xy-6xz+12yz$$

Ex 2 $\quad (1+2x-3x^2)^2 = 1+4x^2+9x^4+2\cdot 1\cdot 2x-2\cdot 1\cdot 3x^2-2\cdot 2x\cdot 3x^2$

$$= 1+4x^2+9x^4+4x-6x^2-12x^3$$

$$= 1+4x-2x^2-12x^3+9x^4,$$

by collecting like terms and rearranging

In Ex 1 we see that the square contains *six* terms; in Ex 2, owing to collection of like terms, the square contains only *five* terms It is easy to see from these examples that *the square of a trinomial can never have more than six terms*

EXAMPLES XVI b.

Write down the square of each of the following expressions

1. $a+2b$	2. $2a-b$	3 $x+3y$	4 $2x-3y$
5 $p-5q$	6 $4-x$	7 $a+7$	8 $cd+1$
9 $2ab-3$	10. $1+x^2$	11 $1+3xy$	12 x^2-2x
13. $a+b-c$	14 $a-b-c$	15 $x+y+2z$	
16 $x-2y+z$	17 $2p-q-r$	18. x^2-x-1	
19. $2x^2-x+1$	20 $k^2-l^2+m^2$	21. $3l^2-5l+2$	
22. $a-b+c+d$	23 $2x+y-3a+b$	24 $m+n-p+2q$	
25 $a-\frac{1}{2}b+\frac{c}{4}$	26 $\frac{x}{3}-3y-\frac{3}{2}$	27. $\frac{3}{2}-m+\frac{2}{3}m^2$	

183 By actual multiplication, we have

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3$$

Also $\quad (a-b)^3=a^3-3a^2b+3ab^2-b^3$

NOTE These formulæ may also be written as follows

$$(a+b)^3=a^3+b^3+3ab(a+b),$$
$$(a-b)^3=a^3-b^3-3ab(a-b)$$

By observing the law of formation of the terms in the foregoing results we can write down the cube of any binomial

EXAMPLE 1 $(2x+y)^3=(2x)^3+3(2x)^2y+3(2x)y^2+y^3$
$=8x^3+12x^2y+6xy^2+y^3$

EXAMPLE 2 $(3x-2a^2)^3=(3x)^3-3(3x)^2(2a^2)+3(3x)(2a^2)^2-(2a^2)^3$
$=27x^3-54x^2a^2+36xa^4-8a^6$

EXAMPLES XVI c

Write down the cube of each of the following expressions

1.	$p+q$	2	$m-n$	3	$a-2b$	4.	$2c+d$
5	$3x+2y$	6	$4x-1$	7	$1-5y$	8	$2ab-3$
9	a^2+3b^2	10	p^2-3q^2	11	$3c^2-2d^3$	12	$4x^3-3x.$
13.	$a-\frac{b}{2}$	14	$\frac{c}{3}+1$	15	$\frac{l^2}{3}-3l$	16	$\frac{x}{6}+2y$

Evolution

184 DEFINITION The **root** of any proposed expression is that quantity which will produce the given expression by being raised to the power denoted by the index of the root

The operation of finding the root is called **Evolution** it is the reverse of Involution It is sometimes spoken of as the *extraction* of the root

Thus $\sqrt[3]{x^6}=x^2$, because $(x^2)^3=x^6$
$\sqrt[3]{-x^6}=-x^2$, because $(-x^2)^3=-x^6$

It should here again be noted that every positive quantity has two square roots equal in magnitude but opposite in sign [Art 95]

EXAMPLE 1 $\sqrt{9a^2x^6}=+3ax^3$, or $-3ax^3$

The two roots are conveniently written $\pm 3ax^3$, and $\pm$ is read "*plus or minus*" The symbol $\pm$ is known as "the double sign"

EXAMPLE 2 $\sqrt{169m^{10}n^8}=\pm 13m^5n^4$

Again, by the Rule of Signs, it is evident that

(1) *Any* EVEN *root of a* POSITIVE *quantity will have the double sign.*

(2) *No* NEGATIVE *quantity can have an* EVEN *root*

(3) *Every* ODD *root of a quantity has the* SAME SIGN *as the quantity itself*

From (2) it follows that such expressions as $\sqrt{-3}$, $\sqrt{-25}$, $\sqrt{-a}$ can have no arithmetical meaning To distinguish them from real positive or negative quantities, such expressions are called **imaginary**, unreal, or impossible

EXAMPLES (i) $\sqrt[4]{a^{12}b^8} = \pm a^3b^2$, because $(a^3b^2)^4$ or $(-a^3b^2)^4 = a^{12}b^8$,

(ii) $\sqrt[3]{-x^9} = -x^3$, because $(-x^3)^3 = -x^9$,

(iii) $\sqrt[5]{c^{20}} = c^4$, because $(c^4)^5 = c^{20}$,

(iv) $\sqrt[4]{81x^{20}} = \pm 3x^5$, because $(3x^5)^4$ or $(-3x^5)^4 = 81x^{20}$

In the present chapter, in dealing with the even root of a positive quantity, we shall confine our attention to the *positive* value

185 From the foregoing examples we may deduce a general Rule for extracting any required root of a simple expression

Find the index of each literal factor by DIVIDING *its index in the given expression by the index of the root required*

If there is a numerical coefficient find its root by Arithmetic, and prefix it, with its proper sign, to the literal expression already obtained

EXAMPLES (i) $\sqrt[3]{-64x^6} = -4x^{\frac{6}{3}} = -4x^2$,

(ii) $\sqrt[6]{a^{24}c^{18}} = a^{\frac{24}{6}}c^{\frac{18}{6}} = a^4c^3$,

(iii) $\sqrt{\dfrac{49x^{10}}{25c^4d^6}} = \dfrac{7x^{\frac{10}{2}}}{5c^{\frac{4}{2}}d^{\frac{6}{2}}} = \dfrac{7x^5}{5c^2d^3}$

EXAMPLES XVI d

Write down the square root of each of the following expressions

1. x^2y^6	2 $4c^4d^8$	3 $16a^4b^{16}$	4 $9x^6y^{12}$
5 $64p^2q^{16}$	6 $81x^{10}$	7 $144x^{24}y^6$	8 $36m^{36}$
9 $\dfrac{1}{81a^{18}}$	10 $\dfrac{64x^{18}}{25}$	11 $\dfrac{16}{m^{16}n^8}$	12 $\dfrac{169y^{26}}{49}$
13 $\dfrac{289}{324p^{12}}$	14 $\dfrac{100}{81x^{10}y^{18}}$	15 $\dfrac{36c^{12}}{25a^{10}}$	16 $\dfrac{324a^{24}}{400b^{40}c^{20}}$

Write down the cube root of each of the following expressions

17 $8a^3b^6$	18 $27c^6d^9$	19 $-64x^3y^9$	20 $343a^{18}$
21. $-\dfrac{125}{b^9c^{12}}$	22 $-\dfrac{27a^{27}}{8b^9}$	23. $\dfrac{m^{15}}{8p^9q^{12}}$	24 $-\dfrac{1}{729a^{27}}$

Write down the value of each of the following expressions

25 $\sqrt[4]{a^8b^{12}}$	26 $\sqrt[5]{x^{15}y^{20}}$	27 $\sqrt[6]{64a^{18}b^{12}}$
28 $\sqrt[5]{-32a^{10}}$	29 $\sqrt[7]{128a^{63}}$	30 $\sqrt[3]{-m^{18}n^{27}p^{36}}$
31. $\sqrt[5]{-\dfrac{243}{x^{15}}}$	32 $\sqrt[7]{-\dfrac{a^7b^{14}}{c^{28}}}$	33. $\sqrt[5]{-\dfrac{32k^{20}l^{15}}{243n^{50}}}$

186 By Art 84, we are able to write down the square of any binomial

Thus $$(2x+3y)^2=(2x)^2+2\cdot 2x\cdot 3y+(3y)^2$$

Conversely, by observing the form of the terms of an expression, it may sometimes be recognised as a complete square, and its square root written down at once

EXAMPLE 1 *Find the square root of* $25x^2-40xy+16y^2$

The expression $=(5x)^2-2\cdot 20xy+(4y)^2$

$=(5x)^2-2(5x)(4y)+(4y)^2$

$=(5x-4y)^2$

Thus the required square root is $5x-4y$

EXAMPLE 2 *Find the square root of* $\frac{64a^2}{9b^2}+4+\frac{32a}{3b}$

The expression $=\left(\frac{8a}{3b}\right)^2+(2)^2+2\left(\frac{16a}{3b}\right)$

$=\left(\frac{8a}{3b}\right)^2+2\left(\frac{8a}{3b}\right)(2)+(2)^2$

$=\left(\frac{8a}{3b}+2\right)^2$

Thus the required square root is $\frac{8a}{3b}+2$

EXAMPLE 3 *Find the square root of* $4a^2+b^2+c^2+4ab-4ac-2bc$

Arrange the terms in descending powers of a, and the other letters alphabetically, then

the expression $=4a^2+4ab-4ac+b^2-2bc+c^2$

$=4a^2+4a(b-c)+(b-c)^2$

$=(2a)^2+2\cdot 2a(b-c)+(b-c)^2$

$=\{2a+(b-c)\}^2$,

whence the required square root is $2a+b-c$

Or we might proceed as follows

the expression $=(2a)^2+b^2+c^2+2\cdot(2a)b-2\cdot(2a)c-2\cdot b\cdot c$,

which is evidently the square root of $2a+b-c$ [Art 182]

EXAMPLES XVI e

By inspection determine the square root of each of the following expressions

1 $x^2+10x+25$ 2 $y^2-18y+81$ 3 $121-22m^2+m^4$

4 $9-12x^3+4x^6$ 5 $9c^2+42c+49$ 6 $16-40y^6+25y^{12}$

7 $a^4-6a^2b^2+9b^4$ 8 $x^6-24x^3yz+144y^2z^2$

9 $m^4n^4+26m^2n^2p^3+169p^6$ 10 $49a^2b^6-112ab^3d^5+64d^{10}$

11 $\frac{1}{4}a^2-3ab+9b^2$ 12 $\frac{m^2}{9}+\frac{mn^2}{3}+\frac{n^4}{4}$

By inspection determine the square root of each of the following expressions

13 $\frac{9a^2}{b^2}-\frac{30ac}{bd}+\frac{25c^2}{d^2}$

14 $\frac{9a^2}{16x^2}+\frac{4x^2}{9a^2}-1$

15 $(x-1)^2+4a^2+4a(x-1)$

16. $100c^{12}-20c^6(a+b)+(a+b)^2$

17 $x^2+2xy-4xz+y^2-4yz+4z^2$

18. $a^2+4b^2+c^2-4ab-2ac+4bc.$

19 $a^2+b^2+9c^2-2ab-6ac+6bc$

20 $p^2-6pq+4pr+9q^2-12qr+4r^2.$

187 When the square root cannot be easily determined by inspection we may use the general method explained in the following examples, which is applicable to all cases *But the pupil is advised to use methods of inspection, wherever possible, in preference to rules*

EXAMPLE 1 *To find the square root of* $a^2+2ab+b^2$

Supposing we know the result to be $a+b$, we have to devise a process for finding the terms a and b

The first term a is obviously the square root of a^2, the first term of the given expression

Now $(a^2+2ab+b^2)-a^2=2ab+b^2=b(2a+b)$

Hence after finding the first term and subtracting its square from the given expression, the remainder when divided by $2a+b$ will give the second term of the root

The work may be arranged as follows

$$\begin{array}{r|l} & a^2+2ab+b^2\,(\,a+b \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

The first part of the divisor is obtained by doubling a, the term of the root already found Dividing $2ab$ by $2a$ we get $+b$, the new term in the root, which has to be annexed to the divisor to complete it

EXAMPLE 2 *Find the square root of* $9x^2-42xy+49y^2$

$$\begin{array}{r|l} & 9x^2-42xy+49y^2\,(\,3x-7y \\ & 9x^2 \\ \hline 6x-7y & -42xy+49y^2 \\ & -42xy+49y^2 \\ \hline \end{array}$$

Explanation The square root of $9x^2$ is $3x$, and this is the first term of the root

By doubling this we obtain $6x$, which is the first term of the divisor Dividing $-42xy$, the first term of the remainder, by $6x$ we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor We next multiply the complete divisor by $-7y$ and subtract the result from the first remainder There is now no remainder, and the root has been found

188 The rule can be extended so as to find the square root of any multinomial The first two terms of the root will be obtained as before When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling *the two terms of the root already found* The full process will be clear from the following example

EXAMPLE *Find the square root of*

$$25x^2a^2-12xa^3+16x^4+4a^4-24x^3a$$

Rearrange the terms in descending powers of x

$$\begin{array}{l|l}
 & 16x^4-24x^3a+25x^2a^2-12xa^3+4a^4\ (4x^2-3xa+2a^2 \\
 & 16x^4 \\
\hline
8x^2-3xa & -24x^3a+25x^2a^2 \\
 & -24x^3a+\ 9x^2a^2 \\
\hline
8x^2-6xa+2a^2 & 16x^2a^2-12xa^3+4a^4 \\
 & 16x^2a^2-12xa^3+4a^4 \\
\hline
\end{array}$$

Explanation When we have obtained two terms in the root, $4x^2-3xa$, we have a remainder

$$16x^2a^2-12xa^3+4a^4$$

Doubling the terms of the root already found, we place the result, $8x^2-6xa$, as the first part of the divisor Dividing $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor, we get $+2a^2$, which we annex both to the root and divisor We now multiply the complete divisor by $2a^2$ and subtract There is no remainder, and the root is found

The work may often be shortened by the use of detached coefficients

EXAMPLES XVI f

Find the square root of each of the following expressions

1. $a^4-4a^3+6a^2-4a+1$
2. $a^4+2a^3+5a^2+4a+4$
3. $x^4-2x^3-x^2+2x+1$
4. $4x^4-4x^3+5x^2-2x+1$
5. $x^4-6x^3+13x^2-12x+4$
6. $4x^4-12x^3+25x^2-24x+16$
7. $8p^3+1+4p^4-4p$
8. $a^4+13a^2+4-12a-6a^3$
9. $a^2-2ax+x^2+2a-2x+1$
10. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$
11. $4c^4+6cd^3+12c^3d+d^4+13c^2d^2$
12. $x^6+y^6+x^2y^4-2xy^5+2x^3y^3-2x^4y^2$
13. $a^6+4a^2b^3-2a^3b^2-4a^5b+b^4+4a^4b^2$
14. $14a^3-4a^4+4a^2+a^6+49-28a$
15. $16m^6-4m^9+m^{10}+16m^7-4m^8$.
16. $10b^2-20b^3+1-4b-24b^5+25b^4+16b^6$, in ascending powers
17. $25x^2-20x^3+16+x^6+10x^4-4x^5-24x$, „ descending „
18. $9-8x^5-22x^3-12x+x^6+20x^4+28x^2$, „ ascending „
19. $4a^6+13a^4-12a^5+16-22a^3-8a+25a^2$, „ descending „
20. $a^6-7a^2x^4+4x^6+10a^4x^2-12ax^5+28a^3x^3-8a^5x$

189 We have seen in Art 182 that the square of a trinomial cannot contain more that 6 terms, as the square roots of longer expressions than these are not often required the method of Arts. 187, 188 need rarely be applied *in full* We may either use methods of inspection as in Art 186, or we may proceed as in the following examples.

EXAMPLE *Find the square root of*

$$x^4-4x^3+10x^2-12x+9$$

Here the square root is a trinomial, the first term of which is x^2, and the last either $+3$ or -3 Also by the process of Art 187, the second term is $-4x^3\div(2\times x^2)$, or $-2x$

Hence the square root (if there is one) is either x^2-2x+3 or x^2-2x-3; and, by considering the sign of the term $-12x$ in the given expression, we see that the last term of the root must be positive

Hence the required root is x^2-2x+3

The result should be verified by expanding $(x^2-2x+3)^2$ and comparing the terms with those of the given expression.

190 When an expression contains powers of a certain letter and also powers of its reciprocal (Art 150) there is an important point to be observed Thus in the expression

$$2x+\frac{1}{x^2}+4+x^3+\frac{5}{x}+7x^2+\frac{8}{x^3},$$

the order of *descending* powers of x is

$$x^3+7x^2+2x+4+\frac{5}{x}+\frac{1}{x^2}+\frac{8}{x^3},$$

and the numerical quantity 4 stands between x and $\frac{1}{x}$

The reason for this arrangement will appear in the *Theory of Indices*

EXAMPLE 1 *Find the square root of*

$$4x^2+5+\frac{6}{x}-12x+\frac{1}{x^2}$$

The terms must first be arranged in descending powers of x, thus

$$4x^2-12x+5+\frac{6}{x}+\frac{1}{x^2}$$

Proceeding as before, the first and last terms of the square root are respectively $2x$, and either $+\frac{1}{x}$ or $-\frac{1}{x}$ Also the second term is

$$-12x\div(2\times 2x),\text{ or }-3$$

Hence the square root is either $2x-3+\frac{1}{x}$ or $2x-3-\frac{1}{x}$

By considering the term $\frac{6}{x}$ in the given expression we see that the last term of the root must be negative

Hence the required root is $2x-3-\frac{1}{x}$

EXAMPLE 2 *Required the square root of* $\frac{9a^2}{b^2}+7+\frac{b^2}{4a^2}-\frac{12a}{b}-\frac{2b}{a}$

First arrange the terms in descending powers of a, thus

$$\frac{9a^2}{b^2}-\frac{12a}{b}+7-\frac{2b}{a}+\frac{b^2}{4a^2}$$

Proceeding as in the last two examples we find the three terms of the root are $\frac{3a}{b}$, -2, $+\frac{b}{2a}$ Thus the root is $\frac{3a}{b}-2+\frac{b}{2a}$

EXAMPLES XVI. g

Find the square root of each of the following expressions

1. $4a^4+4a^3-7a^2-4a+4$
2. $1-10x+27x^2-10x^3+x^4$
3. $a^{10}-4a^9-4a^8+16a^7+16a^6$
4. $67c^2+49+9c^4-70c-30c^3$
5. $1+2a-a^3+\frac{a^4}{4}$
6. $\frac{x^4}{64}+\frac{x^3}{8}-x+1$
7. $4-12b+\frac{27b^3}{2}+\frac{81b^4}{16}$
8. $x^4-3x^3+\frac{11}{12}x^2+2x+\frac{4}{9}$
9. $4a^6-4a^5+\frac{7a^4}{3}-\frac{2a^3}{3}+\frac{a^2}{9}$
10. $x^4-2x^2+3-\frac{2}{x^2}+\frac{1}{x^4}$
11. $4x^4-12x+\frac{25}{x^2}-\frac{24}{x^5}+\frac{16}{x^8}$
12. $\frac{x}{a}-1+\frac{4a^2}{x^2}+\frac{x^2}{4a^2}-\frac{4a}{x}$
13. $3-\frac{6x}{a}-\frac{2a}{3x}+\frac{a^2}{9x^2}+\frac{9x^2}{a^2}$

191 Cube Roots By Art 183 we have

$$\text{the cube root of } a^3+3a^2b+3ab^2+b^3=a+b,$$
$$\text{,, ,, } a^3-3a^2b+3ab^2-b^3=a-b$$

Hence when a perfect cube contains *only four terms* its cube root can be found at once by writing down the cube roots of the *two cube terms*

EXAMPLE 1 The cube root of $27a^3+54a^2b+36ab^2+8b^3$

$$=\sqrt[3]{27a^3}+\sqrt[3]{8b^3}=3a+2b$$

EXAMPLE 2 The cube root of $125x^6-300x^4+240x^2-64$

$$=\sqrt[3]{125x^6}-\sqrt[3]{64}=5x^2-4$$

Again, $(a+b+c)^3=\{a+(b+c)\}^3$

$$=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$$

By expanding $(b+c)^2$ and $(b+c)^3$ we see that the cube of a trinomial cannot contain more than 10 terms Also the second term of the root is obtained by dividing the second term of the given expression by *three times the square of the first term of the root*

EXAMPLE 3 The cube root of $8x^6-36x^5+66x^4-63x^3+33x^2-9x-1$ is $2x^2-3x+1$, the first and last terms being the cube roots of $8x^6$ and 1, while the second term is $-36x^5 \div 3(2x^2)^2$, or $-3x$

EXAMPLES XVI h

Find the cube root of each of the following expressions

1. $x^3-6x^2y+12xy^2-8y^3$
2. $8a^3+12a^2b+6ab^2+b^3$
3. $27x^3-135x^2y+225xy^2-125y^3$
4. $x^6+12x^4y^2+48x^2y^4+64y^6$
5. $a^3-2a^2b+\frac{4}{3}ab^2-\frac{8}{27}b^3$
6. $\frac{a^3}{216}+\frac{a^2x}{6}+2ax^2+8x^3$
7. $1+3a+6a^2+7a^3+6a^4+3a^5+a^6$
8. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$
9. $a^3+3a^2b-3a^2c+3ab^2-6abc+3ac^2+b^3-3b^2c+3bc^2-c^3$
10. $\frac{a^3}{b^3}-\frac{3a^2}{b^2}+\frac{6a}{b}-7+\frac{6b}{a}-\frac{3b^2}{a^2}+\frac{b^3}{a^3}$

(*Miscellaneous Examples in Involution and Evolution*)

As in Art 66, find graphically the square of

11. $x+5$ 12. $a-b$ 13. $2c-d$ 14. $a+b+c$

15. Simplify $2(p+q)^2+(p-q+r)^2-(p+q-r)^2$
16. Express $(a-b)^3+3(a-b)^2b+3(a-b)b^2+b^3$ in the simplest form
17. By expressing the trinomials in factors, find the square root of
$$(x^2+8x+7)(x^2+10x+21)(x^2+4x+3)$$
18. Find the square root of
$$(a^2+a-6)(a^2-4)(a^2+5a+6)$$
19. Write down the square root of $(a-b)^2+2(a^2-b^2)+(a+b)^2$
20. Simplify $(c+d)^3+3(c+d)^2(c-d)+3(c+d)(c-d)^2+(c-d)^3$
21. Shew that $a(a+1)(a+2)(a+3)+1$ is a perfect square, and find its square root
22. If $z=x+y$, shew that $z^3-x^3-y^3=3xyz$
23. Prove that
$$(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)=8x^3$$
24. Find the square root of $4x^4+8x^3+8x^2+4x+1$ Deduce the square root of 48841
25. Find, by inspection, the value of

(i) $(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)+2(b-c)(c-a)+2(c-a)(a-b)$

(ii) $(3x-y)^3+3(3x-y)^2(2y-3x)+3(3x-y)(2y-3x)^2+(2y-3x)^3$

CHAPTER XVII

HARDER CASES OF RESOLUTION INTO FACTORS

Trinomials

192 IN Chapter XIV we considered the resolution into factors of certain trinomial expressions We now proceed to the case of trinomials in which *the coefficient of the highest power is not unity*

By observing the manner in which, in ordinary multiplication, the terms of the product are formed, we may write down the following results

$$(3x+2)(x+4)=3x^2+14x+8, \tag{1}$$

$$(3x-2)(x-4)=3x^2-14x+8, \tag{2}$$

$$(3x+2)(x-4)=3x^2-10x-8, \tag{3}$$

$$(3x-2)(x+4)=3x^2+10x-8 \tag{4}$$

Here we see, as before, that

(i) *If the third term of the trinomial is positive, then the second terms of its factors both have the same sign, and this sign is the same as that of the middle term of the trinomial*

(ii) *If the third term of the trinomial is negative, then the second terms of its factors have opposite signs*

Now consider in detail the result $3x^2-14x+8=(3x-2)(x-4)$

The first term $3x^2$ is the product of $3x$ and x

The third term $+8$ „ „ -2 and -4.

The middle term $-14x$ is the result of adding together the two products $3x\times-4$ and $x\times-2$

Again, consider the result $3x^2-10x-8=(3x+2)(x-4)$

The first term $3x^2$ is the product of $3x$ and x

The third term -8 „ „ $+2$ and -4

The middle term $-10x$ is the result of adding together the two products $3x\times-4$ and $x\times2$, and its sign is negative because the greater of these two products is negative

The above observations lead us to the following method

EXAMPLE 1 *Resolve into factors* $3x^2-14x-5$

The only factors of $3x^2$ are $3x$ and x, and of 5 the only factors are 5 and 1 Hence we may write down $(3x\quad 5)(x\quad 1)$ for a first trial, noticing that 5 and 1 *must have opposite signs* These factors give $3x^2$ and -5 for the first and third terms

The coefficient of the *middle* term would be obtained by the sum of the products 3×1 and 5×1 taken with opposite signs It is clear that these cannot combine to produce -14

Next try $(3x\quad 1)(x\quad 5)$

We have now to take the sum of 3×5 and 1×1 with opposite signs This sum will equal -14 if we arrange the signs so that, of these products, the negative shall be the greater That is, for the second terms of our factors we must have $+1$ and -5 respectively

Thus $$3x^2-14x-5=(3x+1)(x-5)$$

The result should be verified by mentally forming the product of the two factors

193 It will not usually be necessary to put down all these steps at length It will soon be found that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once

EXAMPLE 1 *Resolve into factors* $14x^2+29x-15$, (1)

$14x^2-29x-15$ (2)

In each case we may write down $(7x\quad 3)(2x\quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs

And since $7\times5-3\times2=29$, we have only now to insert the proper signs in each factor

In (1) the positive sign must predominate,

in (2) the negative „ „

Therefore $$14x^2+29x-15=(7x-3)(2x+5)$$
$$14x^2-29x-15=(7x+3)(2x-5)$$

NOTE In each expression 14 admits of factors 14 and 1, and the third term admits of factors 15 and 1, with opposite signs, but these are cases referred to above which would be rejected as unsuitable For on trial it would at once be found that the coefficient of x for the middle term was either much too large or much too small

EXAMPLE 2 *Resolve into factors* $5x^2+17x+6$, (1)

$5x^2-17x+6$ (2)

In (1) we notice that the factors which give 6 are both positive

In (2) „ „ „ „ negative

And therefore for (1) we may write $(5x+\quad)(x+\quad)$

(2) „ „ $(5x-\quad)(x-\quad)$

And, since $5\times3+1\times2=17$, we see that

$$5x^2+17x+6=(5x+2)(x+3)$$
$$5x^2-17x+6=(5x-2)(x-3)$$

EXAMPLE 3 $16x^2-56xy+49y^2=(4x-7y)(4x-7y)$
$$=(4x-7y)^2$$

EXAMPLE 4 $8+6a-5a^2=(4+5a)(2-a)$

194 When the above method becomes tedious owing to the presence of large numbers, with several pairs of factors, the following method may be used

EXAMPLE *Resolve* $15x^2+22x-48$ *into factors*

Multiply and divide the expression by the coefficient of x^2, thus

$$15x^2+22x-48=\tfrac{1}{15}\{(15x)^2+22\cdot15x-48\cdot15\}$$
$$=\tfrac{1}{15}(y^2+22y-720), \text{ if } y \text{ stands for } 15x,$$
$$=\tfrac{1}{15}(y+40)(y-18)$$
$$=\frac{(15x+40)(15x-18)}{5\times3}, \text{ replacing } y \text{ by } 15x,$$
$$=(3x+8)(5x-6)$$

EXAMPLES XVII a

Resolve into factors

1	$2a^2+3a+1$	2	$2a^2+5a+2$	3	$3a^2+5a+2$
4	$3a^2+4a+1$	5	$2a^2+9a+4$	6	$2a^2+7ab+6b^2$
7	$2b^2+7b+3$	8	$2b^2+9b+10$	9	$2b^2+11b+5$
10	$5b^2-7b+2$	11.	$3b^2-11b+6$	12	$3b^2-10b+3$
13	$2c^2+3c-2$	14	$6c^2-c-2$	15	$2c^2-5cd+3d^2$
16	$2c^2-c-1$	17	$2c^2+c-28$	18	$2c^2-17c+8$
19	$4x^2+4x-3$	20	$6x^2+5x-6$	21	$3x^2+13x-30$
22	$2x^2-11xy+15y^2$	23	$4x^2+x-14$	24.	$5x^2+11x+2$
25	$4y^2-12y+5$	26	$12y^2+y-6$	27.	$6y^2+7y-3$
28.	$12x^2-23xy+10y^2$	29	$24a^2+22a-21$	30	$15y^2-77y+10$
31.	$3-5p-12p^2$	32	$6+17p+5p^2$	33	$4+13p+10p^2$.
34	$15+16p-15p^2$	35	$8+p-7p^2$	36	$28+31p-5p^2$

195 We shall now give some harder applications of the foregoing rules

EXAMPLE 1 *Resolve into factors* $(a+2b)^2-16x^2$

This expression, being *the difference between two squares*, is resolved into factors by the rule of Art 167

The sum of $a+2b$ and $4x$ is $a+2b+4x$,
and their difference is $a+2b-4x$,

$$(a+2b)^2-16x^2=(a+2b+4x)(a+2b-4x)$$

EXAMPLE 2 *Resolve into factors* $x^2-(y-z)^2$

$$x^2-(y-z)^2=\{x+(y-z)\}\{x-(y-z)\}$$
$$=(x+y-z)(x-y+z)$$

If the factors contain like terms they should be collected so as to give the result in its simplest form

EXAMPLE 3
$$\begin{aligned}&(3x+7y)^2-(2x-3y)^2\\ &=\{(3x+7y)+(2x-3y)\}\{(3x+7y)-(2x-3y)\}\\ &=(3x+7y+2x-3y)(3x+7y-2x+3y)\\ &=(5x+4y)(x+10y)\end{aligned}$$

EXAMPLES XVII. b.

Resolve into factors

1. $(x+y)^2-z^2$
2. $(x-y)^2-z^2$
3. $(a-b)^2-4c^2$
4. $(a+b)^2-9c^2$
5. $(a-2b)^2-c^2$
6. $(a+2b)^2-c^2$
7. $(c+d)^2-4a^2$
8. $(c-d)^2-1$
9. $(c+2d)^2-9$
10. $m^2-(n+p)^2$
11. $m^2-(n-p)^2$
12. $4m^2-(n+p)^2$
13. $1-(a+b)^2$
14. $4-(a-b)^2$
15. $9-(a+b)^2$

Resolve into factors and simplify

16. $(a+b)^2-b^2$
17. $c^2-(c+d)^2$
18. $x^2-(x-y)^2$
19. $(m-3n)^2-9n^2$
20. $(m+2n)^2-4n^2$
21. $(m-4n)^2-16n^2$
22. $(2a+x)^2-(a-x)^2$
23. $9x^2-(3x-4y)^2$
24. $(3a-2b)^2-(2b-c)^2$
25. $(2x+3y)^2-(a-5y)^2$
26. $(2m-3n)^2-(m+3n)^2$
27. $(a-5b)^2-(5b+1)^2$
28. $4(2a-3b)^2-(3a-7b)^2$
29. $9(a-2b)^2-(4a-7b)^2$

196 By suitably grouping together the terms, compound expressions can often be expressed as the difference of two squares, and so be resolved into factors

EXAMPLE 1 *Resolve into factors* $a^2-2ax+x^2-4b^2$
$$\begin{aligned}a^2-2ax+x^2-4b^2&=(a^2-2ax+x^2)-4b^2\\ &=(a-x)^2-(2b)^2\\ &=(a-x+2b)(a-x-2b)\end{aligned}$$

EXAMPLE 2 *Resolve into factors* $9a^2-c^2+4cx-4x^2$
$$\begin{aligned}9a^2-c^2+4cx-4x^2&=9a^2-(c^2-4cx+4x^2)\\ &=(3a)^2-(c-2x)^2\\ &=(3a+c-2x)(3a-c+2x)\end{aligned}$$

EXAMPLE 3 *Resolve into factors* $2bd-a^2-c^2+b^2+d^2+2ac$

Here the terms $2bd$ and $2ac$ suggest the proper preliminary arrangement of the expression Thus
$$\begin{aligned}2bd-a^2-c^2+b^2+d^2+2ac&=b^2+2bd+d^2-a^2+2ac-c^2\\ &=b^2+2bd+d^2-(a^2-2ac+c^2)\\ &=(b+d)^2-(a-c)^2\\ &=(b+d+a-c)(b+d-a+c)\end{aligned}$$

EXAMPLES XVII. c.

Resolve into factors

1. $a^2+2ab+b^2-c^2$
2. $x^2-c^2-2cd-d^2$
3. $4x^2-4xy+y^2-1$
4. $1-m^2+6mn-9n^2$
5. $c^2-2cd+d^2-9$
6. $c^2-d^2-8d-16$
7. $25-y^2+2yz-z^2$
8. $4p^2-12pq+9q^2-81$
9. $9y^2-16c^2-16cd-4d^2$.
10. $121-25a^2-10ab-b^2$
11. x^4-x^2+2x-1
12. $x^6+2x^3+1-x^4$
13. $25a^2-x^2+b^2-10ab$
14. $x^2-4xy+4y^2-9x^2y^2$
15. $x^4-c^2+9y^2-6x^2y$
16. $a^4-4c^4+9b^4-6a^2b^2$
17. $a^2+2ab+b^2-c^2-2cd-d^2$
18. $a^2-2ab+b^2-c^2+2cd-d^2$
19. $x^2-14x+49-y^2+2yz-z^2$
20. $a^4+2a^3+a^2-100$
21. $9a^2-12a+4-b^2+8bc-16c^2$
22. $49y^6-28y^3+4-36y^2$
23. $1+2x+2yz+x^2-y^2-z^2$
24. $2cd-2xy+x^2+y^2-c^2-d^2$.
25. $m^4+n^4-a^4-b^4+2m^2n^2-2a^2b^2$
26. $a^6-6a^3-a^2-2ab-b^2+9$
27. $9a^2-12ax-p^2-q^2-2pq+4x^2$

*28. $c^2+9(d^2-a^2)+6cd$

197 The following case is specially important

EXAMPLE *Resolve into factors* $x^4+x^2y^2+y^4$

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^4+2x^2y^2+y^4)-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy) \\ &= (x^2+xy+y^2)(x^2-xy+y^2) \end{aligned}$$

198 We shall now explain a general method by which any trinomial expression of the form x^2+px+q or ax^2+bx+c can be expressed as the difference of two squares

By Art 182 we have the following identities

$$x^2+2ax+a^2=(x+a)^2, \qquad x^2-2ax+a^2=(x-a)^2$$

So that if a trinomial is a perfect square, and *its highest power* x^2 *has unity for its coefficient*, we must always have the term without x equal to *the square of half the coefficient of* x If therefore the first two terms (containing x^2 and x) of such a trinomial are given, the square may be completed by adding the square of half the coefficient of x

Thus x^2+6x is made a perfect square if we add to it $\left(\frac{6}{2}\right)^2$, or 9, and it then becomes x^2+6x+9, or $(x+3)^2$

Similarly to make x^2-7x a perfect square we must add $\left(-\frac{7}{2}\right)^2$, or $\frac{49}{4}$, and we then have $x^2-7x+\frac{49}{4}$, or $\left(x-\frac{7}{2}\right)^2$

NOTE The added term is always positive

Example 1 *Find the factors of* x^2+6x+5

The expression may be written $(x^2+6x+9)+5-9$,

that is,
$$\begin{aligned} x^2+6x+5&=(x+3)^2-4\\ &=(x+3+2)(x+3-2)\\ &=(x+5)(x+1)\end{aligned}$$

Example 2 *Find the factors of* $x^2-7x-228$

$$\begin{aligned} x^2-7x-228&=\left(x^2-7x+\tfrac{49}{4}\right)-228-\tfrac{49}{4}\\ &=\left(x-\tfrac{7}{2}\right)^2-\tfrac{961}{4}\\ &=\left(x-\tfrac{7}{2}+\tfrac{31}{2}\right)\left(x-\tfrac{7}{2}-\tfrac{31}{2}\right)\\ &=(x+12)(x-19)\end{aligned}$$

Example 3 *Find the factors of* $3x^2-13x+14$

$$\begin{aligned} 3x^2-13x+14&=3\left(x^2-\tfrac{13}{3}x+\tfrac{14}{3}\right)\\ &=3\left\{x^2-\tfrac{13}{3}x+\left(\tfrac{13}{6}\right)^2+\tfrac{14}{3}-\tfrac{169}{36}\right\}\\ &=3\left\{\left(x-\tfrac{13}{6}\right)^2-\tfrac{1}{36}\right\}\\ &=3\left(x-\tfrac{13}{6}-\tfrac{1}{6}\right)\left(x-\tfrac{13}{6}+\tfrac{1}{6}\right)\\ &=3\left(x-\tfrac{7}{3}\right)(x-2)\\ &=(3x-7)(x-2)\end{aligned}$$

As the process of completing the square is quite general and applicable to all cases, it may conveniently be used when factorization by trial would prove uncertain and tedious For example, if the factors of $24x^2+118x-247$ were required, it would probably be best to apply the general method at once

EXAMPLES XVII. d

Resolve into factors

1. $a^4+a^2b^2+b^4$	2 $m^4+4m^2n^2+16n^4$	3. $a^4+3a^2b^2+4b^4$
4. $p^4+9p^2q^2+81q^4$	5. $625c^4+25c^2d^2+d^4$	6 $x^4+y^4-11x^2y^2$
7. $4m^4-21m^2n^2+n^4$	8. $x^4+25y^4-19x^2y^2$	9 $x^2+32x+247$
10 $a^2+4a-221$	11. $y^2-26y+165$	12. $a^2+32ab+207b^2$
13. $2c^2-41c+144$	14 $27m^2-15mn-112n^2$	15 $12x^2-68x+91$
16. $24p^2+5p-36$	17 $30a^2+37ab-84b^2$	18 $12x^2-102x+48$

199 Miscellaneous cases of resolution into factors.

Example 1 *Resolve into factors* x^6-y^6

$$\begin{aligned} x^6-y^6&=(x^3+y^3)(x^3-y^3)\\ &=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)\end{aligned}$$

Note When an expression can be arranged either as the difference of two squares, or as the difference of two cubes, the rule for the difference of two squares should be used first

EXAMPLE 2. *Resolve the following expressions into factors:*

(i) $25a^2-4b^2+5a+2b$, (ii) $x(2+x^2)-y(2+y^2)$,
(iii) $28x^4y+64x^3y-60x^2y$.

(i) $25a^2-4b^2+5a+2b=(5a+2b)(5a-2b)+(5a+2b)$
$=(5a+2b)(5a-2b+1)$.

(ii) $x(2+x^2)-y(2+y^2)=2x+x^3-2y-y^3$
$=2(x-y)+(x^3-y^3)$
$=2(x-y)+(x-y)(x^2+xy+y^2)$
$=(x-y)(2+x^2+xy+y^2)$.

(iii) $28x^4y+64x^3y-60x^2y=4x^2y(7x^2+16x-15)$
$=4x^2y(7x-5)(x+3)$.

EXAMPLES XVII. e.

(*Trinomials.*)

Resolve into two or more factors:

1. y^2-y-72. 2. $c^2+21c+108$. 3. $m^2+12m-85$.
4. $2z^2+z-15$. 5. $4a^2-8ab-5b^2$. 6. $6p^2-13pq+2q^2$.
7. $8x^4+2x^2-15$. 8. $6m^2+7m-3$. 9. $a^2-22ac+57c^2$.
10. $z^2+34z+289$. 11. $x^2-6y(x+12y)$. 12. $4-x(5-x)$.
13. $6-x(1+x)$. 14. $12a^2-7ab-12b^2$. 15. $28x^2-x-15$.
16. $6x^2+xy-12y^2$. 17. $6x^3-5x^4+x^5$. 18. $x^4-2x^3-63x^2$.
19. $a^2+2a-255$. 20. $6x^3-38x^2-144x$. 21. $72-14x-x^2$.
22. $3(2b^2-1)-7b$. 23. a^4+2a^2-3. 24. $c^4-2c^2d^2-63d^4$.

(*Miscellaneous.*)

25. $250p^3+2$. 26. $100a^2b^4-4$. 27. $729+c^3d^3$.
28. $9x^3-4xy^2$. 29. $(a+x)^2-1$. 30. $16-(b-c)^2$.
31. $l^2+l-272$. 32. $p^4+q^4-7p^2q^2$. 33. a^4+3a^2+4.
34. $16x^4+4x^2y^2+y^4$. 35. $a^8x^8-64a^2y^6$. 36. $729a^7b-ab^7$.
37. $500x^2y-20y^3$. 38. $(a+b)^4-1$. 39. $(c+d)^3-1$.
40. $1-(x-y)^3$. 41. $x^2-6x-247$. 42. $a^2-22a-279$.
43. $250(a-b)^3+2$. 44. $(c+d)^3+(c-d)^3$. 45. $8x^3+(y-2x)^3$.
46. x^2-4y^2+x-2y. 47. a^2-b^2-a-b. 48. $(a+b)^2+a+b$.
49. a^3+b^3+a+b. 50. a^2-9b^2+a+3b.
51. $4(x-y)^3-(x-y)$. 52. $x^4y-x^2y^3-x^3y^2+xy^4$.

53. Express in factors the square root of
$$(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21).$$

54. By means of the Remainder Theorem find the factors of
(i) $3x^3-9a^2x+6a^3$, (ii) $x^3-37x-84$,
(iii) $6a^3+a^2-19a+6$, (iv) $a^4-a^3b-7a^2b^2+ab^3+6b^4$.

Some Applications of Factors.

200 The formulæ for resolution of expressions into factors are often as useful in their converse as in their direct application

EXAMPLE 1 *Multiply* $a+b-c$ *by* $a-b+c$

The expressions may be written as follows

$$a+(b-c) \text{ and } a-(b-c)$$

Thus we have the sum and difference of a and $b-c$.

Hence the product $=a^2-(b-c)^2=a^2-b^2+2bc-c^2$

EXAMPLE 2 *Find the product of* $2x-7y+3z$ *and* $2x-7y-3z$

The product $=(\overline{2x-7y}+3z)(\overline{2x-7y}-3z)$

$$=(2x-7y)^2-9z^2=4x^2-28xy+49y^2-9z^2$$

EXAMPLE 3 *Find the product of*

$$x+2,\ x-2,\ x^2-2x+4,\ x^2+2x+4$$

Taking the first factor with the third, and the second with the fourth, the product $=\{(x+2)(x^2-2x+4)\}\{(x-2)(x^2+2x+4)\}$

$$=(x^3+8)(x^3-8)=x^6-64$$

EXAMPLE 4 *Shew that* $(2x-3y+1)^3-(1-3x+2y)^3$ *is divisible by* $5(x-y)$

An expression of the form A^3-B^3 has a divisor of the form $A-B$

the given expression is divisible by $(2x-3y+1)-(1-3x+2y)$,

that is, by $5x-5y$, or $5(x-y)$

201 Identities An identity (Art 98) asserts that two expressions are always equal for all values of the letters involved in it, and the proof of this equality is called "proving the identity" The method of procedure is to choose one of the expressions given, and to shew by successive transformations that it can be made to assume the form of the other

NOTE We use the sign $\equiv$ to denote *identical equality*

EXAMPLE 1 *Prove the identity*

$$17(5x+3a)^2-2(40x+27a)(5x+3a)\equiv 25x^2-9a^2$$

Since each term of the first expression contains the factor $5x+3a$, the first side $\equiv(5x+3a)\{17(5x+3a)-2(40x+27a)\}$

$$\equiv(5x+3a)(85x+51a-80x-54a)$$

$$\equiv(5x+3a)(5x-3a)\equiv 25x^2-9a^2$$

EXAMPLE 2 *If* $x+y+z=0$, *prove that* $x^3+y^3+z^3=3xyz$.

Since $x+y+z=0$, we have $z=-(x+y)$,

hence $x^3+y^3+z^3=x^3+y^3-(x+y)^3$

$$=(x+y)\{x^2-xy+y^2-(x+y)^2\}$$

$$=(x+y)\{x^2-xy+y^2-x^2-2xy-y^2\}$$

$$=(-z)\times(-3xy)=3xyz.$$

EXAMPLES XVII *f*

By the use of factors find the product of

1 $x+y-z$, $x-y+z$ 2 $x-y-z$, $x+y+z$

3 $2a+b+c$, $2a+b-c$ 4 $a-3b+1$, $a-3b-1$

5 $1+a-a^2$, $1-a-a^2$ 6 a^2+2a+3, a^2-2a-3

7. $x+y-c+d$, $x+y+c-d$ 8 $x-y+a-b$, $x-y-a+b$

9 $c+d$, $c-d$, c^2+d^2 10. $(c+d)^2$, $(c-d)^2$, $(c^2+d^2)^2$

11. $a-b$, $a+b$, $a^4+a^2b^2+b^4$ 12. $x^2-2(x-1)$, $x^2+2(x+1)$

13 $a-3$, $a+3$, a^2-3a+9, a^2+3a+9

14 Prove that $(7x^2+4x+8)^2-(x^2-9x+13)^2$ is divisible by $(3x-1)(2x+5)$, and find the quotient

15 Shew that $(2a+3b-c)^3+(3a+7b+c)^3$ is divisible by $5(a+2b)$

16. Shew that the product of $2x^2+x-6$ and $6x^2-5x+1$ is divisible by $3x^2+5x-2$

17. Shew that $(x+1)^2$ exactly divides $(x^3+x^2+4)^2-(x^3-2x+3)^2$

Prove the following identities

18 $ax(x^2-a^2)+a^3(x+a)\equiv a(x^3+a^3)$

19 $(a+b)^3-(a-b)^2(a+b)\equiv 4ab(a+b)$

20. $c^4-d^4-(c-d)^3(c+d)\equiv 2cd(c^2-d^2)$

21 $(m-n)(m+n)^3-m^4+n^4\equiv 2mn(m^2-n^2)$

22 $(x+y)^4-3xy(x+y)^2\equiv(x+y)(x^3+y^3)$

23 $3ab(a-b)^2+(a-b)^4\equiv(a-b)(a^3-b^3)$

24 $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)\equiv(a+b)(x+a)(x^2-ax+a^2)$

25 $(b-c)^3+(c-a)^3+(a-b)^3\equiv 3(b-c)(c-a)(a-b)$ [See Art 201, Ex 2]

202 Solution of Quadratic Equations by Factors

DEFINITION An equation which contains the square of the unknown quantity, but no higher power, is called a quadratic equation, or an equation of the second degree

Thus $x^2-4x=32$, $3x^2+4x-15=0$ are quadratic equations

EXAMPLE 1 *Find values of x which satisfy the equations*

$$\text{(i) } (x-2)(x-7)=0, \qquad \text{(ii) } 2x^2+7x=15$$

(i) Any value of x which makes either of the factors zero will make the product zero, that is, such a value will satisfy the equation

Now $x-2=0$ only when $x=2$, and $x-7=0$ only when $x=7$

Thus 2 and 7 are roots of the equation

(ii) By transposition, $2x^2+7x-15=0$,

and since $2x^2+7x-15=(2x-3)(x+5)$,

the equation may be written $(2x-3)(x+5)=0$

This is satisfied either when $2x-3=0$, or when $x+5=0$, that is, when $x=\frac{3}{2}$, or when $x=-5$

EXAMPLE 2 *Solve the equation* $2\left(\frac{x}{3}-1\right)=(x+2)(x-3)$

Simplifying and removing brackets, we have

$$\frac{2x}{3}-2=x^2-x-6,$$

clearing of fractions, $\quad 2x-6=3x^2-3x-18$

Now bring all the terms over to one side of the equation,

thus $\quad -3x^2+5x+12=0$

Change the sign of every term so as to have the *square term* positive,

thus $\quad 3x^2-5x-12=0,$

$$(3x+4)(x-3)=0$$

Hence $\quad 3x+4=0,\ \text{or}\ x-3=0,$

that is, $\quad x=-\frac{4}{3},\ \text{or}\ 3$

NOTE Before using factors the equation must be simplified and arranged so that *all the terms are on one side with the square term positive*

203 In each of the above cases we have found *two* roots It will be proved in a later chapter that every quadratic equation has two roots Sometimes the roots are equal

Thus in the equation $\quad x^2-6x+9=0,$

we have $\quad (x-3)(x-3)=0,$

whence $x=3$ is the only solution But in this and similar cases it is convenient to say that the quadratic has *two equal roots*

204 The following example deserves special notice

EXAMPLE *Find the values of* x *which satisfy the equation*

$$3x(x-2)=x^2-4$$

We have $\quad 3x(x-2)=(x+2)(x-2)$

If $x-2\neq 0$* we may remove this factor from each side of the equation,

thus $\quad 3x=x+2,$

whence $\quad x=1$

But if $x-2=0$, each side of the equation reduces to zero, and the equation is satisfied

Hence from $x-2=0$ we get another root, viz $x=2$

Thus the roots are 1, 2

If in the course of simplification any *factor which contains the unknown* is observed to be common to both sides of an equation, it must not be rejected, since every such linear factor equated to zero will give one root of the equation

* The sign $\neq$ stands for "is not equal to"

EXAMPLES XVII. g.

Write down the roots of the following equations

1. $(x-1)(x-2)=0$ 2. $(x+3)(x+4)=0$ 3. $x(x-3)=0$
4. $2x(5-x)=0$ 5. $x^2+8x=0$ 6. $5x^2=6x$
7. $(2x-1)(x+4)=0$ 8. $(3x-2)(2x+3)=0$ 9. $(5x-6)^2=0$

Solve the following equations, and verify the solutions in Nos 10–25

10. $x^2-7x+6=0$ 11. $x^2-3x=28$ 12. $x^2=2x+99$
13. $x^2-x=132$ 14. $x^2+8x+16=0$ 15. $23x=120+x^2$
16. $3x^2-10x+3=0$ 17. $6x^2-13x+6=0$ 18. $6x^2+x=2$
19. $2x^2-5x-12=0$ 20. $x(3x+2)=5$ 21. $2x^2-15=x$
22. $2x^2-7x=39$ 23. $2x(x+1)=15+x$
24. $15-11x=8x(1+x)$ 25. $18+5x^2=33x$
26. $4x^2=\frac{4}{15}x+3$ 27. $x^2-2=\frac{23}{12}x$ 28. $x^2-\frac{3x}{4}+\frac{1}{8}=0$
29. $(x+1)(2x+3)=4x^2-22$ 30. $(3x-5)(2x-5)=x^2+2x-3$

Find, by inspection, *one* root of the following equations

31. $x-1=3(x^2-1)$ 32. $2x(x+7)=x^2-3x$ 33. $2x^2-32=x+4$
34. $(3x+5)^2+2x(3x+5)=0$ 35. $\frac{7}{8}(x-\frac{1}{3})+\frac{5}{11}(3x-1)=0$

(In the following problems negative answers are to be neglected)

36. Find two numbers, differing by 3, such that the sum of their squares is 117

37. Find two consecutive numbers such that their product is 182

38. Find a number which when increased by 17 is equal to 60 times the reciprocal of the number

39. The sum of a number and its square is six times the next highest number find it

40. Find two consecutive odd numbers whose product is 255

41. The units' digit of a number is the square of the tens' digit, and the sum of the digits is 12, find the number

42. If on New Year's day each one of a family sends a card to each of the rest, and the postman delivers 156 cards, how many are there in the family?

43. The adjacent sides of a rectangular court yard differ in length by 11 yards, if its area is 840 square yards, find its dimensions

44. The perimeter of one square exceeds that of another by 100 feet, and the area of the larger square exceeds three times the area of the smaller by 325 square feet find the lengths of their sides

MISCELLANEOUS EXAMPLES IV.

Exercises for Revision

A

1. Find the factors of

(i) $x^2+x-132$, (ii) $2a^2+3ab-2b^2$, (iii) $(b+c)^2-9a^2$.

2. Find the remainder when $3x^4-2x^2+3x-15$ is divided by $x-2$.

3 Without removing brackets, add together

$$a^2+3a(x+y), \quad -a(x+y)+(x+y)^2-4,$$
$$2a^2-(x+y)^2+5, \quad \text{and} \quad -a^2-5a(x+y)-2$$

4. Solve the equations

$$\text{(i)} \quad \frac{3x-1}{4}-\frac{1}{2}(x+1)=x+1-\frac{1}{6}(5x+3),$$

$$\text{(ii)} \quad 7x+\frac{5y+9x}{11}=17, \quad 9x+\frac{11y+9x}{17}=21$$

5 I bought a certain number of articles at 7 for $6d$, if they had been 13 for $1s$ I should have spent $6d$ more how many did I buy?

6. A pays a debt of £6 in shillings and half crowns If he had half as many shillings again, and three times as many half crowns he could pay his debt twice over How many shillings and half crowns has he?

7. Plot the graphs of $y=\frac{x}{2}+\frac{3}{4}$, $y=x+\frac{1}{2}$ from $x=-5$ to $x=5$ Find from the graphs the values of x and y which satisfy both equations.

B

8. Multiply x^3+x^2+3x+5 by x^2-x-2, and divide

$$3p^5+16p^4-33p^3+14p^2 \text{ by } p^2+7p$$

9. What value of x will make the product of $x+1$ and $2x+1$ less than the product of $x+3$ and $2x+3$ by 14?

10 A man walks at the rate of a miles an hour for p hours; he then rides for q hours at the rate of b miles an hour How far has he travelled, and how long would it have taken to ride the same distance at c miles an hour?

Also work out the result, supposing $p=7$, $q=3$, $a=4$, $b=9$, $c=11$

11. Resolve into factors

(i) $4a^2+12ab+9b^2$, (ii) $a^4+2a^3+a^2-1$,

(iii) $a^2-ab-ac+bc$, (iv) a^2-2a-b^2+1

12. Solve the equations

$$\text{(i)} \quad 3x-4-\frac{4(7x-9)}{15}=\frac{4}{5}\left(6+\frac{x-1}{3}\right),$$

$$\text{(ii)} \quad \frac{2x-5}{3}-\frac{x-y}{2}=4, \quad \frac{y}{2}-\frac{2x-y}{5}=3\frac{1}{2}.$$

13. Of the candidates in a certain examination 36 per cent failed. If there had been 7 more candidates, of whom one passed, the failures would have been 37.5 per cent. How many candidates were there?

14. A farmer buys 20 sheep and 15 cows for £330. Had he bought half as many sheep again but at £1 a head less, and 3 fewer cows at 75 per cent. of the price he paid, he would have spent £81 less. What did he pay for each sheep and each cow?

C

15. Multiply $3ab+4bc-5ac$ by $3ab-4bc+5ac$ by using factors.

16. Find the square root of $(2x^2-5x-3)(4x^2+12x+5)(2x^2-x-15)$.

17. Find the value of $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(b+c-a)^2$.

18. A man who owes x half-crowns tenders a £5 note in payment, and receives as change $2x$ florins, $3x$ sixpences, and 4 shillings. Find the amount of the debt.

19. Resolve the following expressions into factors:

(i) $a^4+a^2b^2+b^4$, (ii) $6(x-1)^2-x$, (iii) $10x^3-40xy^2$.

20. Solve the equations:

(i) $0.2x-0.5y=0.25$, $2.5x-y=1.025$, (ii) $\frac{5}{x}-3y=1$, $\frac{3}{x}+5y=21$.

21. If 1 cwt. of sugar costs £1. 6s. 8d., draw a graph to find the price of any number of pounds. Find the cost of 26 lbs. How many pounds can be bought for 4s. 10d.?

D

22. What values of x will make the following statements true?

(i) $3(x-5)=4(x-5)$, (ii) $(x-1)(3x-4)=(2x-1)(x-1)$,

(iii) $7(x-2)=x(x-2)$; (iv) $(2x+5)(x-2)=(x+12)(2x+5)$.

23. State in words the general truth expressed by the formula

$$b^2\equiv a^2-(a+b)(a-b).$$

Use it to find the square of 9999, by taking $a=10000$, $b=9999$.

24. Using Detached Coefficients, multiply

$$2x^4-8x^3+5x^2+5 \text{ by } x^2-4x-5.$$

25. Shew that $x+y$ and $x-y$ are factors of the expression

$$(x^2-2y^2)^3+(2x^2-y^2)^3.$$

26. Solve the equations:

(i) $5x^2+4x-1=0$, (ii) $y-1.2x=0.25y+1.8x=1.4$,

(iii) $2x-3y+z+1=5x-3z-6=3x+2y-4z=0$.

27. By buying oranges at 20 for a shilling and selling them at 6 for 5d., a hawker gained 4s. 8d. How many did he buy?

28 Mary is twenty-four, Ann was half the age Mary is, when Mary was the age Ann is now How old is Ann?

29. A basket of 65 oranges is bought for 4*s* 2*d* Draw a graph to shew the price for any other number How many could be bought for 3*s* 4*d*? Find the price (to the nearest penny) which must be paid for 36 and for 78 oranges respectively

E

30. Use factors to find the product of $2m^2+8$, $m+2$, $3m-6$

31 Solve the equations

$$\text{(i)}\ \frac{2(x-1)}{5}+\frac{15}{2}\left(1-\frac{x}{3}\right)+\frac{19}{10}=\frac{9}{5}\left(\frac{x}{6}-\frac{1}{3}\right),$$

$$\text{(ii)}\ x+3y-z=6,\quad 3x-2y+z=5,\quad 2x+4y-3z=1$$

32 Without unnecessary work calculate the coefficient of x^3 in the product of

$$x^4-29x^3-20x^2+5 \text{ and } 2x^3-x^2-x+1$$

33 Write down the values of

$$\text{(i)}\ (1+\tfrac{1}{2}x+\tfrac{1}{3}x^2)^2,\qquad \text{(ii)}\ (\tfrac{1}{2}a^2-\tfrac{2}{3}b^3)^3$$

34 If $x+3$ divides $3x^2+x-6a$ without remainder, find the value of a

35 A dealer buys a tons of coal at b shillings per ton If he sells it at c pence per cwt without gain or loss, shew that $3b=5c$

36. Assuming that 13 dollars are approximately equal to 54 shillings, draw a graph to shew the relation between dollars and shillings for any sum up to £5

Read off (i) the number of shillings in 7 dollars,

(ii) the number of dollars in £3

Shew that £4 7*s* is very nearly equal to 21 dollars

37. The highest and lowest marks gained in an examination are 297 and 132 respectively These have to be reduced in such a way that the maximum for the paper (200) shall be given to the first candidate, and that there shall be a range of 150 marks between the first and last Draw a graph from which the reduced marks may be read off and find what marks should be given to candidates who gain 200, 262, 163 marks in the examination

Find the equation between x, the actual marks gained, and y, the corresponding marks when reduced

CHAPTER XVIII

Highest Common Factor

205. When a factor divides two or more algebraical expressions it is said to be a **common factor** of those expressions.

Thus, as 5 is a *common factor* of 10, 15, and 20,

so a^2b „ „ a^2b^2, a^3bc, a^2bc^2.

206. Definition. The **Highest Common Factor** of two or more algebraical expressions is the *expression of highest dimensions* which divides each of them without remainder. The abbreviation H.C.F. is used for the words *highest common factor*.

The terms *highest common divisor* (H.C.D.) and *greatest common measure* (G.C.M.) are sometimes used instead of highest common factor. But the latter term cannot strictly be used with regard to algebraical expressions.

For example, a, x, and a^2x are common factors of a^2x and a^2x^3, and the *highest* of these common factors is a^2x.

Now suppose $a=\frac{1}{2}$ and $x=\frac{1}{3}$, then $a^2x=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$.

So that, in this case, the factor a^2x, although it is the *highest* common factor, is numerically *less* than either of the factors a, x. This shews that the algebraically *highest* common factor is not always the *greatest* common factor.

207. In the case of *simple expressions* the highest common factor can be written down by inspection.

Example. *Write down the H.C.F. of* a^3b^3, a^2b^4c, $a^4b^3c^2$.

The highest power of a, which is a common factor, is a^2,

„ „ b, „ „ is b^3,

and c is not a common factor.

Hence the expression of *highest dimensions*, which is a common factor of a^3b^3, a^2b^4c, and $a^4b^3c^2$, is a^2b^3.

208. If the expressions have numerical coefficients, find by Arithmetic their greatest common measure, and prefix it as a coefficient to the algebraical highest common factor.

Example. *Find the H.C.F. of* $21a^4x^3$, $35a^2x^4$, $28a^3x$.

The greatest common measure of 21, 35, and 28 is 7, and the highest common factor of a^4x^3, a^2x^4, and a^3x is a^2x; therefore the H.C.F. is $7a^2x$.

EXAMPLES XVIII a.

Find the highest common factor of

1. $xy,\ x^3y^2$ 2 $2x^3,\ 4x^2y$ 3. $3ab^3,\ 2a^2b$ 4. $cd^5,\ 4c^3d$
5. $2pq^2,\ 8p^4q^3$ 6. $5,\ 15p^4q$ 7. $3a^2b,\ 9abc$ 8. $a^2b^2c,\ a^3bc^5$
9. $7x^3yz^4,\ 21x^2y^5z^3$ 10. $a^2b,\ ab^2,\ a^2b^2$ 11. $x^2y,\ x^3y,\ x^2y^2$.
12. $3x^2y^2,\ 6x^3,\ 9x^4y$ 13. $p^3,\ 2p^2q^2,\ 3p^3q$ 14 $2cd,\ 4c^2,\ 6abc$
15. $12ab^2c,\ 8a^2b^4,\ 4a^3bc^3$ 16 $15a^3,\ 45a^2b,\ 25a^4$
17. $9x^3y^3z,\ 6xy^3z^3,\ 3x^2y^4z^4$ 18 $51p^4q^2r^3,\ 34p^2r^5,\ 17p^3r^4$
19. $25bc^4d^3,\ 35b^3c^5,\ 45c^3d$ 20 $35x^5y^3z^4,\ 49x^2yz^3,\ 14xy^2z^2$

[*Arts* 209, 210 *and Examples* 1–24 *in the next Exercise may be taken immediately after* Chap XIV, *in illustration of Easy Factors*]

209 The highest common factor of *compound* expressions which are given as the product of factors, or which can be easily resolved into factors, may readily be found by a similar method

EXAMPLE 1 *Find the H C F of* $3a(a+b)$ *and* $6(a^2-b^2)$

It will be easy to pick out the common factors if the expressions are arranged as follows

$$3a(a+b)=3a(a+b),$$
$$6(a^2-b^2)=3\times 2(a+b)(a-b),$$

therefore the H C F is $3(a+b)$

EXAMPLE 2 *Find the H C F of* $3a^2+9ab,\ a^3-9ab^2,\ a^3+6a^2b+9ab^2$.

Resolving each expression into its factors, we have

$$3a^2+9ab=3a(a+3b),$$
$$a^3-9ab^2=a(a^2-9b^2)=a(a+3b)(a-3b),$$
$$a^3+6a^2b+9ab^2=a(a^2+6ab+9b^2)=a(a+3b)(a+3b),$$

therefore the H C F is $a(a+3b)$

210 When two or more expressions contain different powers of the same *compound* factor, it should be noticed that the highest common factor must contain the *highest power* of the compound factor which is common to all the given expressions

Thus the H C F of $a(a-b)^2$, $b(a-b)^3$, and $ab(a-b)^5$ is $(a-b)^2$

EXAMPLE *Find the highest common factor of*

$$ax^2+2a^2x+a^3,\quad 2ax^2-4a^2x-6a^3,\quad 3(ax+a^2)^2$$

We have
$$ax^2+2a^2x+a^3=a(x^2+2ax+a^2)$$
$$=a(x+a)^2 \qquad (1),$$
$$2ax^2-4a^2x-6a^3=2a(x^2-2ax-3a^2)$$
$$=2a(x+a)(x-3a) \qquad (2),$$
$$3(ax+a^2)^2=3a^2(x+a)^2 \qquad (3).$$

Therefore from (1), (2), (3), by inspection, the H C F is $a(x+a)$.

EXAMPLES XVIII. b.

Find the highest common factor of

1. $a(a+b)$, $b(a+b)$
2. c^2-d^2, $c(c-d)$
3. $x(x+y)$, $x^2y(x+y)$
4. $x(2x+1)$, $4x^2-1$
5. $mn(m-2n)$, $2n^2(m-2n)$
6. $p^2q(2p+3q)$, $4p^2-9q^2$.
7. $c^2d(c+d)^2$, $c^4-c^2d^2$
8. a^3-3a^2b, $3ab^2-9b^3$
9. $ab^2(b+c)$, $a^2b(b+c)^2$
10. $n^4-4n^2m^2$, m^2n+2m^3
11. a^3+8, a^2+5a+6
12. $(x-3)^3$, x^3-27
13. $ax(a-x)^3$, $2a^2x(a-x)^2$
14. $d^3(c-d)^2$, $d^2(c^2+cd-2d^2)$
15. $m(m-n)^2$, $(m-n)(m^2-n^2)$
16. $x^3y^2+x^2y^3$, $x^4(x^2-y^2)$
17. x^4-27a^3x, $(x-3a)^2$
18. $4a^2+2ab$, $12a^2b-3b^3$
19. c^3-c, $(c-1)^2$, c^3-1
20. $d+2$, d^2-4, d^3+8
21. $p^2+7p-18$, $p^2+10p+9$
22. $m^2-3m-18$, m^2+5m+6
23. $x^2+14x+33$, x^3+10x^2-11x
24. x^4-27x, $x^4+2x^3-15x^2$
25. $15a^2+8a+1$, $12a^2+a-1$
26. $14c^2+5c-1$, $8c^3+8c^2+2c$
27. $a^2-ab-2b^2$, $a^2+3ab+2b^2$
28. $c^2-4cd+3d^2$, $c^2-2cd+d^2$
29. $x^2+3xy+2y^2$, $x^2+5xy+6y^2$
30. $2a^2-9a+4$, $3a^2-7a-20$
31. $4a^2b^3-9b$, $2a^2b^2-ab-3$
32. $2x^4-7x^3+3x^2$, $3x^3-7x^2-6x$
33. $2ab^3-2a^3b$, $3(ab-a^2)^2$
34. $16p^2-8pq+q^2$, $(4pq-q^2)^2$
35. x^2-x-2, x^2+x-6, $3x^2-13x+14$
36. x^3-16x, $2x^3+9x^2+4x$, $2x^3+x^2-28x$
37. $2x^2+5x+2$, $3x^2+8x+4$, $2x^2+3x-2$
38. $2x+cx+2c+c^2$, $8c+c^4$, $4+4c+c^2$

211. When one or more of the expressions cannot readily be put into factors, we may sometimes proceed as in the following example.

EXAMPLE. *Find the H.C.F. of* $3x^2+x-14$ *and* $x^3+2x^2-3x-10$.

We have $\qquad 3x^2+x-14=(3x+7)(x-2)$.

Of these two factors $3x+7$ obviously does not divide the second expression. Hence the H.C.F. (if there is one) must be $x-2$.

Applying the Remainder Theorem to the second expression, we find

$$2^3+2\cdot2^2-3\cdot2-10=8+8-6-10=0.$$

Thus $x-2$ divides both expressions. Hence the H.C.F. is $x-2$.

EXAMPLES XVIII. b. (*Continued*)

Find the highest common factor of

39. $x^2-4x-21$, x^3+3x^2-3x-9
40. $5x^2-3x-8$, x^4-2x^3-4x-7.
41. a^3-125, $a^4-5a^3+a^2+4a-45$
42. x^2+x-6, $x^2+3x-10$, x^3-x^2-5x-2
43. a^2+a-12, a^2-2a-3, $a^3-4a^2-2a+15$.
44. $2a^2+11a-21$, $3a^2+25a+28$, $a^4+5a^3-14a^2-5a-35$

212 When the expressions are not easily resolved into factors their H C F may be found by a method similar to the 'Division Method' used in Arithmetic.

The method depends on the following principles

(i) *If an expression contains a certain factor, any multiple of the expression is divisible by that factor*

(ii) *If two expressions have a common factor, it will divide their sum and their difference, and it will also divide the sum and difference of any multiples of them*

These statements will be formally proved in Art 218

EXAMPLE *Find the highest common factor of*

$$4x^3-3x^2-24x-9 \text{ and } 8x^3-2x^2-53x-39$$

x	$4x^3-3x^2-24x-9$	$8x^3-2x^2-53x-39$	2
	$4x^3-5x^2-21x$	$8x^3-6x^2-48x-18$	
$2x$	$2x^2-3x-9$	$4x^2-5x-21$	2
	$2x^2-6x$	$4x^2-6x-18$	
3	$3x-9$	$x-3$	
	$3x-9$		

Therefore the H C F is $x-3$

Explanation The given expressions are arranged according to descending powers of x The expressions so arranged having their first terms of the same degree, we take for divisor that whose highest power has the smaller coefficient Arrange the work in parallel columns as above The first quotient 2 is placed to the right of the dividend, when the first remainder $4x^2-5x-21$ is made the divisor we put the quotient x to the *left* Again, when the second remainder $2x^2-3x-9$ is in turn made the divisor, the quotient 2 is placed to the *right*, and so on As in Arithmetic, the last divisor $x-3$ is the highest common factor required

The process succeeds because at any stage every common factor of the original expressions is a factor of the dividend and divisor at that stage This follows from the principles above quoted Hence the H C F. will be the same as that for the *last* divisor and dividend, that is the last divisor is the H C F

With detached coefficients the work would stand as follows:

1	4 - 3 - 24 - 9	8 - 2 - 53 - 39	2
	4 - 5 - 21	8 - 6 - 48 - 18	
2	2 - 3 - 9	4 - 5 - 21	2
	2 - 6	4 - 6 - 18	
3	3 - 9	1 - 3	
	3 - 9		

213 This method is only useful to determine a *compound* factor of the highest common factor *Simple* factors of the given expressions must be first removed from them, and the highest common factor of these, if any, must be reserved and multiplied into the *compound* factor found by the division method

EXAMPLE *Find the highest common factor of*

$$24x^4-2x^3-60x^2-32x \quad \textit{and} \quad 18x^4-6x^3-39x^2-18x$$

We have $24x^4-2x^3-60x^2-32x=2x(12x^3-x^2-30x-16)$,

and $18x^4-6x^3-39x^2-18x=3x(6x^3-2x^2-13x-6)$

Removing the simple factors $2x$ and $3x$, and *reserving* their common factor x, we continue as follows

$$\begin{array}{r|r|r|r}
2x & 6x^3-2x^2-13x-6 & 12x^3-x^2-30x-16 & 2 \\
 & 6x^3-8x^2-8x & 12x^3-4x^2-26x-12 & \\ \hline
2 & 6x^2-5x-6 & 3x^2-4x-4 & x \\
 & 6x^2-8x-8 & 3x^2+2x & \\ \hline
 & 3x+2 & -6x-4 & -2 \\
 & & -6x-4 &
\end{array}$$

Therefore the H C F is $x(3x+2)$

214 In some cases certain modifications of the arithmetical method are necessary

EXAMPLE *Find the highest common factor of*

$$3x^3-13x^2+23x-21 \quad \textit{and} \quad 6x^3+x^2-44x+21$$

$$\begin{array}{r|r|r}
3x^3-13x^2+23x-21 & 6x^3+x^2-44x+21 & 2 \\
 & 6x^3-26x^2+46x-42 & \\ \hline
 & 27x^2-90x+63 &
\end{array}$$

The remainder, if made a divisor, as it stands, would give a *fractional* quotient Noticing that $27x^2-90x+63=9(3x^2-10x+7)$, we take out the factor 9 This will not affect the result, because the two original expressions have no *simple* factors, and therefore, in rejecting the 9 we are not rejecting a *common* factor of those expressions

Resuming the work, we have

$$\begin{array}{r|r|r|r}
x & 3x^3-13x^2+23x-21 & {}^*3x^2-10x+7 & x \\
 & 3x^3-10x^2+7x & 3x^2-7x & \\ \hline
-1 & -3x^2+16x-21 & -3x+7 & -1 \\
 & -3x^2+10x-7 & -3x+7 & \\ \hline
 & 2\,)\,6x-14 & & \\
 & 3x-7 & &
\end{array}$$

Therefore the H C F is $3x-7$

The factor 2 is removed for the same reason as the factor 9 above

*The work may here be completed as follows $3x^2-10x+7$ must contain the H C F But $3x^2-10x+7=(3x-7)(x-1)$

By the Remainder Theorem we find that $x-1$ does not divide either of the given expressions Hence $3x-7$ is the H C F

EXAMPLES XVIII. c.

Find the highest common factor of

1. $x^3-7x^2+14x-8,\ x^3-6x^2+11x-6$
2. $2a^3-5a^2+5a-6,\ 2a^3-7a^2+6a-9.$
3. $c^3-7c^2+11c-5,\ c^3-8c^2+13c-6$
4. $4a^3-11a^2+25a+7,\ 2a^3-5a^2+11a+7$
5. $b^3-6b^2-86b+35,\ b^3-5b^2-99b+40$
6. $4x^3+9x^2-2x-1,\ 4x^3+10x^2-2$ ·7. $a^3-18a+35,\ a^3-21a+20$

·8. $b^3-67b+24,\ b^3-76b+96$ ·9 $2x^3-8x+30,\ 9x^3-12x^2+75.$

10. $6m^4-9m^3-39m^2+36m,\ 2m^4-13m^3-28m^2+32m.$
11. $3x^4+6x^3-12x^2-24x,\ 4x^4+14x^3+8x^2-8x$
12. $3x^3y-24xy^3+9y^4,\ 3x^4-8x^3y+xy^3$ 13 $c^4+12c-5,\ c^4+2c^2+8c+5.$
14. $3a^4-11a^3+15a^2-6a,\ a^4-6a^3+12a^2-9a$
15. $y^3-y^2-100,\ y^4+5y^3-76y+20$
16. $6a^3-a^2b+6ab^2-35b^3,\ 3a^3+7a^2b-47ab^2+45b^3$

215 Sometimes the process is more convenient when the expressions are arranged in *ascending* powers

EXAMPLE *Find the highest common factor of*

$$(1)\ 3-4a-16a^2-9a^3, \qquad (2)\ 4-7a-19a^2-8a^3$$

To avoid a fractional quotient we must here *introduce* a suitable simple factor into one of the two given expressions As neither of them contains a simple factor we shall not thereby multiply one expression by a factor which is contained in the other, and, therefore, the H.C F will not be affected

Multiply (1) by 4 and use (2) as divisor

$$\begin{array}{r|l|l|l}
 & 4-7a-19a^2-8a^3 & 12-16a-64a^2-36a^3 & 3 \\
 & 5 & \underline{12-21a-57a^2-24a^3} & \\
4 & 20-35a-95a^2-40a^3 & a\,)\,5a-7a^2-12a^3 & \\
 & \underline{20-28a-48a^2} & {}^{*}5-7a-12a^2 & 5 \\
 & -7a-47a^2-40a^3 & \underline{5+5a} & \\
 & \underline{-5} & -12a-12a^2 & -12a \\
7a & 35a+235a^2+200a^3 & \underline{-12a-12a^2} & \\
 & \underline{35a-49a^2-84a^3} & & \\
 & 284a^2\,)\,\underline{284a^2+284a^3} & & \\
 & \qquad 1+a & &
\end{array}$$

Therefore the H C F is $1+a$

After the first division the factor a is removed as explained in Art 214; then, at successive stages, the factors 5 and -5 are introduced, and finally the factor $284a^2$ is removed

*At this stage we may proceed as follows

The remainder $=5-7a-12a^2=(1+a)(5-12a)$, and since $5-12a$ does not divide either of the given expressions, the H C F is $1+a$

216 From the last two examples it appears that we may multiply or divide either of the given expressions, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the given expressions

217 The H C F of more than two expressions must *be a factor of the H C F of any two of them* Therefore the H C F of more than two expressions may be obtained as follows

(1) *Take any two of the given expressions and find their H C F*

(2) *Take this result and a third expression, and find their H C F; and so on*

The H C F last found must be the H C F required, because it is the highest factor contained in *all* of the expressions

EXAMPLES XVIII. d

Find the highest common factor of

1. $x^4-2x^3+x^2-1,\ x^4-3x^2+1$
2. $8x^3+6x^2-12x-9,\ 24x^4-20x^2-24$
3. $y^4-2y^3-4y-7,\ y^4+y^3-3y^2-y+2$
4. $c^3+4c^2d-63d^3,\ c^4-7c^2d^2+441d^4$
5. $6x^4-2x^3-17x^2+2x-5,\ 6x^5-4x^4-11x^3-3x^2-3x-1$
6. $x^7-6x^6+13x^5-12x^4+4x^3,\ x^7-5x^6+8x^5-4x^4$
7. $2x^4+5x^3y-6x^2y^2-9xy^3,\ 6x^3y-17x^2y^2+14xy^3-3y^4$
8. $24a^4b+72a^3b^2-6a^2b^3-90ab^4,\ 6a^4b^2+13a^3b^3-4a^2b^4-15ab^5$
9. $4m^5+14m^4+20m^3+70m^2,\ 8m^7+28m^6-8m^5-12m^4+56m^3$
10. $2x^4-3x^2-14,\ 6x^4+10x^3-17x^2-35x-14$
11. $2x^4-5x^3-5x^2+18,\ 4x^4-18x^3+81x-81$
12. $y^4-15y^3+75y^2-145y+84,\ y^4-17y^3+101y^2-247y+210$
13. $6-8a-32a^2-18a^3,\ 20-35a-95a^2-40a^3$
14. $9x^2-15x^3-45x^4-12x^5,\ 42x-49x^2-203x^3-84x^4$
15. $3x^5-5x^3+2,\ 2x^5-5x^2+3$
16. $1+x+x^3-x^5,\ 1-x^4-x^6+x^7$.
17. $x^3+x^2-7x+2,\ 2x^3-x^2-7x+2,\ x^3-4x^2+4x$
18. $a^3-4a^2+4a-3,\ a^3-a^2-7a+3,\ a^3-5a^2+9a-9$
19. $2y^3-7y^2+7y-2,\ 4y^3-13y^2+11y-2,\ y^4-3y^3+6y-4$

218 The statements of Art 212 may be proved as follows

(1) Suppose F is a factor which divides an expression A, then F will divide any multiple of A

For if F is contained a times in A, then $A=aF$

$$mA=maF,\ \text{that is, F is a factor of } mA$$

(ii) If F divides *two* expressions A and B, then clearly it will divide $A \pm B$

Now suppose $A = aF$, $B = bF$,

then
$$mA \pm nB = maF \pm nbF$$
$$= F(ma \pm nb)$$

Thus F divides $mA \pm nB$, where mA and nB are any multiples of A and B

219 We may now give a general proof of the rule for finding the H C F. of any two compound algebraical expressions

Let A and B be the two expressions *after the simple factors have been removed* Let them be arranged in descending or ascending powers of some common letter; also let the highest power of that letter in B be not less than the highest power in A

Divide B by A; let p be the quotient, and C the remainder Suppose C to have a *simple* factor m Remove this factor, and so obtain a new divisor D Further, suppose that in order to make A divisible by D it is necessary to multiply A by a *simple* factor n Let q be the next quotient and E the remainder Finally, divide D by E; let r be the quotient, and suppose that there is no remainder. Then E will be the H.C F required.

The work will stand thus

$$\begin{array}{l} A)\,B\,(p \\ \quad\;\; \underline{pA} \\ m)\,\underline{C} \\ \qquad \overline{D})\,nA\,(q \\ \qquad\quad\;\; \underline{qD} \\ \qquad\qquad E)\,D\,(r \\ \qquad\qquad\quad\; \underline{rE} \end{array}$$

First, to shew that E is a common factor of A and B

By examining the steps of the work, it is clear that E divides D, therefore also qD, therefore $qD + E$, therefore nA; therefore A, since n is a *simple* factor.

Again. E divides D, therefore mD, that is, C And since E divides A and C, it also divides $pA + C$, that is, B Hence E divides both A and B

Secondly, *to shew that E is the highest common factor*

If not, let there be a factor X of higher dimensions than E

Then X divides A and B, therefore $B - pA$, that is, C; therefore D (since m is a *simple* factor); therefore $nA - qD$, that is, E

Thus X divides E; which is impossible since, by hypothesis, X is of higher dimensions than E.

Therefore E is the highest common factor.

CHAPTER XIX.

FRACTIONS

220 IN Arithmetic the fraction $\frac{4}{5}$ denotes *four times the fifth part of the unit* But if we divide 4 units by 5 we get a result which is four times as great as the fifth part of 1 unit

Hence the fraction $\frac{4}{5}$ is *the result of dividing* 4 *units by* 5

In Algebra we define the fraction $\frac{a}{b}$ as *the result of dividing* a *by* b, *where* a *and* b *may have any values whatever*

The statement of Art 68 may now be written

$$\textit{fraction} \times \textit{divisor} = \textit{dividend}$$

221 *The value of a fraction is not altered if we multiply or divide the numerator and denominator by the same quantity*

To prove that $$\frac{a}{b}=\frac{ma}{mb}$$

Let the fraction $\frac{a}{b}$ be denoted by x, then, by definition, $bx=a$

Multiply both sides by m, then $mbx=ma$

Divide both sides by mb, then $$x=\frac{ma}{mb},$$

that is, $$\frac{a}{b}=\frac{ma}{mb}$$

Conversely, $$\frac{ma}{mb}=\frac{a}{b}$$

222 To reduce a fraction to its lowest terms *divide numerator and denominator by every factor which is common to both, that is, by their highest common factor*

EXAMPLE 1 *Reduce* $\frac{6a^2c}{9ac^2}$ *to its lowest terms*

The H C.F of the numerator and denominator is $3ac$

Thus $$\frac{6a^2c}{9ac^2}=\frac{3ac\times 2a}{3ac\times 3c}=\frac{2a}{3c}$$

EXAMPLE 2 *Reduce* $\dfrac{7x^2yz}{28x^3yz^2}$ *to its lowest terms.*

The H C F of the numerator and denominator is $7x^2yz$.

Thus $$\frac{7x^2yz}{28x^3yz^2}=\frac{7x^2yz\times 1}{7x^2yz\times 4xz}=\frac{1}{4xz}$$

EXAMPLE 3 $$\frac{51c^3d^5}{17c^2d^3}=\frac{17c^2d^3\times 3cd^2}{17c^2d^3\times 1}=3cd^2$$

NOTE Dividing numerator and denominator by a common factor is called *cancelling* that factor It is important to remember that when a factor is cancelled, its place is really taken by *unity* The beginner should be careful not to begin cancelling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary

EXAMPLES XIX. a.

Reduce to lowest terms

1. $\dfrac{6x^3y}{9xy^2}$ 2 $\dfrac{4cd^4}{6c^2d^3}$ 3 $\dfrac{3a^4b^3}{9a^5b}$ 4. $\dfrac{3m^5n}{m^3n^2}$

5. $\dfrac{m^4n^3}{5mn^3}$ 6 $\dfrac{18p^4q^2}{27q^2}$ 7 $\dfrac{21c^2d^2}{28d^2e^2}$ 8. $\dfrac{x^2yz^3}{x^3y^2z}$

9. $\dfrac{12a^2b^3c^2}{8ab^2c}$ 10 $\dfrac{15abx^2}{65a^3b^2x}$ 11 $\dfrac{25cd^4x^3}{40c^2d^3}$ 12 $\dfrac{3a^2x^3y^5}{57x^4y}$

13. $\dfrac{39m^3n^2}{52l^2m^4n^2}$ 14. $\dfrac{48ax^3y^2}{64a^2xy^6}$ 15 $\dfrac{38b^2cd^4}{57bc^3d^2}$ 16. $\dfrac{42kl^2n^3}{210k^4l^2n}$

223 When the numerator, or denominator, is a compound expression, whose factors may be written down by inspection, the fraction may be simplified by a similar method

EXAMPLE 1 *Reduce to its lowest terms* $\dfrac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}$

$$\frac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}=\frac{6a^2x^2\times 4ac^2}{6a^2x^2(3a-2x)}=\frac{4ac^2}{3a-2x}$$

EXAMPLE 2 *Simplify* $\dfrac{6x^2-8xy}{6x^2+xy-12y^2}$

$$\frac{6x^2-8xy}{6x^2+xy-12y^2}=\frac{2x(3x-4y)}{(3x-4y)(2x+3y)}=\frac{2x}{2x+3y}$$

EXAMPLE 3 *Reduce to its lowest terms* $\dfrac{x^3y-xy^3}{x^3-y^3}$

$$\frac{x^3y-xy^3}{x^3-y^3}=\frac{xy(x^2-y^2)}{(x-y)(x^2+xy+y^2)}=\frac{xy(x+y)(x-y)}{(x-y)(x^2+xy+y^2)}=\frac{xy(x+y)}{x^2+xy+y^2}$$

NOTE In simplifying fractions, a factor must not be removed unless it divides both numerator and denominator, *each taken as a whole.*

EXAMPLES XIX. b.

(*Examples 1–20 may be taken immediately after Chap.* XIV. *in illustration of Easy Factors.*)

Reduce to lowest terms:

1. $\dfrac{ab}{a^2b^2-ab}$	2. $\dfrac{x^2+xy}{xy+y^2}$	3. $\dfrac{xy+2y^2}{x^2+2xy}$
4. $\dfrac{c^2-2c}{4c^3-8c^2}$	5. $\dfrac{3a^2+3ab}{4ab+4b^2}$	6. $\dfrac{2a^3+6a^2b}{6a^2b+18ab^2}$
7. $\dfrac{b^2+3ab}{2ab^3+6a^2b^2}$	8. $\dfrac{6xy-3x^2}{4xy^2-2x^2y}$	9. $\dfrac{pqr-qr^2}{pmr-mr^2}$
10. $\dfrac{a^2-4b^2}{a^2-2ab}$	11. $\dfrac{6x^2+3xy}{4x^2-y^2}$	12. $\dfrac{(2x+y)^2}{4x^3-xy^2}$
13. $\dfrac{2x^2y^2-8}{3x^2y+6x}$	14. $\dfrac{5p^2+30p}{p^2+p-30}$	15. $\dfrac{(m+2n)^2}{m^3-4mn^2}$
16. $\dfrac{x^2y-5xy}{x^2y-4xy-5y}$	17. $\dfrac{a^2-5a+6}{a^2-6a+9}$	18. $\dfrac{x^2-4ax-21a^2}{(x^2+3ax)^2}$
19. $\dfrac{x^3+y^3}{x^2-xy-2y^2}$	20. $\dfrac{x^4-5x^2+4}{x^2-3x+2}$	21. $\dfrac{x^2+4x-45}{3x^2-14x-5}$
22. $\dfrac{2a^2-3a-14}{2a^2+11a+14}$	23. $\dfrac{3x^3-24}{2x^2+6x-20}$	24. $\dfrac{9c^2d^2-3cd^3+d^4}{27c^3+d^3}$

224. When the numerator and denominator cannot easily be put into factors, their H.C.F. may be found by the rules given in Chap. XVIII.

EXAMPLE. *Reduce to lowest terms* $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}$.

First Method. The H.C.F. of numerator and denominator is $3x-7$.

Dividing numerator and denominator by $3x-7$, we obtain as respective quotients x^2-2x+3 and $5x^2-x-3$.

$$\text{Thus}\quad \frac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\frac{x^2-2x+3}{5x^2-x-3}.$$

Second Method. By Art. 212, the H.C.F. of numerator and denominator must be a factor of their sum $18x^3-51x^2+21x$, that is, of $3x(3x-7)(2x-1)$. If there be a common divisor it must clearly be $3x-7$; hence arranging numerator and denominator so as to shew $3x-7$ as a factor,

$$\text{the fraction}=\frac{x^2(3x-7)-2x(3x-7)+3(3x-7)}{5x^2(3x-7)-x(3x-7)-3(3x-7)}$$

$$=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}$$

$$=\frac{x^2-2x+3}{5x^2-x-3}.$$

225 If *either* numerator *or* denominator can readily be resolved into factors we may use the following method

EXAMPLE *Reduce to lowest terms* $\dfrac{x^3+3x^2-4x}{7x^3-18x^2+6x+5}$

The numerator $=x(x^2+3x-4)=x(x+4)(x-1)$

Of these factors the only one which can be a common divisor is $x-1$ Hence, arranging the denominator,

$$\text{the fraction}=\frac{x(x+4)(x-1)}{7x^2(x-1)-11x(x-1)-5(x-1)}$$

$$=\frac{x(x+4)(x-1)}{(x-1)(7x^2-11x-5)}=\frac{x(x+4)}{7x^2-11x-5}$$

EXAMPLES XIX. c

Reduce to lowest terms

1. $\dfrac{2x^2+3x+1}{x^3+2x^2-x-2}$ 2. $\dfrac{2a^3+a^2-8a+5}{7a^2-12a+5}$ 3 $\dfrac{c^3-2c+1}{3c^3+7c-10}$

4. $\dfrac{2x^3+3x^2+49}{x^4+3x^2+14x}$ 5. $\dfrac{x^3-2x^2+3x-6}{x^4-x^3-x^2-2x}$ 6 $\dfrac{x^4-x^3+7x+2}{3x^4-7x^3+11x^2-1}$

7. $\dfrac{x^4-11x^2+7x-3}{2x^5+7x^4-x^3+x}$ 8. $\dfrac{2a^4+a^3b-4a^2b^2-3ab^3}{4a^4+a^3b-2a^2b^2+ab^3}$

9 $\dfrac{4x^3+3x^2-20x-15}{5x^4+2x^3-25x^2-10x}$ 10 $\dfrac{7a^3-4a^2-21a+12}{5a^3+2a^2-15a-6}$

11 $\dfrac{x^3-6x^2+11x-6}{x^3-2x^2-x+2}$ 12. $\dfrac{3y^3-8y^2+1}{y^3-8y+3}$ 13 $\dfrac{a^4+9a-20}{5a^5+9a^4-64a}$

14 $\dfrac{6x^3-x^2+16}{8x^3-14x^2+19x-4}$ 15 $\dfrac{3c^5-5c^3+2}{2c^5-5c^3+3}$ 16 $\dfrac{3-4x-16x^2-9x^3}{4-7x-19x^2-8x^3}$

Multiplication and Division of Fractions

226 To multiply together two or more fractions: *multiply together all the numerators to form a new numerator, and all the denominators to form a new denominator*

To prove that $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$

Let x represent the fraction $\dfrac{a}{b}$, and y the fraction $\dfrac{c}{d}$

Then $bx=a$, and $dy=c$

$bx\times dy=ac$, or $xy\times bd=ac$

Dividing both sides by bd, $xy=\dfrac{ac}{bd}$,

that is, $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$

Similarly $$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf},$$

and so for any number of fractions

In practice the application of this rule is modified by cancelling, in the course of the work, factors which are common to numerator and denominator

EXAMPLE $$\frac{2am^3}{x^4y^2}\times\frac{3b^2y^2}{5m^2}\times\frac{am^3x^2}{6b^2m^4}=\frac{2am^3\times 3b^2y^2\times am^3x^2}{x^4y^2\times 5m^2\times 6b^2m^4}=\frac{a^2}{5x^2},$$

the result being obtained by removing like factors from numerator and denominator

227 To divide by a fraction: *multiply by its reciprocal*

Since division is the inverse of multiplication, we may define the quotient x when $\frac{a}{b}$ is divided by $\frac{c}{d}$, to be such that

$$x\times\frac{c}{d}=\frac{a}{b}$$

Multiplying by $\frac{d}{c}$ we have $x\times\frac{c}{d}\times\frac{d}{c}=\frac{a}{b}\times\frac{d}{c}$,

But $\because \frac{c}{d}\times\frac{d}{c}=1$, $\quad x=\frac{a}{b}\times\frac{d}{c}$,

that is, $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c},$$

which proves the rule

EXAMPLE $$\frac{7a^3}{4x^3y^2}\times\frac{6c^3x}{5ab^2}\div\frac{28a^2c^2}{15b^2xy^2}=\frac{7a^3}{4x^3y^2}\times\frac{6c^3x}{5ab^2}\times\frac{15b^2xy^2}{28a^2c^2}=\frac{9c}{8x},$$

all the other factors cancelling each other

NOTE When several fractions are connected by the signs $\times$, $\div$, each sign applies only to the fraction which immediately follows it

EXAMPLES XIX. d.

Simplify the following expressions

1 $\frac{ab}{xy}\times\frac{x^2y^2}{ab^2}$ 2 $\frac{ab^3}{4c^2d}\times\frac{2cd^3}{ab}$ 3 $\frac{3ab^2}{5b^3c}\times\frac{15b^2c^2}{9a^2b}$

4 $\frac{7y^2}{5xy^3}\times\frac{25y^2}{14xy}$ 5 $\frac{14b^2c}{12ad}\times\frac{6a^2d^3}{7ab^2}$ 6 $\frac{39a^2b^3}{21c^2d^3}\times\frac{28d^2c^3}{13ab^2}$

7 $\frac{2x^2y}{3y^2z}\times\frac{z^4}{3x^3}\div\frac{4yz^3}{9x^2}$ 8 $\frac{3xy^2}{4yz}\times\frac{5z^3x}{7x^2y}\div\frac{45x^3y^3z}{14xy^3z^2}$

9 $\frac{13c^2}{15a}\times\frac{16a^4}{39d^3}\div\frac{48a^2b^2c}{27d^2}$. 10 $\frac{x^3}{ay^2z}\times\frac{4y^3}{a^2xz^2}\times\frac{a^3y^2z^3}{128x^2y^4z}$.

11 $\frac{160m^4}{15n}\div\frac{156m^2np^5}{25p^3}\times\frac{36n^2p^3}{65m^2n}$ 12 $\frac{x^2}{a^2z^2}\div\frac{c^3x}{az^3}\times\frac{a^4y^3}{xz}\times\frac{c^3}{a^3y^3}$

228 The same method is followed when the numerators and denominators of the fractions are compound expressions

EXAMPLE 1 *Simplify* $\dfrac{3a^2-2a}{4a^3}\times\dfrac{2a^3+10a^2}{18a-12}$

$$\text{The expression}=\frac{a(3a-2)}{4a^3}\times\frac{2a^2(a+5)}{6(3a-2)}$$
$$=\frac{a+5}{12},$$

by cancelling factors common to both numerator and denominator

EXAMPLE 2 $$\frac{xy-ay}{9x^2-4a^2}\div\frac{y^2}{3ax+2a^2}=\frac{y(x-a)}{(3x+2a)(3x-2a)}\times\frac{a(3x+2a)}{y^2}$$
$$=\frac{a(x-a)}{y(3x-2a)}$$

EXAMPLE 3 *Simplify* $\dfrac{x^4-8x}{2x^2+5x-3}\times\dfrac{2x-1}{x^2+2x+4}\div\dfrac{x^2-2x}{x+3}$

$$\text{The expression}=\frac{x(x^3-8)}{(2x-1)(x+3)}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$$
$$=\frac{x(x-2)(x^2+2x+4)}{(2x-1)(x+3)}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$$
$$=1$$

NOTE When all the factors of numerator and denominator cancel each other, it is a common mistake with beginners to give the result as 0 A little reflection will shew that *the result of such a multiplication can never be zero*

EXAMPLES XIX e

Simplify

1 $\dfrac{x^2}{2a+3}\times\dfrac{6a^2+9a}{4x^3}$ 2 $\dfrac{a^2-2}{6ab}\times\dfrac{18b^3}{5a^4-10a^2}$ 3 $\dfrac{a^2-4}{x^2-1}\times\dfrac{x^3-x}{a^3+2a^2}$

4 $\dfrac{b^2-100}{a^2-b^2}\div\dfrac{b+10}{a-b}$ 5. $\dfrac{x^3-x}{2x^3+6x^2}\times\dfrac{x^2+3x}{x^2-1}$ 6 $\dfrac{9c^2-16d^2}{c^2-25}\times\dfrac{c^2-5c}{3c-4d}$

7 $\dfrac{m^2-4n^2}{mn(m+2n)^2}\div\dfrac{2m-4n}{m^3n^3}$ 8. $\dfrac{a^2b^2-9}{4a^3-a}\times\dfrac{2a^2+a}{ab+3}$

9. $\dfrac{a^3-b^3}{a^2-2ab+b^2}\times\dfrac{a^2-b^2}{a^2+ab}$ 10 $\dfrac{c^2-16}{c^2-8c+16}\div\dfrac{2c+8}{3c-9}$

11. $\dfrac{x^2+3x+2}{x^2-4x-12}\times\dfrac{x^2-7x+6}{x^2-4}$ 12 $\dfrac{x^2+4x}{x^3-9x}\div\dfrac{x^2+2x-8}{x^2+x-6}$

13. $\dfrac{x^2+7x+12}{x^2+9x+20}\times\dfrac{x^2+3x+2}{x^2+5x+6}$ 14. $\dfrac{x^3+9x^2+20x}{x^2+5x+4}\div\dfrac{x^2+7x+10}{x^2+3x+2}$

Find the value of

15. $\frac{y^2-2y-24}{y^2-16}\times\frac{y^2-y-12}{y^2+6y+9}$

16. $\frac{a^2-4a-21}{a^2-49}\div\frac{a^3+27}{a^2+9a+14}$

17. $\frac{m^2-m-12}{m^3-64}\div\frac{m^2+m-6}{m^2+4m+16}$

18. $\frac{2y-y^2-6y^3}{2-7y+6y^2}\div\frac{4y+12y^2+9y^3}{4-9y^2}$

19. $\frac{b^4-27b}{2b^2+5b}\times\frac{4b^2-25}{2b^2-11b+15}$

20. $\frac{3x^2-7x+2}{2x^2-5x-3}\times\frac{x^2-9}{9x^2-6x+1}$

21. $\frac{2x-3}{2x^2+13x-24}\times\frac{4x^2-3x-7}{4x-7}\times\frac{x^2+5x-24}{x^2-2x-3}$

22. $\frac{4x^2+x-14}{6x-14}\times\frac{4x^2}{x^2-4}\times\frac{x-2}{4x-7}\div\frac{2x^2+4x}{3x^2-x-14}$

23. $\frac{x^3+8y^3}{x^2-3xy+2y^2}\times\frac{2x^2-3xy-2y^2}{x^2-2xy+4y^2}\div\frac{2x^2+5xy+2y^2}{x^2-2xy+y^2}$

24. $\frac{x^2-5x+6}{x^2+5x+4}\div\frac{x^2-4x+3}{2x^2+3x+1}\times\frac{x^2+3x-4}{2x^2-3x-2}$

25. $\frac{2a^2+3ab-2b^2}{a^2+2ab+4b^2}\times\frac{a^3-8b^3}{a^2+3ab+2b^2}\div\frac{2a^2-5ab+2b^2}{a^2+2ab+b^2}$

26. $\frac{x^2-9}{5x^3y^3}\div\left(\frac{x+3}{10x^4}\times\frac{2x-6}{xy^4}\right)$

27. $\frac{c^3-8}{c+3}\div\left(\frac{c-2}{4c}\times\frac{8c^4}{c^2+3c}\right)$.

28. $\frac{a^4-x^4}{a^2-2ax+x^2}\div\left(\frac{a^3+x^3}{ax^2-x^3}\times\frac{a^2x^2+x^4}{a^2-ax+x^2}\right)$

29. $\frac{a^3+a^2b+ab^2}{a^2-3ab-4b^2}\div\frac{a^2+6ab-27b^2}{a^2+8ab-9b^2}\times\frac{a^2-7ab+12b^2}{a^3-b^3}$

30. $\frac{(b+c)^2-a^2}{b^2+bc-ab}\times\frac{(c+a)^2-b^2}{(a+b)^2-c^2}\times\frac{b^2+ab-bc}{ac+a^2-ab}$

31. $\frac{x^2+2xy+y^2-a^2}{y^2-c^2+2cx-x^2}\times\frac{y^2-2xy+x^2-c^2}{(y-c)^2-x^2}\div\frac{x+y+a}{y+x-c}$

32. $\frac{a^4+a^2b^2+b^4}{a^2-4ab-21b^2}\times\frac{a^2+2ab-3b^2}{a^3-b^3}\div\frac{1}{a-7b}$

33. $\frac{m^3+4m^2n+4mn^2}{3m^2n-5mn^2-2n^3}\div\frac{(m+2n)^3}{27m^3+n^3}\times\frac{m^2-4n^2}{9m^2-3mn+n^2}$

34. $\frac{1+8x^3}{(2-x)^2}\times\frac{4x-x^3}{1-4x^2}\div\frac{(1-2x)^2+2x}{2-5x+2x^2}$.

35. $\frac{x^2(x-4)^2}{(x+4)^2-4x}\div\frac{(x^2-4x)^3}{(x+4)^2}\times\frac{64-x^3}{16-x^2}$

CHAPTER XX.

LOWEST COMMON MULTIPLE

229 DEFINITION The lowest common multiple of two or more algebraical expressions is *the expression of lowest dimensions* which is divisible by each of them without remainder The abbreviation L C M is used for the words *lowest common multiple.*

230 In the case of *simple expressions*, the lowest common multiple can be written down by inspection, as follows

EXAMPLE 1 *Find the lowest common multiple of* a^4, a^3, a^2, a^6

As no other letter than a occurs in the given expressions, the required expression of lowest dimensions is the lowest power of a divisible by all, that is a^6

EXAMPLE 2 *Find the lowest common multiple of* a^3b^4, ab^5, a^2b^7

The lowest power of a that is divisible by a^3, a, a^2 is a^3, and

" " b " " b^4, b^5, b^7, is b^7

Therefore the expression of lowest dimensions divisible by a^3b^4, ab^5 and a^2b^7 is a^3b^7

231 If the expressions have numerical coefficients, find by Arithmetic their least common multiple, and prefix it as a coefficient to the algebraical lowest common multiple

EXAMPLE *Find the L C M of* $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$

The least common multiple of 21, 35 and 28 is 420,

the lowest power of a divisible by a^4, a^2, a^3 is a^4,

" " x " x^3, x^4, x is x^4,

" " y " y, y^4 is y^4

the L C M is $420a^4x^4y^4$

EXAMPLES XX. a.

Find the lowest common multiple of

1. a^2, abc
2. $2a^2$, a^3b
3. $3x^2y$, $2xy^3$
4. $4xyz$, $3x^2z$
5. $3a^2bc^2$, $5bc^3$
6. $15c^4d$, $5cd^3$
7. xy, yz, zx
8. x^2y, xy^3, yz^3
9. ab^2, bxy
10. $2a$, $3b$, $4c$
11. x^3, $2y^3$, $3z^2$
12. $7a^2$, $2ab$, $3b^3$.
13. p^2qr, pq^2r, pqr^2
14. $4a^2bc$, $8a^3b^2$, $12bc^3$
15. $17a^3$, $85b^3$, $68a^2b^5$
16. $13c^3d^2$, $39cd^4$, $3c^5d$
17. $27m^2n^2p^2$, $81n^2x^2$, $6pm^3$
18. $32a^4b^3c$, $48abc^5$, $16a^4c^2$
19. $7a^2b$, $4ac^2$, $6ac^3$, $21bc$.
20. $8a^2b^2$, $24a^4b^2c^2$, $18abc^3$.

[Art. 232 and Examples 1–20 in the next Exercise may be taken immediately after Chap. XIV *in illustration of Easy Factors.]*

232. The lowest common multiple of compound expressions which are given as the product of factors, or which can be easily resolved into factors, can be found in a similar way.

Example 1. *Find the L.C.M. of*

$$6x^2(a-x)^2,\quad 8a^3(a-x)^3,\quad \textit{and}\quad 12ax(a-x)^5.$$

The least common multiple of 6, 8, and 12 is 24, and the lowest common multiple of the algebraic factors is $a^3x^2(a-x)^5$.

Therefore the L.C.M. is $24a^3x^2(a-x)^5$.

Example 2. *Find the L.C.M. of*

$$(x^2+2x)^2,\quad 2x^4+3x^3-2x^2,\quad \textit{and}\quad 2x^3-3x^2-14x.$$

Resolving the expressions into factors, we have

$$(x^2+2x)^2=\{x(x+2)\}^2 \qquad =x^2(x+2)^2,$$
$$2x^4+3x^3-2x^2=x^2(2x^2+3x-2)=x^2(x+2)(2x-1),$$
$$2x^3-3x^2-14x=x(2x^2-3x-14)=x(x+2)(2x-7).$$

Therefore the L.C.M. is $x^2(x+2)^2(2x-1)(2x-7)$.

EXAMPLES XX. b.

Find the lowest common multiple of

1. $a^3,\ a^2-2a$.
2. $y^2+y,\ y^3-y$.
3. $a^2b+ab^2,\ a^2+ab$.
4. $7c^2(c+1),\ 28c^3$.
5. $2p^2+p,\ 4p^2q$.
6. $p^2-4,\ p^2+2p$.
7. $x^2-4,\ x^3-8$.
8. $3cd,\ 6c^3+12c^2d$.
9. $(x-1)^2,\ x^3-1$.
10. $a^2+2a+1,\ a^2+3a+2$.
11. $x^2-21x+108,\ x^2-81$.
12. $x^2-4,\ x^2+2x$.
13. $x^3-x,\ (x-1)^2$.
14. $x^2+x-2,\ x^2-4x+3$.
15. $x^2-4x-21,\ x^2-9x+14$.
16. $a^2-ab-2b^2,\ a^2-5ab-6b^2,\ a^2-2ab-3b^2$.
17. $m^2-9m-22,\ m^2-8m-33,\ m^2+5m+6$.
18. $c^2-2cd-15d^2,\ c^2-18cd+65d^2,\ c^2-10cd-39d^2$.
19. $x^2-18x+45,\ x^2-26x+165,\ x^2-14x+33$.
20. $x^2-19x+78,\ x^2-21x+104,\ x^2-14x+48$.
21. $x^2-xy-2y^2,\ 2x^2-5xy+2y^2,\ 2x^2+xy-y^2$.
22. $3x^2-13x+14,\ x^2-4,\ 3x^2-x-14$.
23. $2x^2-5x-3,\ x^2+x-12,\ 2x^2+9x+4$.
24. $3m^2+5m+2,\ m^2-m-2,\ 3m^2-4m-4$.
25. $4x^2-10x-6,\ 3x^3-10x^2+3x,\ 12x^2+2x-2$.
26. $9a^2-36x^2,\ 4a^2-4ax+x^2,\ 2a^2+3ax-2x^2$.
27. $15a^3x(a+x)^3,\ 20ax^3(a-x)^3,\ 36a^2x^2(a^2-x^2)^2$.
28. $x^4+x^2y^2+y^4,\ x^3y+y^4,\ (x^2-xy)^3$.
29. $3a^3-18a^2x+27ax^2,\ 4a^4+24a^3x+36a^2x^2,\ 6a^4-54a^2x^2$.
30. $(2c^2-3cd)^2,\ (4c-6d)^3,\ 8c^3-27d^3$.

233 When the expressions are not easily separated into factors, we may proceed as follows

Let A and B be two expressions, and X their H.C.F.,

then $$A=mX, \quad \text{and} \quad B=nX,$$

where m and n are expressions which have no common factor

$$\text{L.C.M. of A and B}=mnX=\frac{mX \cdot nX}{X}=\frac{A\,B}{X}.$$

Hence the L.C.M. of two expressions may be found by dividing their product by their H.C.F.

Also we conclude that the product of two expressions is the same as the product of their H.C.F. and L.C.M.

234 We may also use the method of the following example

EXAMPLE. *Find the L.C.M. of*

$$x^3-2x^2-13x-10 \quad \text{and} \quad x^3-x^2-10x-8$$

The highest common factor, found by the method of Art. 212, is found to be x^2+3x+2

By division, we obtain

$$x^3-2x^2-13x-10=(x^2+3x+2)(x-5)=(x+1)(x+2)(x-5),$$
$$x^3-x^2-10x-8=(x^2+3x+2)(x-4)=(x+1)(x+2)(x-4).$$

Therefore the L.C.M. is $(x+1)(x+2)(x-4)(x-5)$

EXAMPLES XX. c.

Find the lowest common multiple of

1. $a^3-3a^2-10a-24, \quad a^3-2a^2-9a-18$
2. $x^3-3x^2-4x+12, \quad x^3-5x^2-8x+4$
3. $d^4+3d^3-d-3, \quad d^3+d^2-5d+3$
4. $x^3-3x^2y-18xy^2+40y^3, \quad x^3-4x^2y-11xy^2+30y^3$
5. $x^3+4x^2y-3xy^2-18y^3, \quad x^3-2x^2y-9xy^2+18y^3, \quad (xy+3y^2)^2.$
6. $c^3-6c^2+6c-5, \quad c^3+6c^2-6c+7, \quad c^2-2c-35$
7. $x^2-y^2, \quad x^3-y^3, \quad x^3-x^2y-xy^2-2y^3$
8. $12x^3-44x^2+51x-18, \quad 4x^3-4x^2-39x-36$
9. $12x^4-32x^3+15x^2-9x, \quad 12x^5-40x^4-39x^3-9x^2.$

Find the H.C.F. and the L.C.M. of

10. $x^3-5x^2-9x-9, \quad x^3+x^2-3x-9$
11. $x^4-3x^3-3x^2-3x+2, \quad x^3-x^2, \quad x^3+x^2$
12. $9a^4b^2-4a^2b^4, \quad (3a^2b+2ab^2)^2, \quad 3a^3b-10a^2b^2-8ab^3.$

CHAPTER XXI

Addition and Subtraction of Fractions

235. To find the algebraical sum of a number of fractions, we must first reduce them to a common denominator. For this purpose it is usually best to take the *lowest common denominator* (L.C.D.), which is the L.C.M. of the denominators of the given fractions.

The process is exactly the same as in Arithmetic.

Example 1. *Express the fractions* $\frac{2a}{3b}, \frac{3b}{5c}, \frac{c}{6a}$ *with their lowest common denominator.*

$$\frac{2a}{3b}=\frac{2a\times 10ac}{3b\times 10ac}=\frac{20a^2c}{30abc},$$

$$\frac{3b}{5c}=\frac{3b\times 6ab}{5c\times 6ab}=\frac{18ab^2}{30abc},$$

$$\frac{c}{6a}=\frac{c\times 5bc}{6a\times 5bc}=\frac{5bc^2}{30abc}.$$

The L.C.D. is $30abc$, and the successive multipliers are obtained by dividing the L.C.D. by $3b$, $5c$, $6a$ respectively.

Example 2. *Express with lowest common denominator*

$$\frac{5x}{2a(x-a)} \text{ and } \frac{4a}{3x(x^2-a^2)}$$

The lowest common denominator is $6ax(x-a)(x+a)$.

We must therefore multiply the numerators by $3x(x+a)$ and $2a$ respectively.

Hence the equivalent fractions are

$$\frac{15x^2(x+a)}{6ax(x-a)(x+a)} \text{ and } \frac{8a^2}{6ax(x-a)(x+a)}$$

EXAMPLES XXI. a

Express with lowest common denominator.

1. $\frac{a^2}{2}, \frac{2c^2}{5}$.
2. $\frac{ab}{9}, \frac{cd}{12}$.
3. $\frac{ab}{2}, \frac{bc}{a}$.
4. $\frac{a}{x^2}, \frac{3a}{2x}$.
5. $\frac{a}{x}, \frac{2}{y}, \frac{c^2}{z}$.
6. $\frac{x}{a}, \frac{a}{x}, \frac{1}{2x^2}$.
7. $\frac{a}{2bx}, \frac{b}{cx^2}, \frac{c}{3ax}$.
8. $\frac{x-3}{4}, \frac{x+4}{3}$.
9. $\frac{2a-b}{2a^2}, \frac{a-2b}{3ab}$.
10. $\frac{a}{x+a}, \frac{x}{x-a}$.
11. $\frac{c}{c-d}, \frac{d}{c-d}, \frac{1}{cd}$.
12. $\frac{2a}{3x(x-a)}, \frac{3x}{2a(x^2-a^2)}$.
13. $\frac{x^2}{a^2-ab}, \frac{xy}{a^2+ab}, \frac{y^2}{5(a^2-b^2)}$.
14. $\frac{y+2}{y^2-2y-3}, \frac{y+1}{y^2-y-6}$.
15. $\frac{4c^2}{2c^3+13c^2-7c}, \frac{3c^3}{c^3-49c}$.
16. $\frac{x^2-xy}{4(x^2-2xy+y^2)}, \frac{x^2+xy+y^2}{6(x^2-y^2)}$.

Addition and Subtraction of Fractions.

236 *To prove that* $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$.

By Art 221, we have $\frac{a}{b}=\frac{ad}{bd}$, and $\frac{c}{d}=\frac{bc}{bd}$,

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad}{bd}\pm\frac{bc}{bd}=\frac{ad\pm bc}{bd}$$

If b and d have any common factors, bd is not the L C D of the denominators To avoid working with fractions not in their simplest form, some modification of the above is necessary Hence the following rule

To add or subtract fractions: *reduce them to the lowest common denominator, find the algebraical sum of the new numerators, and retain the common denominator*

EXAMPLE 1 *Simplify* $\frac{7x}{8}+\frac{5x}{12}-\frac{2x}{3}$

The expression $=\frac{21x+10x-16x}{24}$

$=\frac{15x}{24}=\frac{5x}{8}$

The L C D is 24, and the successive multipliers for the numerators are 3, 2, and 8

NOTE Answers should always be given in lowest terms

When no denominator is expressed the denominator 1 may be understood And if a fraction is not in its lowest terms it should be simplified before combining it with other fractions

EXAMPLE 2 *Find the value of* $3x-\frac{a^2}{4y}+\frac{x^2y}{4xy^2}$

The expression $=\frac{3x}{1}-\frac{a^2}{4y}+\frac{x}{4y}=\frac{12xy-a^2+x}{4y}$

This result cannot be simplified, for no factor of the denominator will divide the *whole* of the numerator

EXAMPLE 3 *Find the value of* $\frac{x-2y}{xy}+\frac{3y-a}{ay}-\frac{3x-2a}{ax}$

The lowest common denominator is axy

Thus the expression $=\frac{a(x-2y)+x(3y-a)-y(3x-2a)}{axy}$

$=\frac{ax-2ay+3xy-ax-3xy+2ay}{axy}$

$=0,$

since the terms in the numerator destroy each other.

EXAMPLES XXI b

Find the value of

1 $\frac{a}{2}+\frac{a}{3}-\frac{a}{6}$

2 $\frac{b}{4}+\frac{5b}{12}-\frac{b}{3}$

3 $\frac{x}{5}-\frac{1}{4}+\frac{x}{10}$

4 $\frac{x}{ab}+\frac{y}{bc}+\frac{z}{ca}$

5 $a+\frac{b}{ca}-\frac{c}{ab}$

6 $\frac{x}{yz}-3+\frac{z}{xy}$

7 $\frac{2x^3}{ax}+\frac{3y}{yz}$

8 $3x-\frac{4x^3y}{8xy^2}$

9 $\frac{6a^2}{9a}-\frac{b^2}{a^2}$

10. $\frac{c^3}{3c^2}-\frac{a^2}{c}+2$

11. $\frac{3x^2}{6x}-\frac{y^2}{x^2}+\frac{1}{x}$

12 $\frac{3ab}{5x}-\frac{ab}{2x}-\frac{ab}{10x}$.

13 $\frac{2y+1}{5}-\frac{3y-2}{10}+\frac{y}{2}$

14 $\frac{b+4}{6}-\frac{b}{3}+\frac{b-3}{12}$

15 $-\frac{2x+3}{35}+\frac{x-1}{7}+\frac{x+3}{5}$

16 $\frac{3c-4}{6}+\frac{2c-5}{15}-\frac{c+7}{20}$

17 $\frac{3a-5}{a}-\frac{a-4}{a}+\frac{a^2-3a}{2a^2}$

18 $\frac{a+3x}{2a}+\frac{a-x}{6a}-\frac{2x+a}{3a}$

19 $\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}$

20 $1+\frac{a-b}{2b}+\frac{3ab-b^2}{b^2}$.

21. $\frac{2+a}{51a}-\frac{a-5}{34a}+\frac{a+2}{17a}$

22 $\frac{2x^2-3x}{x}-\frac{2x^2+x^3}{x^2}+\frac{x^3-x^4}{x^3}$

23 $\frac{x}{3z}+\frac{xy-2x^2}{2xy}-\frac{xy-yz}{2yz}$

24 $3-\frac{a}{2x}+\frac{ay+3x}{2xy}$

237 We shall now consider the addition and subtraction of fractions whose denominators are compound expressions

EXAMPLE 1 *Simplify* $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$

The lowest common denominator is $(x-2a)(x-a)$

Hence, multiplying the numerators by $x-a$ and $x-2a$ respectively, we have

$$\text{the expression}=\frac{(2x-3a)(x-a)-(2x-a)(x-2a)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-(2x^2-5ax+2a^2)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-2x^2+5ax-2a^2}{(x-2a)(x-a)}$$

$$=\frac{a^2}{(x-2a)(x-a)}$$

NOTE In finding the value of an expression like $-(2x-a)(x-2a)$, the beginner should first express the product in brackets, and then remove the brackets, as in the above example After a little practice he will be able to take both steps together

EXAMPLE 2 *Find the value of* $\dfrac{3x+2}{x^2-16}+\dfrac{x-5}{(x+4)^2}$

The lowest common denominator is $(x-4)(x+4)^2$

Hence the expression $=\dfrac{(3x+2)(x+4)+(x-5)(x-4)}{(x-4)(x+4)^2}$

$$=\frac{3x^2+14x+8+x^2-9x+20}{(x-4)(x+4)^2}$$

$$=\frac{4x^2+5x+28}{(x-4)(x+4)^2}$$

EXAMPLE 3 *Simplify* $\dfrac{3y+6}{y^2-y-6}-\dfrac{12}{y^2-2y-3}$

The expression $=\dfrac{3(y+2)}{(y+2)(y-3)}-\dfrac{12}{(y-3)(y+1)}$

$$=\frac{3}{y-3}-\frac{12}{(y-3)(y+1)}=\frac{3(y+1)-12}{(y-3)(y+1)}$$

$$=\frac{3y-9}{(y-3)(y+1)}=\frac{3(y-3)}{(y-3)(y+1)}=\frac{3}{y+1}$$

EXAMPLES XXI c

Find the value of

1. $\dfrac{1}{a+b}+\dfrac{1}{a-b}$

2. $\dfrac{3}{x-2}-\dfrac{2}{x+2}$

3. $\dfrac{x}{a-x}-\dfrac{y}{a-y}$

4. $\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}$

5. $\dfrac{y+2}{y-2}-\dfrac{y-2}{y+2}$

6. $\dfrac{1}{x+3}+\dfrac{3}{x^2-9}$

7. $\dfrac{b}{a+b}+\dfrac{b^2}{a^2-b^2}$

8. $\dfrac{5a}{a^2-16}-\dfrac{1}{a+4}$

9. $\dfrac{c^2}{c^2-9d^2}-\dfrac{c-3d}{c+3d}$

10. $\dfrac{3}{x^2-4}+\dfrac{1}{(x-2)^2}$

11. $\dfrac{x}{(x-y)^2}-\dfrac{y}{x^2-y^2}$

12. $\dfrac{3b}{(b+1)^2}-\dfrac{2}{b+1}$

13. $\dfrac{x+2y}{x-2y}-\dfrac{x(x+4y)}{x^2-4y^2}$

14. $\dfrac{y^2}{y-y^3}-\dfrac{y}{1+y^3}$

15. $\dfrac{2a^2}{a^2-b^2}-\dfrac{2a^2}{a^2+ab}$

16. $\dfrac{pq}{25p^2-q^2}+\dfrac{2p^2q}{10p^2q+2pq^2}$

17. $\dfrac{c^2-4d^2}{c^2-2cd}-\dfrac{c^2+2cd-8d^2}{c^2-4d^2}$

18. $\dfrac{3}{x^2-4}+\dfrac{1}{(x-2)^2}$

19. $\dfrac{1}{a(x^2-a^2)}-\dfrac{1}{x(x+a)^2}$

20. $\dfrac{1+x+x^2}{1-x^3}+\dfrac{x-x^2}{(1-x)^3}$

21. $\dfrac{2x-7}{(x-3)^2}-\dfrac{2(x+2)}{x^2-9}$

22. $\dfrac{4(m-1)}{m^2+3m+2}+\dfrac{4(m-3)}{m^2-m-6}$

23. $\dfrac{y^2+2y}{y^2+y-2}-\dfrac{y}{y+1}$

238 The following examples furnish additional practice in the simplification of fractions

EXAMPLE 1 *Simplify* $\dfrac{a^2-2a}{a^2-a-2}-\dfrac{3a}{6a-4}+\dfrac{5a}{6a^2+2a-4}$

$$\text{The expression}=\frac{a(a-2)}{(a+1)(a-2)}-\frac{3a}{2(3a-2)}+\frac{5a}{2(3a^2+a-2)}$$

$$=\frac{a}{a+1}-\frac{3a}{2(3a-2)}+\frac{5a}{2(3a-2)(a+1)}$$

$$=\frac{2a(3a-2)-3a(a+1)+5a}{2(3a-2)(a+1)}$$

$$=\frac{3a^2-2a}{2(3a-2)(a+1)}=\frac{a(3a-2)}{2(3a-2)(a+1)}=\frac{a}{2(a+1)}$$

nes the work will be simplified by first combining two of the fractions, instead of finding the lowest common multiple of all the denominators at once

EXAMPLE 2 *Simplify* $\dfrac{3}{8(a-x)}-\dfrac{1}{8(a+x)}-\dfrac{a-2x}{4(a^2+x^2)}$

Taking the first two fractions together,

$$\text{the expression}=\frac{3(a+x)-(a-x)}{8(a^2-x^2)}-\frac{a-2x}{4(a^2+x^2)}$$

$$=\frac{a+2x}{4(a^2-x^2)}-\frac{a-2x}{4(a^2+x^2)}$$

$$=\frac{(a+2x)(a^2+x^2)-(a-2x)(a^2-x^2)}{4(a^4-x^4)}$$

$$=\frac{a^3+2a^2x+ax^2+2x^3-(a^3-2a^2x-ax^2+2x^3)}{4(a^4-x^4)}$$

$$=\frac{4a^2x+2ax^2}{4(a^4-x^4)}=\frac{ax(2a+x)}{2(a^4-x^4)}$$

EXAMPLES XXI c (*Continued*)

Find the value of

24. $\dfrac{6}{x^2-2x-8}+\dfrac{1}{x^2+5x+6}$

25. $\dfrac{7}{y^2+y-12}-\dfrac{6}{y^2+2y-8}$.

26. $\dfrac{7}{x^2+13x+30}+\dfrac{1}{x^2+5x+6}$

27. $\dfrac{8}{y^2+10y+9}+\dfrac{5}{y^2-3y-4}$

28. $\dfrac{x-3}{x^2-3x-4}-\dfrac{x-1}{x^2-x-2}$

29. $\dfrac{a^2-2ab}{a^2+ab-6b^2}-\dfrac{ab-7b^2}{a^2-ab-42b^2}$.

30. $\dfrac{p+2}{2}-\dfrac{p}{p+2}-\dfrac{p^3-2p^2}{2p^2-8}$

31. $\dfrac{a+x}{2(a-x)}+\dfrac{a-x}{2(a+x)}-\dfrac{2ax}{a^2-x^2}$.

32. $\dfrac{3}{2+2x}-\dfrac{4}{3-3x}+\dfrac{5}{4-4x^2}$

33. $\dfrac{5}{2-2x}-\dfrac{4}{3+3x}+\dfrac{3}{4-4x^2}$.

Find the value of

34. $\dfrac{x^2}{ab}+\dfrac{(x-a)^2}{a(a-b)}-\dfrac{(x-b)^2}{b(a-b)}$

35. $\dfrac{a^2}{xy}+\dfrac{(a+x)^2}{x(x-y)}-\dfrac{(a+y)^2}{y(x-y)}$

36. $\dfrac{1}{3a-1}+\dfrac{2}{a-1}+\dfrac{1}{a}$

37. $\dfrac{3}{x}-\dfrac{3}{x-y}+\dfrac{1}{4x-2y}$

38. $\dfrac{3x^2-8}{x^3-1}-\dfrac{5x+7}{x^2+x+1}+\dfrac{2}{x-1}$

39. $\dfrac{3m^2-5}{m^3-1}-\dfrac{4m+5}{m^2+m+1}+\dfrac{1}{m-1}$.

40. $\dfrac{3x-5}{3x^2-2x-5}-\dfrac{3x+5}{3x^2+2x-5}+\dfrac{2x^2}{x^2-1}$

41. $\dfrac{1}{2x^2+3x-2}-\dfrac{1}{3x^2+7x+2}-\dfrac{1}{6x^2-x-1}$

42. $\dfrac{3}{x^2+x-2}-\dfrac{5}{2x^2+3x-2}-\dfrac{1}{2x^2-3x+1}$

43. $\dfrac{1}{2(1+y)}+\dfrac{1}{2(1-y)}+\dfrac{1}{1+y^2}$

44. $\dfrac{1}{2-x}-\dfrac{1}{2+x}-\dfrac{2x}{4+x^2}$

45. $\dfrac{1}{3a+2}+\dfrac{1}{3a-2}-\dfrac{6a}{9a^2+4}$

46. $\dfrac{4a}{(a-1)^2}+\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}$

47. $\dfrac{2}{3(3m-5)}+\dfrac{2}{3(3m+5)}-\dfrac{4m}{9m^2+25}$

48. $\dfrac{2}{12+3a^2}+\dfrac{3}{4+2a}+\dfrac{3}{4-2a}$.

49. $\dfrac{5y}{2(y+1)(y-3)}-\dfrac{15(y-1)}{16(y-3)(y-2)}-\dfrac{9(y+3)}{16(y+1)(y-2)}$

50. $\dfrac{c+3d}{4(c+d)(c+2d)}+\dfrac{c+2d}{(c+d)(c+3d)}-\dfrac{c+d}{4(c+2d)(c+3d)}$

51. $\dfrac{5(2b-3)}{11(6b^2+b-1)}+\dfrac{7b}{6b^2+7b-3}-\dfrac{12(3b+1)}{11(4b^2+8b+3)}$

52. $\dfrac{3}{8(a+b)}-\dfrac{1}{8(a-b)}+\dfrac{a+2b}{4(a^2+b^2)}$

Changes of Sign in Addition of Fractions.

239 An algebraical fraction has been defined as the quotient obtained by *dividing the numerator by the denominator*

Thus $\dfrac{-a}{-b}$ denotes the division of $-a$ by $-b$, and the result is obtained by dividing a by b and prefixing the proper sign, in this case $+$

Therefore $$\frac{-a}{-b}=+\frac{a}{b} \qquad (1).$$

Similarly, $$\frac{-a}{b}=-\frac{a}{b} \qquad (2),$$

and $$\frac{a}{-b}=-\frac{a}{b} \qquad (3).$$

By comparing the right side with the left side of these identities respectively, we see that

(1) *If the signs of* **both** *numerator and denominator of a fraction be changed, the sign of the whole fraction will be* **unchanged**

(2) *If the sign of the* **numerator** **alone** *be changed, the sign of the whole fraction will be* **changed**

(3) *If the sign of the* **denominator** **alone** *be changed, the sign of the whole fraction will be* **changed**

240 When the numerator or denominator is a compound expression, the alteration of sign applies to each term of the expression

EXAMPLE 1 $\dfrac{b-a}{y-x}=\dfrac{-(b-a)}{-(y-x)}=\dfrac{-b+a}{-y+x}=\dfrac{a-b}{x-y}$

EXAMPLE 2 $\dfrac{x-x^2}{2y}=-\dfrac{-(x-x^2)}{2y}=-\dfrac{-x+x^2}{2y}=-\dfrac{x^2-x}{2y}$

EXAMPLE 3 *Reduce to its lowest terms* $\dfrac{x^2y-y^3}{y^2-xy}$

$$\frac{x^2y-y^3}{y^2-xy}=\frac{y(x^2-y^2)}{y(y-x)}=\frac{x^2-y^2}{y-x}=-\frac{x^2-y^2}{x-y}=-(x+y)$$

EXAMPLE 4 *Simplify* $\dfrac{a}{x+a}+\dfrac{2x}{x-a}+\dfrac{a(3x-a)}{a^2-x^2}$

Here it is evident that the lowest common denominator of the first two fractions is x^2-a^2, therefore it will be convenient to alter the sign of the denominator in the third fraction

$$\text{Thus the expression}=\frac{a}{x+a}+\frac{2x}{x-a}-\frac{a(3x-a)}{x^2-a^2}$$

$$=\frac{a(x-a)+2x(x+a)-a(3x-a)}{x^2-a^2}$$

$$=\frac{ax-a^2+2x^2+2ax-3ax+a^2}{x^2-a^2}$$

$$=\frac{2x^2}{x^2-a^2}$$

EXAMPLES XXI. d

Find the value of

1. $\dfrac{1}{1+a}+\dfrac{1}{a-1}+\dfrac{3a}{1-a^2}$

2. $\dfrac{4}{1+x}-\dfrac{3}{1-x}-\dfrac{7x}{x^2-1}$

3. $\dfrac{2y}{4-y^2}-\dfrac{1}{y-2}-\dfrac{1}{2+y}$

4. $\dfrac{5}{3+z}-\dfrac{2}{3-z}+\dfrac{6(1-z)}{z^2-9}$

5. $\dfrac{x}{a(a-b)}+\dfrac{3}{ab}+\dfrac{1}{b(b-a)}$

6. $\dfrac{1}{p(p-2)}+\dfrac{3}{2p}+\dfrac{1}{2(2-p)}$

Find the value of

7. $\frac{2-x}{1-2x}-\frac{2+x}{1+2x}-\frac{1-6x}{4x^2-1}$.

8. $\frac{3-y}{1-3y}-\frac{3+y}{1+3y}-\frac{1-16y}{9y^2-1}$.

9. $\frac{1}{2(x-y)}-\frac{1}{2(x+y)}+\frac{y}{y^2-x^2}$.

10. $\frac{5}{1+2m}+\frac{3m}{2m-1}-\frac{4-13m}{1-4m^2}$.

11. $\frac{1}{2(1+a)}-\frac{1}{2(a-1)}+\frac{1}{1+a^2}$.

12. $\frac{5}{18x+54}-\frac{1}{54-18x}-\frac{x-2}{3x^2+27}$.

13. $\frac{2x-1}{x^2+x}+\frac{2x+1}{x^2-x}+\frac{4x+2}{x-x^3}$.

14. $\frac{2a+1}{a^2-a}+\frac{2a-1}{a^2+a}+\frac{2-4a}{a-a^3}$.

15. $\frac{1-2a}{1+2a}+\frac{1-2a}{1-2a}-\frac{1-20a^2}{4a^2-1}$.

16. $\frac{3-2m}{3+2m}+\frac{2m+3}{2m-3}+\frac{12}{4m^2-9}$.

17. $\frac{3}{x+1}-\frac{1}{x+3}+\frac{3}{1-x}-\frac{1}{3-x}$.

18. $\frac{4}{y+1}-\frac{1}{4-y}+\frac{4}{1-y}-\frac{1}{y+4}$.

19. $\frac{a-1}{a-2}-\frac{a+1}{a+2}-\frac{4}{4-a^2}+\frac{2}{2-a}$.

20. $\frac{b-1}{b+2}-\frac{b+1}{b-2}-\frac{12}{4-b^2}+\frac{6}{2+b}$.

21. $\frac{x^2+1}{x^2-1}-\frac{x}{1-x^2}-\frac{1}{x-1}$.

22. $\frac{1}{y+1}+\frac{y}{1-y^2}-\frac{1+y^2}{y^2+1}$.

23. $\frac{2a-5}{a^2-5a+6}+\frac{2}{2a-a^2}+\frac{3}{3a-a^2}$.

24. $\frac{8-3a^2}{1-a^3}-\frac{5a+7}{a^2+a+1}+\frac{2}{a-1}$.

25. $\frac{4b+5}{1+b+b^2}-\frac{1}{b-1}+\frac{3b^2-5}{1-b^3}$.

26. $\frac{a}{(a-x)^2}+\frac{3a}{x^2+ax-2a^2}+\frac{1}{2a+x}$.

241 If the sign of *only one* of the factors in a product is changed *the sign of the product as a whole is altered*.

Thus $(a-b)(d-c)=(a-b)\times\{-(c-d)\}=-(a-b)(c-d)$.

In the following example the application of this principle alters the sign of the denominator of each fraction, and therefore the sign of each fraction must also be changed.

EXAMPLE. *Simplify* $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

Here in finding the L.C.M. of the denominators it must be observed that there are not *six* different compound factors to be considered; for three of them differ from the other three respectively only in sign.

Thus
$$(a-c)=-(c-a),$$
$$(b-a)=-(a-b),$$
$$(c-b)=-(b-c).$$

Hence, replacing the second factor in each denominator by its equivalent, we may write the expression in the form

$$-\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)} \qquad (1)$$

Now the L C M is $(b-c)(c-a)(a-b)$,

$$\text{and the expression} = \frac{-(b-c)-(c-a)-(a-b)}{(b-c)(c-a)(a-b)}$$

$$= \frac{-b+c-c+a-a+b}{(b-c)(c-a)(a-b)}$$

$$= 0$$

NOTE In the expression (1) the letters in the several factors occur in what is known as **cyclic order**, that is, b follows a, c follows b, a follows c, just as if the letters were arranged round a circle and the letters taken in order round the circumference

242 If the sign of each of *two* factors in a product is changed the sign of the product is unaltered, thus

$$(a-x)(b-x) = \{-(x-a)\}\{-(x-b)\} = (x-a)(x-b)$$

Similarly $$(a-x)^2 = (x-a)^2$$

In other words, in the simplification of fractions we may change the sign of each of *two* factors in a denominator without altering the sign of the fraction, thus

$$\frac{1}{(b-a)(c-b)} = \frac{1}{(a-b)(b-c)}$$

EXAMPLE *Simplify* $\frac{1}{a-x} - \frac{1}{a+x} - \frac{2x}{a^2+x^2} + \frac{4x^3}{(x^2-a^2)^2}$

Here it should be evident that the first two denominators give L C M a^2-x^2, which readily combines with a^2+x^2 to give L C M a^4-x^4 Hence it will be convenient to proceed as follows

$$\text{The expression} = \frac{a+x-(a-x)}{a^2-x^2} - \frac{2x}{a^2+x^2} + \frac{4x^3}{(x^2-a^2)^2}$$

$$= \frac{2x}{a^2-x^2} - \frac{2x}{a^2+x^2} + \frac{4x^3}{(x^2-a^2)^2}$$

$$= \frac{4x^3}{a^4-x^4} + \frac{4x^3}{(a^2-x^2)^2}$$

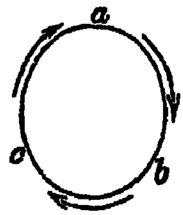
$$= \frac{4x^3(a^2-x^2)+4x^3(a^2+x^2)}{(a^2+x^2)(a^2-x^2)^2}$$

$$= \frac{8a^2x^3}{(a^2+x^2)(a^2-x^2)^2}$$

NOTE It should be observed that, in adding the last two fractions, $+\frac{4x^3}{(x^2-a^2)^2}$ is written $+\frac{4x^3}{(a^2-x^2)^2}$, because *two* factors, $(x^2-a^2)(x^2-a^2)$, of the denominator are thereby changed in sign, therefore the sign of the fraction *as a whole* is not changed

243 The observance of the principle of cyclic order is especially important in a large class of examples in which the differences of three letters are involved

Thus we are observing cyclic order when we write $b-c$, $c-a$, $a-b$, moving round in order in the direction of the arrows, or $c-b$, $b-a$, $a-c$, moving in the opposite direction We are violating cyclic order by the use of arrangements such as $b-c$, $a-c$, $a-b$, or $a-c$, $b-a$, $b-c$ It will always be found that the work is rendered shorter and easier by following cyclic order from the beginning, and adhering to it throughout the simplification



EXAMPLES XXI. e

Find the value of

1. $\dfrac{1}{(a+5)(a-2)}+\dfrac{1}{(1-a)(2-a)}$

2. $\dfrac{1}{x-2}+\dfrac{2}{(2-x)^2}-\dfrac{x}{x^2+4}$

3. $\dfrac{1}{1-x}+\dfrac{x}{(x-1)^2}$

4. $\dfrac{1}{2y^2-y-3}+\dfrac{1}{(1-2y)(1+y)}$

5. $\dfrac{1-c}{c-2}+\dfrac{c-3}{c-4}-\dfrac{1}{(2-c)^2}$

6. $\dfrac{2x-1}{(4-x)^2}-\dfrac{2(x+7)}{x^2-16}$

7. $\dfrac{1}{(x-3)(x-4)}+\dfrac{1}{(4-x)(5-x)}+\dfrac{2}{(x-5)(3-x)}$

8. $\dfrac{a-2}{(a-3)(a-4)}+\dfrac{2(a-3)}{(a-2)(4-a)}+\dfrac{a-4}{(2-a)(3-a)}$

9. $\dfrac{2}{(2+x)^2}+\dfrac{2}{(x-2)^2}-\dfrac{1}{x+2}-\dfrac{1}{2-x}$

10. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

11. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}+\dfrac{a-b}{(c-a)(c-b)}$

12. $\dfrac{1+x}{(x-y)(x-z)}+\dfrac{1+y}{(y-z)(y-x)}+\dfrac{1+z}{(z-x)(z-y)}$

13. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}$

14. $\dfrac{m^2nr}{(m-n)(m-r)}+\dfrac{n^2rm}{(n-r)(n-m)}+\dfrac{r^2mn}{(r-m)(r-n)}$.

15. $\dfrac{a+b-c}{(a-b)(a-c)}+\dfrac{b+c-a}{(b-c)(b-a)}+\dfrac{c+a-b}{(c-a)(c-b)}$

16. $\dfrac{q+r}{(x-y)(x-z)}+\dfrac{r+p}{(y-z)(y-x)}+\dfrac{p+q}{(z-x)(z-y)}$.

CHAPTER XXII

MISCELLANEOUS FRACTIONS.

244 DEFINITION A fraction which contains a fractional expression in its numerator or denominator, or in both of them, is called a **Complex Fraction**

Thus $\dfrac{a}{\frac{b}{c}}$, $\dfrac{\frac{a}{b}}{x}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, $\dfrac{y}{1-\frac{1}{x}}$ are complex fractions

The above fractions may also be written as follows

$$a\Big/\frac{b}{c},\quad \frac{a}{b}\Big/x,\quad \frac{a}{b}\Big/\frac{c}{d},\quad y\Big/\left(1-\frac{1}{x}\right)$$

245 An algebraical fraction has been defined as the result obtained by *dividing the numerator by the denominator*

Thus $$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

246 The following special cases should be noted, so that the results may be written down at sight

$$\frac{1}{\frac{a}{b}}=1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a}\qquad \frac{a}{\frac{1}{b}}=a\div\frac{1}{b}=a\times\frac{b}{1}=ab$$

Similarly, $\dfrac{\frac{1}{a-x}}{\frac{1}{a+x}}=\dfrac{a+x}{a-x}$, at once, as in Art 245

247 **Simplification of Complex Fractions.**

EXAMPLE 1 *Simplify* $\dfrac{x+\frac{a^2}{x}}{x-\frac{a^4}{x^3}}$

The expression $=\left(x+\dfrac{a^2}{x}\right)\div\left(x-\dfrac{a^4}{x^3}\right)=\dfrac{x^2+a^2}{x}\div\dfrac{x^4-a^4}{x^3}$

$$=\frac{x^2+a^2}{x}\times\frac{x^3}{x^4-a^4}=\frac{x^2}{x^2-a^2}$$

EXAMPLE 2. *Simplify* $\dfrac{\frac{4}{x}+\frac{x}{2}-3}{\frac{x}{6}-\frac{1}{3}-\frac{4}{3x}}$.

The value of the fraction will not be altered if we multiply numerator and denominator by the same quantity. Hence multiplying above and below by $6x$, which is the L.C.M. of the denominators, we have

$$\text{the expression} = \frac{24+3x^2-18x}{x^2-2x-8} = \frac{3(x^2-6x+8)}{x^2-2x-8}$$

$$= \frac{3(x-4)(x-2)}{(x-4)(x+2)} = \frac{3(x-2)}{x+2}.$$

EXAMPLE 3. *Simplify* $\dfrac{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}$.

The numerator $= \dfrac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)(a^2-b^2)} = \dfrac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}$.

Similarly the denominator $= \dfrac{4ab}{(a+b)(a-b)}$.

Hence the fraction $= \dfrac{4a^2b^2}{(a^2+b^2)(a^2-b^2)} \div \dfrac{4ab}{(a+b)(a-b)}$

$$= \frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)} \times \frac{(a+b)(a-b)}{4ab} = \frac{ab}{a^2+b^2}.$$

EXAMPLES XXII. a.

Find the value of

1. $\dfrac{1}{a+\frac{b}{c}}$.　2. $\dfrac{1}{\frac{x}{y}-z}$.　3. $\dfrac{x+1}{1-\frac{1}{x^2}}$.　4. $\dfrac{c}{\frac{1}{1-d}}$.

5. $\dfrac{\frac{1}{b}+\frac{1}{a}}{\frac{a}{b}-\frac{b}{a}}$.　6. $\dfrac{1-\frac{x}{a}}{\frac{x^2}{a}-a}$.　7. $\dfrac{\frac{3}{a}-\frac{a}{3}}{\frac{1}{a}-\frac{1}{3}}$.　8. $\dfrac{\frac{2x}{y}-\frac{5}{3}}{\frac{5y}{2x}+3}$.

9. $\dfrac{4-\frac{8b}{3a}}{\frac{3a}{b}-2}$.　10. $\dfrac{\frac{a^2c}{b}-\frac{b}{c}}{a+\frac{b}{c}}$.　11. $\dfrac{\frac{x}{y}+\frac{a}{b}}{\frac{x}{b}+\frac{a}{y}}$.　12. $\dfrac{\frac{x}{a}-\frac{y}{b}}{\frac{b}{y}-\frac{a}{x}}$.

13 $\dfrac{\frac{1}{a}-\frac{2}{a^3}+1}{1+\frac{4}{a}-\frac{5}{a^2}}$

14 $\dfrac{\frac{6}{x}-5+x}{\frac{1}{2}+\frac{1}{2x}-\frac{6}{x^2}}$

15 $\dfrac{\frac{1}{a}+\frac{2}{a^2}-\frac{15}{a^3}}{a-\frac{25}{a}}$

16 $\dfrac{x-3-\frac{30}{x-2}}{x-1-\frac{20}{x-2}}$

17 $\dfrac{\frac{m}{1+m}+\frac{1-m}{m}}{\frac{m}{1+m}-\frac{1-m}{m}}$

18. $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}$

248 The fraction in the following example is called a **Continued Fraction**, and in cases of this kind we begin the work from the lowest fraction, and simplify step by step

EXAMPLE *Find the value of* $\dfrac{1}{4-\dfrac{3}{2+\dfrac{x}{1-x}}}$

$$\text{The expression}=\frac{1}{4-\dfrac{3}{\dfrac{2-2x+x}{1-x}}}=\frac{1}{4-\dfrac{3(1-x)}{2-x}}$$

$$=\frac{1}{\dfrac{8-4x-3+3x}{2-x}}=\frac{1}{\dfrac{5-x}{2-x}}=\frac{2-x}{5-x}$$

EXAMPLES XXII. b.

Find the value of

1 $1-\dfrac{1}{1+\frac{1}{a}}$

2. $2-\dfrac{1}{1-\frac{1}{x}}$

3 $a+\dfrac{a}{1-\frac{1}{a}}$

4 $1-\dfrac{a-b}{a+b+\frac{b^2}{a-b}}$

5. $a-\dfrac{a-2b}{2-\frac{a+b}{a-b}}$

6 $x-\dfrac{y}{1+\dfrac{1}{1+\frac{y}{x}}}$

7. $a-\dfrac{1}{c+\dfrac{1}{c+\frac{1}{a}}}$

8 $1-\dfrac{x+y}{x+y-\dfrac{x+\frac{y^2}{x}}{1-\frac{y}{x}}}$

9 $\dfrac{1}{x+\dfrac{1}{1+\frac{x+1}{3-x}}}$

10 $\dfrac{\frac{2}{b+c}-\frac{1}{b}}{c+\frac{bc}{c-2b}}+\dfrac{\frac{2}{b+c}-\frac{1}{c}}{b+\frac{bc}{b-2c}}$

Find the value of

11. $\dfrac{\dfrac{2yz}{y+z}-y}{\dfrac{1}{z}+\dfrac{1}{y-2z}}+\dfrac{\dfrac{2yz}{y+z}-z}{\dfrac{1}{y}+\dfrac{1}{z-2y}}$

12. $2-\dfrac{3}{1+\dfrac{5a}{2+\dfrac{4a^2-1}{a+1}}}$

13. $2-\dfrac{2}{1-\dfrac{3}{2-\dfrac{3}{1-x}}}$

14. $\dfrac{y^3+x^3}{y-\dfrac{x}{1+\dfrac{x}{y-x}}}-\dfrac{y^3-x^3}{y+\dfrac{x}{1-\dfrac{x}{y+x}}}$

249 It is sometimes convenient to express a fraction in an equivalent form, partly integral and partly fractional

EXAMPLE 1 $\dfrac{a+5}{a-2}=\dfrac{(a-2)+7}{a-2}=1+\dfrac{7}{a-2}$

EXAMPLE 2 $\dfrac{3x-2}{x+5}=\dfrac{3(x+5)-15-2}{x+5}=\dfrac{3(x+5)-17}{x+5}=3-\dfrac{17}{x+5}$

EXAMPLE 3 *Shew that* $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$

$$\begin{array}{r|l}
x-3 & 2x^2-7x-1\ (2x-1 \\
 & \underline{2x^2-6x\quad\ } \\
 & -x-1 \\
 & \underline{-x+3} \\
 & -4
\end{array}$$

Here by actual division we obtain $2x-1$ as quotient, and -4 as remainder

Thus $$\frac{2x^2-7x-1}{x-3}=2x-1-\frac{4}{x-3}$$

250 If the numerator be of lower dimensions than the denominator, we may still perform the division, and express the result in a form which is partly integral and partly fractional

EXAMPLE *Prove that* $\dfrac{2x}{1+3x^2}=2x-6x^3+18x^5-\dfrac{54x^7}{1+3x^2}$

By division

$$\begin{array}{r|l}
1+3x^2 & 2x\qquad\ (2x-6x^3+18x^5 \\
 & \underline{2x+6x^3} \\
 & -6x^3 \\
 & \underline{-6x^3-18x^5} \\
 & 18x^5 \\
 & \underline{18x^5+54x^7} \\
 & -54x^7
\end{array}$$

whence the required result follows

Here the division may be carried on to any number of terms in the quotient, and we can stop at any term we please by annexing to the quotient the fraction whose numerator is the remainder last found, and whose denominator is the divisor.

Thus, if we carried on the quotient to four terms, we should have

$$\frac{2x}{1+3x^2}=2x-6x^3+18x^5-54x^7+\frac{162x^9}{1+3x^2}.$$

The terms in the quotient may be fractional; thus if x^2 is divided by x^3-a^3, it will be found that the first four terms of the quotient are $\frac{1}{x}+\frac{a^3}{x^4}+\frac{a^6}{x^7}+\frac{a^9}{x^{10}}$, and that the remainder is $\frac{a^{12}}{x^{10}}$.

251. Further illustrative examples in fractions.

EXAMPLE 1. *Divide* $a^2+9b^2+\frac{65b^4}{a^2-9b^2}$ *by* $a+3b+\frac{13b^2}{a-3b}$.

$$\text{The quotient}=\frac{(a^2+9b^2)(a^2-9b^2)+65b^4}{a^2-9b^2}\div\frac{(a+3b)(a-3b)+13b^2}{a-3b}$$

$$=\frac{a^4-81b^4+65b^4}{a^2-9b^2}\times\frac{a-3b}{a^2-9b^2+13b^2}$$

$$=\frac{a^4-16b^4}{a^2-9b^2}\times\frac{a-3b}{a^2+4b^2}$$

$$=\frac{a^2-4b^2}{a+3b}.$$

EXAMPLE 2. *Simplify* $\frac{1}{4(x-2)}-\frac{1}{4(x+2)}-\frac{1}{x^2+4}+\frac{8}{x^4+16}$.

Here we see that the L.C.M. of the first two denominators contains the factor x^2-4, which readily combines with x^2+4 to give x^4-16, which again combines with x^4+16 to give x^8-256. Hence the following compact arrangement will be found convenient.

$$\text{The expression}=\frac{x+2-(x-2)}{4(x^2-4)}- \qquad +$$

$$=\frac{1}{x^2-4}-\frac{1}{x^2+4}+$$

$$=\frac{x^2+4-(x^2-4)}{x^4-16}+$$

$$=\frac{8}{x^4-16}+\frac{8}{x^4+16}$$

$$=\frac{16x^4}{x^8-256}.$$

EXAMPLES XXII. c

Express the following fractions in a form partly integral and partly fractional, as in Art 249

1. $\dfrac{x+8}{x+2}$
2. $\dfrac{x+8}{x-2}$
3. $\dfrac{a-7}{a+3}$
4. $\dfrac{2x+5}{x+1}$
5. $\dfrac{6x-7}{x-3}$
6. $\dfrac{3x+9}{x+2}$
7. $\dfrac{4x+12}{2x-1}$
8. $\dfrac{9x+6}{3x-2}$

Prove the following identities

9. $\dfrac{x+6}{x+4}+\dfrac{x-2}{x-4}\equiv 2+\dfrac{4x}{x^2-16}$
10. $\dfrac{5x+31}{x+6}-\dfrac{2x-9}{x-5}\equiv 3-\dfrac{11}{x^2+x-30}$
11. $\dfrac{a^3-b^3}{(a-b)^2}\equiv a+2b+\dfrac{3b^2}{a-b}$
12. $\dfrac{x^3-y^3}{x+y}\equiv x^2-xy+y^2-\dfrac{2y^3}{x+y}$
13. Shew that $\dfrac{8x^4-6x^3+x-15}{2x^3+x-6}\equiv 4x-5+\dfrac{15}{x+2}$

Perform the following divisions, giving four terms in the quotient, and the remainder at that stage

14. $x\div(1+x)$
15. $(1+x)\div(1-x)$
16. $1\div(1-x+x^2)$
17. $1\div(1-x)^2$
18. $a\div(a-b)$
19. $x^2\div(x+3)$.
20. Divide $x+\dfrac{16x-27}{x^2-16}$ by $x-1+\dfrac{13}{x+4}$
21. Multiply $x+2a-\dfrac{a^2}{2x+3a}$ by $2x-a-\dfrac{2a^2}{x+a}$
22. Multiply $4x^2+14x+\dfrac{98x-27}{2x-7}$ by $\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}$

252 The following Exercise will furnish practice in all the processes connected with fractions

EXAMPLES XXII. d

Simplify the following fractions

1. $\dfrac{2}{4x^2-1}+\dfrac{1}{(2x+1)^2}$
2. $\dfrac{x^2+5x+4}{x^3+4x^2+5x+2}$
3. $\dfrac{x^4-y^4}{a^2b+ab^2}\times\dfrac{a+b}{(x+y)^2}-\dfrac{(x-y)^2}{ab}$
4. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}$
5. $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}-\dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$
6. $\dfrac{(x+1)^3-(x-1)^3}{3x^3+x}$

7. $\dfrac{x^6+x^5-x^4-x}{x^6-1}$

8. $\dfrac{2}{x^2+x}+\dfrac{2x-1}{x^2-x+1}-\dfrac{2x^3-1}{x^4+x}$.

9. $\dfrac{a^4-b^4}{a^2+ab+b^2}\times\dfrac{a^3-b^3}{(a-b)(a-2b)}\times\dfrac{a^2-4b^2}{(a+b)(a+2b)}\times\dfrac{1}{(a-b)(a^2+b^2)}$

10. $\dfrac{1}{x-3}-\dfrac{1}{x+4}+\dfrac{1+\dfrac{x}{x+1}}{x-\dfrac{12}{x+1}}$

11. $\dfrac{2x^3-17x^2+29x-12}{4x^3-36x^2+27x+27}$

12. $\left(1-\dfrac{2a}{\dfrac{z^2}{a}+a}\right)\left(1-\dfrac{3a}{\dfrac{z^2}{a}-a}\right)-\left(1-\dfrac{a}{z-a}\right)\left(1+\dfrac{a}{z+a}\right)$

13. $\dfrac{3}{2(x-1)}-\dfrac{1}{2(x+1)}+\dfrac{x-2}{x^2+1}-\dfrac{2(x^3+2)}{x^4-1}$

14. $\dfrac{1-x}{1-x+x^2}-\dfrac{\dfrac{1}{x}\left(\dfrac{1}{x}-2\right)}{\dfrac{1}{x^3}+1}$

15. $\dfrac{\dfrac{2}{x+1}-\dfrac{1}{x}}{1+\dfrac{x}{1-2x}}+\dfrac{\dfrac{2}{x+1}-1}{x+\dfrac{x}{x-2}}$

16. $\dfrac{\left(1+\dfrac{1}{x}\right)\times\left(1-\dfrac{1}{x}\right)^2}{x-\dfrac{1}{x}}$

17. $\dfrac{1+\dfrac{y^2+z^2-x^2}{2yz}}{1-\dfrac{x^2+y^2-z^2}{2xy}}$

18. $\dfrac{a}{(a-b)(a-c)}-\dfrac{2b}{(b-c)(b-a)}+\dfrac{3c}{(c-a)(c-b)}$

19. $\dfrac{2x^2-17x+21}{15x^2+16x-15}-\dfrac{2x^2-11x+12}{3x^2-10x-25}\times\dfrac{5x^2-23x+12}{x^2-2x-35}$

20. $\dfrac{x^2-(y-z)^2}{(z+x)^2-y^2}+\dfrac{y^2-(z-x)^2}{(x+y)^2-z^2}-\dfrac{(x-y)^2-z^2}{(y+z)^2-x^2}$

21. $\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}-\dfrac{x^4-y^4}{2xy(x-y)}$

22. $\dfrac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\dfrac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}$.

23. $\dfrac{x}{1+\dfrac{x}{1-x+\dfrac{x}{1+x}}}-\dfrac{1+x+x^2}{1+3x+3x^2+2x^3}$

24. $\left(\dfrac{\dfrac{x}{y}+2}{\dfrac{x}{y}+1}+\dfrac{x}{y}\right)+\left(\dfrac{x}{y}+2-\dfrac{\dfrac{x}{y}}{\dfrac{x}{y}+1}\right)$

25. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}}-\dfrac{\dfrac{1}{y}-\dfrac{1}{x}}{x^2-y^2}$.

Simplify the following fractions

26 $\dfrac{1}{6x-2}-\dfrac{1}{2(x-\frac{1}{3})}-\dfrac{1}{1-3x}$

27 $\dfrac{x}{9}+\dfrac{2}{3}+\dfrac{4}{x-6}-\dfrac{2}{3}\,\dfrac{1}{1-\dfrac{6}{x}}$

28. $\dfrac{1}{(1-a)^2}+\dfrac{2}{1-a^2}+\dfrac{1}{(1+a)^2}$

29 $\dfrac{(x^3-2x)^2-(x^2-2)^2}{(x-1)(x+1)(x^2-2)^2}$

30. $\dfrac{(x-y)^4-xy(x-y)^2-2x^2y^2}{(x-y)(x^3-y^3)+2x^2y^2}$

31 $\dfrac{x^3-1}{x^4+1}+\dfrac{x^3+1}{x^3-1}-\dfrac{2x^4}{x^4+x^2+1}$

32 $$\frac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}-\frac{\dfrac{1}{a^2}+\dfrac{1}{ab}+\dfrac{1}{b^2}}{\dfrac{1}{b}-\dfrac{1}{a}}$$

33. $\dfrac{3}{2c-3}-\dfrac{2c+15}{4c^2+9}-\dfrac{2}{2c+3}+\dfrac{18(2c+15)}{81-16c^4}$

34 $$\left(2-\frac{3n}{m}+\frac{9n^2-2m^2}{m^2+2mn}\right)-\left(\frac{1}{m}-\frac{1}{m-2n-\dfrac{4n^2}{m+n}}\right)$$

35 $$\frac{\dfrac{1-b}{1+b}-\dfrac{1-a}{1+a}}{1+\dfrac{(1-a)(1-b)}{(1+a)(1+b)}}$$

36 $$\frac{x+2-\dfrac{1}{x+2}}{x+2-\dfrac{4}{x+5}}\times\frac{x+4-\dfrac{4}{x+4}}{x+4-\dfrac{1}{x+4}}$$

37 $$\frac{(ac+bd)^3-(ad+bc)^3}{(a-b)(c-d)}-\frac{(ac+bd)^3+(ad+bc)^3}{(a+b)(c+d)}$$

38. $\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}-\dfrac{3}{4(x^2-1)}$

39. $\dfrac{1}{3x^2-4xy+y^2}+\dfrac{1}{x^2-4xy+3y^2}-\dfrac{3}{3x^2-10xy+3y^2}$

40. $\dfrac{x+2y}{\frac{2}{7}x-y}-\dfrac{3x^2+63xy+70y^2}{2x^2+3xy-35y^2}$

41. $$\frac{x+\dfrac{y}{2}}{2x^2+xy+\dfrac{y^2}{2}}-\frac{x^2-\dfrac{y^2}{2}}{4\left(x^3-\dfrac{y^3}{8}\right)}$$

42 $\dfrac{1}{4x-8}-\dfrac{1}{4x+8}-\dfrac{1}{x^2+4}+\dfrac{8}{x^4+16}$

43 Shew that $\dfrac{z^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2b^2}\left(x-\dfrac{za^2}{a^2+b^2}\right)^2=\left(\dfrac{x}{a}\right)^2+\left(\dfrac{z-x}{b}\right)^2$

44. Find the value of $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}$, when $x=\dfrac{ab}{a+b}$.

CHAPTER XXIII

Harder Equations and Problems

253 In solving some fractional equations it is convenient to clear of fractions in two or more steps, instead of multiplying at once by the L C M of all the denominators

Example Solve $\frac{x-7}{4}+\frac{x+10}{21}+1=\frac{5x-7}{8}-\frac{9x+6}{35}$

Multiplying throughout by 8, we have

$$2x-14+\frac{8x+80}{21}+8=5x-7-\frac{72x+48}{35},$$

transposing,

$$\frac{8x+80}{21}+\frac{72x+48}{35}=3x-1$$

To clear of fractions we multiply by $3\times7\times5$, or 105

Thus

$$40x+400+216x+144=315x-105,$$
$$544+105=315x-256x,$$
$$649=59x,$$
$$x=11$$

254 The following examples explain how to deal with equations which contain fractions whose denominators involve the unknown quantity

Example 1 Solve $\frac{6x-3}{2x+7}=\frac{3x-2}{x+5}$

The L C M of the denominators is $(2x+7)(x+5)$, and by multiplying both sides of the equation by this expression we have

$$(6x-3)(x+5)=(3x-2)(2x+7),$$

or

$$6x^2+27x-15=6x^2+17x-14,$$
$$10x=1,$$
$$x=\frac{1}{10}$$

Note It will be observed that the first step of the above solution is obtained by *multiplying the numerator on the left by the denominator on the right, and the numerator on the right by the denominator on the left* This process is called *multiplying across*, and can always be directly applied to equations which can be reduced to the form in which the above example is given

When two or more fractions have the same denominator, they should be taken together and simplified.

EXAMPLE 2 *Solve* $\dfrac{24-5x}{x-2}+\dfrac{8x-49}{4-x}=\dfrac{28}{x-2}-13$

By transposition, we have

$$\frac{8x-49}{4-x}+13=\frac{28-(24-5x)}{x-2};$$

$$\frac{3-5x}{4-x}=\frac{4+5x}{x-2}$$

Multiplying across, we have

$$3x-5x^2-6+10x=16-4x+20x-5x^2;$$

that is,
$$-3x=22,$$

$$x=-\frac{22}{3}$$

EXAMPLE 3 *Solve* $\dfrac{x-8}{x-10}+\dfrac{x-4}{x-6}=\dfrac{x-5}{x-7}+\dfrac{x-7}{x-9}$

Here the L C M of the denominators is $(x-10)(x-6)(x-7)(x-9)$, and if we were to clear of fractions by multiplying by this expression the work would be very laborious The following method greatly simplifies the solution

The equation may be written in the form

$$\frac{(x-10)+2}{x-10}+\frac{(x-6)+2}{x-6}=\frac{(x-7)+2}{x-7}+\frac{(x-9)+2}{x-9},$$

whence we have

$$1+\frac{2}{x-10}+1+\frac{2}{x-6}=1+\frac{2}{x-7}+1+\frac{2}{x-9},$$

which gives
$$\frac{1}{x-10}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-9}$$

Transposing,
$$\frac{1}{x-10}-\frac{1}{x-7}=\frac{1}{x-9}-\frac{1}{x-6} \qquad (1)$$

Simplifying each side separately, we have

$$\frac{x-7-(x-10)}{(x-10)(x-7)}=\frac{x-6-(x-9)}{(x-9)(x-6)},$$

or
$$\frac{3}{(x-10)(x-7)}=\frac{3}{(x-9)(x-6)}.$$

Hence, since the two fractions are equal and their numerators are equal, their denominators must also be equal,

that is,
$$(x-10)(x-7)=(x-9)(x-6),$$

$$x^2-17x+70=x^2-15x+54.$$

$$16=2x,$$

$$x=8$$

If at the stage marked (1) we had simplified *without transposition* we should have obtained

$$\frac{2x-16}{(x-10)(x-6)}=\frac{2x-16}{(x-7)(x-9)},$$

or $$(2x-16)(x-7)(x-9)=(2x-16)(x-10)(x-6) \qquad (1)$$

If $2x-16$ is not equal to 0, we may divide by this factor, in which case

$$x^2-16x+63=x^2-16x+60, \qquad (2)$$

which is an impossible result.

Hence we conclude that $2x-16$ must be equal to 0,

whence $x=8$, as before

EXAMPLE 4 *Solve* $\dfrac{5x-64}{x-13}+\dfrac{x-6}{x-7}=\dfrac{4x-55}{x-14}+\dfrac{2x-11}{x-6}$

We have $$5+\frac{1}{x-13}+1+\frac{1}{x-7}=4+\frac{1}{x-14}+2+\frac{1}{x-6},$$

$$\therefore \frac{1}{x-13}+\frac{1}{x-7}=\frac{1}{x-14}+\frac{1}{x-6}$$

Simplifying each side separately we have

$$\frac{2x-20}{x^2-20x+91}=\frac{2x-20}{x^2-20x+84},$$

whence *either* $2x-20=0$, in which case $x=10$,

or $x^2-20x+91=x^2-20x+84$, which is impossible

NOTE It will now be seen that when the two sides of a *simple* equation have a common factor containing the unknown, the equation can be solved at once by removing this factor and equating it to zero

255 In Art 106 it was explained how, by means of the fundamental axioms of addition, subtraction, multiplication, and division, we arrive at the solution of an equation by a series of operations which change the form of the equation step by step until the unknown stands alone on one side of the equation with the answer on the other It is important that each operation should lead to an *equivalent equation*, that is, one which is satisfied by the same value or values of the unknown as the original equation, and by no others This general result will always be secured if our transformations are such that we can legitimately work back from the answer, reversing every step *without fallacy*, until we arrive at the original equation Any step which is not thus strictly reversible requires special examination, for it may *miss a solution*, or, on the other hand, it may *introduce an extraneous value of the unknown which does not satisfy the original equation*

256 The conclusions of the preceding article may be illustrated as follows

(i) In the second solution of Example 3 we have seen that by equating the factor $2x-16$ to zero we obtain the solution of the equation, viz $x=8$ But if we attempt to simplify by dividing both sides of the equation by $2x-16$, we *miss this solution*, and the resulting equation is not *an equivalent one* The step from (1) to (2) is not reversible because $2x-16$ is in reality *a zero factor* which cannot be legitimately used either in multiplication or division

(ii) Consider the equation

$$\frac{x^2-2x-12}{x^2-4}+\frac{3}{x-2}=1\tfrac{1}{3} \qquad (1)$$

Multiplying throughout by $3(x^2-4)$, or $3(x+2)(x-2)$, we have

$$3(x^2-2x-12)+9(x+2)=4x^2-16, \qquad (2)$$

which reduces to $\qquad x^2-3x+2=0,$

or $\qquad (x-1)(x-2)=0$

It is readily seen that $x=1$ satisfies the equation, but if we substitute 2 for x we get $-\frac{12}{0}+\frac{3}{0}=1\frac{1}{3}$ Now as we have not yet attached any meanings to expressions like $-\frac{12}{0}$, $\frac{3}{0}$ this result is unintelligible, and $x=2$ cannot be accepted as a solution of the equation

The step from (1) to (2) was obtained by using $3(x+2)(x-2)$ as a multiplier, *which is not a legitimate operation if* $x-2$ *is equal to* 0 In other words, when x has the value 2, this step is not reversible, for the derived equation is not equivalent to the original equation Equation (2) contains the correct solution of equation (1), but it also contains *an extraneous solution which does not satisfy equation* (1)

We may arrive at the same conclusion as follows

Simplifying the first side of the given equation, we have

$$\frac{x^2-2x-12+3(x+2)}{x^2-4}=\frac{x^2+x-6}{x^2-4}=\frac{(x+3)(x-2)}{(x+2)(x-2)}$$

If we divide numerator and denominator by $x-2$, the equation reduces to

$$\frac{x+3}{x+2}=\frac{4}{3}, \text{ whence } x=1$$

But this division by $x-2$ is only legitimate as long as $x-2$ *is not equal to* 0

Hence $x=2$ is not a solution

257 If in the course of simplification we multiply or divide every term of an equation by any *constant* factor, the step will clearly be reversible, but from the foregoing examples we conclude that multiplication or division is not always a reversible process when the operating factor contains the unknown quantity whose value we are seeking

EXAMPLES XXIII. a.

Solve the equations

1. $\frac{3x-2}{8}+9-\frac{13x-3}{27}=\frac{5x-12}{18}-\frac{2-5x}{4}$

2. $\frac{3x}{4}-\frac{7x-5}{31}-(x-1)+\frac{5}{17}(2x+1)-\frac{x-2}{2}=0$

3. $\frac{3-15x}{33}-\frac{6-5x}{4}=\frac{7x-1}{3}-\frac{4+13x}{22}-\frac{x}{2}$

4. $x-\frac{4x-7}{57}-\frac{1}{6}(x-4)=\frac{2x-1}{3}-\frac{4-5x}{38}$

5. $\frac{2}{3x-1}=\frac{1}{5x-11}$ 6. $\frac{5}{3x+4}=\frac{4}{5(x-3)}$ 7. $\frac{x+1}{2x-1}-\frac{x-3}{2(x-4)}=0$

8. $\frac{3}{5+8x}=\frac{5}{47-6x}$ 9. $\frac{3x+1}{x-5}=\frac{6x-7}{2x+1}$ 10. $\frac{2x-3}{x-2}-\frac{21-8x}{7-4x}=0.$

11. $\frac{1}{12}+\frac{5x-5}{12x+8}=\frac{6x+7}{9x+6}$ 12. $\frac{2}{x-2}+\frac{3}{x}=\frac{5}{x-4}$

13. $\frac{x-5}{2}+\frac{2x-1}{3x+2}=\frac{5x-1}{10}-1\frac{2}{5}$ 14. $\frac{8x+57}{12}-\frac{15-2x}{x+8}=\frac{2(x+2)}{3}+4.$

15. $\frac{5x-17}{13-4x}+\frac{2x-11}{14}-\frac{23}{42}=\frac{3x-7}{21}$

16. $\frac{3}{x-1}-\frac{1}{x+3}=\frac{2\frac{1}{2}}{x-2}-\frac{1}{3x+6}$ [*Simplify each side separately.*]

17. $\frac{1}{3x+12}+\frac{1}{6(x+4)}=\frac{3}{2x+10}-\frac{1}{x+6}$

18. $\frac{1}{x-10}-\frac{1}{x-5}=\frac{1}{x-7}-\frac{1}{x-2}$ 19. $\frac{1}{x-6}-\frac{1}{x-3}=\frac{1}{x-5}-\frac{1}{x-2}$

20. $\frac{x-5}{x-6}-\frac{x-6}{x-7}=\frac{x-1}{x-2}-\frac{x-2}{x-3}$ 21. $\frac{x+8}{x+9}+\frac{x+4}{x+5}=\frac{x+9}{x+10}+\frac{x-3}{x+4}.$

22. $\frac{x-2}{x-3}+\frac{x-3}{x-4}=\frac{x-1}{x-2}-\frac{x-4}{x-5}$ 23. $\frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}-\frac{8-x}{6-x}$

24. $\frac{5x+36}{x+7}=\frac{5x+17}{x+3}$ 25. $\frac{17x-54}{x-3}=\frac{17x-87}{x-5}$

26. $\frac{4x-3}{x-1}+\frac{3x-8}{x-3}=\frac{7x+2}{x}$ 27. $\frac{5x-1}{x}+\frac{3x-5}{x-1}=\frac{8x-19}{x-2}$

28. $\frac{2x-27}{x-14}+\frac{x-7}{x-8}=\frac{x-12}{x-13}+\frac{2x-17}{x-9}$ 29. $\frac{x^2-7x+10}{x^2-7x+12}=\frac{x^2+3x-10}{x^2+3x-8}.$

30. $\frac{5x-21}{x-4}+\frac{8x-10}{2x-3}=\frac{6x-23}{2x-7}+\frac{6x-5}{x-1}$

Equations with Literal Coefficients

258 All the equations hitherto discussed have had *numerical* quantities as coefficients When an equation involves *literal* coefficients it must be remembered that they represent *known quantities*, and will appear in the solution

Example 1 *Solve* $m(x-2n)=n(n-x)+m^2$

Removing brackets, $mx-2mn=n^2-nx+m^2$,

transposing, $mx+nx=m^2+2mn+n^2$,

that is, $x(m+n)=(m+n)^2$,

$$x=m+n$$

Example 2 *Solve* $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}$

Simplifying the left side, we have

$$\frac{a(x-b)-b(x-a)}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

that is, $$\frac{ax-ab-bx+ab}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

or $$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

$$\frac{x}{(x-a)(x-b)}=\frac{1}{x-c}$$

Multiplying across, $x^2-cx=x^2-ax-bx+ab$,

or $ax+bx-cx=ab$,

$$(a+b-c)x=ab,$$

$$x=\frac{ab}{a+b-c}$$

Example 3 *Find the values of* x *which satisfy the equation*

$$x^2-6ax=b^2-9a^2$$

This is a *quadratic*, and may be solved as in Art 202

By transposition, $x^2-6ax+9a^2-b^2=0$,

that is, $(x-3a)^2-b^2=0$,

or $(x-3a-b)(x-3a+b)=0$,

either $x-3a-b=0$, whence $x=3a+b$,

or $x-3a+b=0$, whence $x=3a-b$

Thus the required roots are $3a+b$, and $3a-b$

Note It is important to remember that the equation must be expressed in its simplest form, with all the terms on one side, before solution by means of factors is attempted

EXAMPLES XXIII b

Solve the following equations:

1. $ax+b^2=a^2-bx$ 2. $x^2-a^2=(2a-x)^2$

3. $(a-b)(x-c)=(a-b)(x-c)$ 4. $(x+b)(a+b)=(x-b)(a-b)$

5. $(x-a)(x+a+b)=(x+b)(x+3a)$

6. $(b+c)(x-a)-(c+a)(x-b)=(a+b)(x-c)$

7. $(a-b)(x+a-b)+(a-b)(x-a-b)=2a(2c-x)$

8. $(a-x)(b-x)=(a-b-x)(a+b-x)+a^2$

9. $(p+q-x)(p-q-x)+(p-x)(q+x)+p^2=0$

10. $(ax-b)(bx+a)=a(bx^2-a)$ 11. $n(q+x)-pn=l(q+x)-pl$

12. $\dfrac{x-a}{2}-\dfrac{x-b}{3}=\dfrac{a+3x}{3}-\dfrac{2x-b}{2}$ 13. $\dfrac{a-x}{a}+\dfrac{2a-x}{2a}=\dfrac{3a-x}{3a}$.

14. $\dfrac{1}{5}(x+m)+\dfrac{2}{3}(x-n)=\dfrac{1}{5}(5x-4m)+\dfrac{1}{3}(2x-n)$

15. $\dfrac{x}{bc}+\dfrac{x}{ca}+\dfrac{x}{ab}=a+b+c$ 16. $\dfrac{x-a}{a}+\dfrac{x-b}{b}+\dfrac{x-c}{c}=1$

17. $\dfrac{x-a}{a+b}+\dfrac{x-b}{a-b}=1$ 18. $x+\dfrac{a}{b-a}=\dfrac{bx}{a+b}$ 19. $\dfrac{a+2b}{x-c}=\dfrac{a-2b}{x+c}$.

20. $\dfrac{x+p}{p+q}-\dfrac{x+q}{p-q}=\dfrac{(p+q)^2}{p^2-q^2}$ 21. $\dfrac{x}{x+b-a}-\dfrac{b}{x+b-c}=1$

22. $\dfrac{6x-a}{4x-b}=\dfrac{3x+b}{2x+a}$ 23. $\dfrac{7a-x}{b-3a}-4=\dfrac{3x-5a}{3b-a}$

24. $\dfrac{x-bc}{a}+\dfrac{x-ca}{b}+\dfrac{x-ab}{c}=2(a+b+c)$

25. $\dfrac{x-a}{a-b}-\dfrac{x+b}{a-b}=\dfrac{2a(x-b)}{a^2-b^2}-\dfrac{a-b}{a+b}$

26. $\dfrac{x-c+d}{c-d}-\dfrac{x}{c+d}=\dfrac{2c(x-c-d)}{c^2-d^2}-1$

27. $\left(\dfrac{x}{a}-3\right)\left(\dfrac{3x}{a}-1\right)-\dfrac{1}{a^2}(x-2a)(2x-a)=\left(\dfrac{x}{c}-1\right)^2-1$

Solve the following quadratic equations:

28. $3x^2+5ax-2a^2=0$ 29. $x^2+6ab=2ax-3bx$

30. $2x^2-2mx=nx-mn$ 31. $x^2-4d^2=c(2x-c)$

32. $\dfrac{1}{2x-5c}+\dfrac{5}{2x-c}=\dfrac{2}{c}$. 33. $\dfrac{5}{x-2a}-\dfrac{8}{x-a}=\dfrac{1}{a}$.

34. $\dfrac{p}{x-p}-\dfrac{2p}{x+2p}=\dfrac{1}{10}$ 35. $\dfrac{x}{cd}-\dfrac{4}{d}=\dfrac{d}{cx}-\dfrac{4c}{dx}$.

Simultaneous Equations with Literal Coefficients.

259 EXAMPLE 1 *Solve the equations*

$$\frac{x}{p}+\frac{y}{q}=1, \quad p(x-p)-q(y+q)=2p^2+q^2$$

After simplification these equations may be written

$$qx+py=pq, \qquad (1) \qquad px-qy=3p^2+2q^2 \qquad (2)$$

To eliminate y, multiply (1) by q and (2) by p,

thus
$$q^2x+pqy=pq^2,$$
$$p^2x-pqy=3p^3+2pq^2$$

By addition, $(p^2+q^2)x=3p^3+3pq^2=3p(p^2+q^2)$,

$$x=3p$$

Substituting this value of x in (1), we obtain $y=-2q$

In the following example the coefficients in the first equation are a, b, and c, in the second equation the coefficients of corresponding terms are the same letters distinguished by accents, namely a', b', and c' (read "a dash," "b dash," "c dash") There is no necessary connection between the values of a and a', which are as different as p and q in the preceding example, but the notation here first introduced is convenient as it aids the eye in recognising letters which have a common property Thus a, a' have a common property as being coefficients of x, b, b' as being coefficients of y

Sometimes instead of accents letters are used with a numerical *suffix*, such as a_1, a_2, a_3, b_1, b_2, b_3, etc (read "a one," "a two," "a three," etc)

EXAMPLE 2 *Solve the equations*

$$ax+by+c=0, \qquad (1)$$
$$a'x+b'y+c'=0 \qquad (2)$$

Multiplying (1) by b', and (2) by b, we have

$$ab'x+bb'y=-b'c,$$
$$a'bx+bb'y=-bc',$$

by subtraction, $(ab'-a'b)x=bc'-b'c$,

$$x=\frac{bc'-b'c}{ab'-a'b} \qquad (3)$$

Again, multiplying (1) by a', and (2) by a, we have

$$aa'x+a'by=-a'c,$$
$$aa'x+ab'y=-ac',$$

by subtraction, $(a'b-ab')y=ac'-a'c$,

$$y=\frac{ac'-a'c}{a'b-ab'},$$

or, changing signs in the terms of the denominator so as to have the same denominator as in (3),

$$y=\frac{a'c-ac'}{ab'-a'b}, \quad \text{and} \quad x=\frac{bc'-b'c}{ab'-a'b}$$

260 Since every linear equation in two unknowns, x and y, can by suitable reduction be expressed in the form $ax+by+c=0$, the values of x and y found in the last example may be used as formulæ for writing down the solution of other simultaneous equations

Thus, to solve the equations $x+2y=13$, $3x+y=14$, we may put $a=1$, $b=2$, $c=-13$, $a'=3$, $b'=1$, $c'=-14$

$$x=\frac{bc'-b'c}{ab'-a'b}=\frac{2(-14)-1(-13)}{1\times 1-3\times 2}=\frac{-15}{-5}=3,$$

$$y=\frac{a'c-ac'}{ab'-a'b}=\frac{3(-13)-1(-14)}{1\times 1-3\times 2}=\frac{-25}{-5}=5$$

EXAMPLES XXIII. c.

Solve the equations

1 $ax+by=l$, $ax-by=m$

2 $ax+by=a^2+ab$, $x+y=2a$

3 $3bx+ay=5ab$, $ay-bx=ab$

4 $2bx-ay=ab$, $bx+2ay=3ab$

5 $ax+by=2ab$, $bx-ay=b^2-a^2$

6 $a(x-a)=b(y-a)$, $b(x+b)=a(y+b)$.

7. $cx+dy=c^2+cd$, $dx+cy=cd+d^2$

8 $dx-cy=d^2$, $(c-d)x+dy=c^2$

9 $qx-py+q^2=0$, $(p+q)x+qy=p^2$.

10 $\frac{x}{a}+\frac{y}{b}=2$, $ax+b^2=a^2+by$

11. $\frac{x}{2a}+\frac{y}{2b}=\frac{1}{a+b}$, $ax-by=a-b$

12 $lx+my=n$, $l'x+m'y=n'$

13 $\frac{x}{c}+\frac{y}{c+d}=c$, $\frac{x}{c-d}-\frac{y}{d}=-d$

14 $\frac{x-a}{2}+\frac{y-b}{3}=a$, $\frac{x-b}{3}+\frac{y-a}{2}=b$

15 $a_1x-b_1y=c_1$, $a_2x-b_2y=c_2$

16 $(a^2+b^2)(x-1)=ab(2x-y)$, $4x=y+2$

17 $m(x+y)+n(x-y)=2mn$, $m(x+y)-n(x-y)=mn$.

18. $\frac{m}{x}+\frac{n}{y}=a$, $\frac{n}{x}+\frac{m}{y}=b$

19. $\frac{p}{x}+\frac{q}{y}=0$, $px+qy=r$

20 $lx=my$, $\frac{a}{x}-\frac{b}{y}=c$

21 $\frac{2x-b}{a}=\frac{2y+a}{b}=\frac{3x+y}{a+2b}$

22. $\frac{px+qy}{qx+py}=\frac{1}{2}=\frac{p^2-q^2}{qx+py}$

23. $y=\frac{x+a}{2}+\frac{b}{3}$, $x=\frac{y+b}{2}+\frac{a}{3}$

24. $\frac{x-a}{c-a}+\frac{y-b}{c-b}=1$, $\frac{x+a}{c}+\frac{y-a}{a-b}=\frac{a}{c}$

Harder Problems.

261. EXAMPLE 1 *A man can walk from* A *to* B *and back in a certain time at 4 miles an hour If he walks 3 miles an hour from* A *to* B, *and returns at 5 miles an hour, he takes 10 minutes longer for the double journey. Find the distance from* A *to* B

Let x be the distance in miles from A to B

At 4 miles per hour he will *go* and *return* in $\frac{2x}{4}$ hours, or $\frac{x}{2}$ hours

At 3 „ „ „ from A to B in $\frac{x}{3}$ „

and at 5 „ „ „ „ B to A in $\frac{x}{5}$ „

Hence $$\frac{x}{3}+\frac{x}{5}=\frac{x}{2}+\frac{1}{6},$$

or $$10x+6x=15x+5,$$
$$x=5$$

Thus the distance from A to B is 5 miles

EXAMPLE 2 *Divide £720 into two parts such that, if they are put out to interest at $3\frac{1}{2}$ % and 5 % respectively, they may together yield the same annual income as if the whole were invested at $4\frac{1}{2}$ %*

Let x be the number of pounds invested at $3\frac{1}{2}$ %, then $720-x$ is the number invested at 5 %

The interest on £x for 1 year at $3\frac{1}{2}$ % is £$\frac{x\times 3\frac{1}{2}}{100}$,

„ „ £$(720-x)$ „ 1 „ 5 % is £$\frac{(720-x)\times 5}{100}$,

and „ „ £720 „ 1 „ $4\frac{1}{2}$ % is £$\frac{720\times 4\frac{1}{2}}{100}$.

Hence $$\frac{x\times 3\frac{1}{2}}{100}+\frac{(720-x)\times 5}{100}=\frac{720\times 4\frac{1}{2}}{100},$$

or $$\frac{7x}{2}+3600-5x=\frac{720\times 9}{2},$$

whence $$7x+7200-10x=6480;$$
$$3x=720,$$
$$x=240$$

Thus the required investments are £240 at $3\frac{1}{2}$ % and £480 at 5 %.

The following solution, by the use of two unknowns, is slightly simpler than the above

Let £x be invested at $3\frac{1}{2}$ % and £y at 5 % Then $x+y=720$ (1)

Also $\frac{x\times 3\frac{1}{2}}{100}+\frac{y\times 5}{100}=\frac{720\times 4\frac{1}{2}}{100}$, whence $7x+10y=6480$ (2)

From (1) and (2) we obtain $x=240$, $y=480$

EXAMPLE 3 *A grocer buys 25 lbs. of tea and 30 lbs of coffee for £5 2s 6d By selling the coffee at a loss of 5 per cent, and the tea at a gain of 10 per cent, he makes a profit of 4s 3d, what was the prime cost of tea and coffee per lb?*

Let the price per lb of tea and coffee be represented by x shillings and y shillings respectively,

then the total prime cost was $(25x+30y)$ shillings

Therefore $\qquad 25x+30y=102\frac{1}{2},$

which reduces to $\qquad 10x+12y=41 \qquad (1)$

The gain upon the tea is $\frac{1}{10}\times 25x$ shillings, and the

loss upon the coffee is $\frac{1}{20}\times 30y$ shillings,

thus the net gain is $\left(\frac{5x}{2}-\frac{3y}{2}\right)$ shillings

$$\frac{5x}{2}-\frac{3y}{2}=4\frac{1}{4},$$

which reduces to $\qquad 10x-6y=17 \qquad (2)$

From (1) and (2) we get $x=2\frac{1}{2}$, $y=1\frac{1}{3}$

Thus the tea cost 2s 6d per lb, and the coffee 1s 4d per lb

EXAMPLE 4 *A man buys oranges at 6d a dozen, and twice as many at 11d a score, he sells the whole of them at 8d a dozen, and makes a profit of 5s How many oranges did he buy?*

Let x denote the number he bought at 12 for 6d, then $2x$ is the number he bought at 20 for 11d

The *cost price*, in pence, of x at 12 for 6d $=x\,\frac{6}{12}$ or $\frac{x}{2}$;

and ,, ,, $2x$ at 20 for 11d $=2x\,\frac{11}{20}$ or $\frac{11x}{10}$

Therefore the total cost price $=\left(\frac{x}{2}+\frac{11x}{10}\right)$ pence,

and the *selling price* of $3x$ at 12 for 8d $=\left(3x\,\frac{8}{12}\right)$ pence, or $2x$ pence

Therefore the *gain* $=\left\{2x-\left(\frac{x}{2}+\frac{11x}{10}\right)\right\}$ pence

Hence $\qquad 2x-\left(\frac{x}{2}+\frac{11x}{10}\right)=60,$

or $\qquad 2x-\frac{x}{2}-\frac{11x}{10}=60,$

that is, $\qquad 20x-5x-11x=600,$

$$4x=600,$$

$$x=150$$

Thus he bought 150 at 6d a dozen, and the total number of oranges was $3x$, or 450

EXAMPLES XXIII. d.

1. A man can walk from A to B and back in a certain time at the rate of 4 miles an hour, if he walks $3\frac{1}{2}$ miles an hour from A to B, and $4\frac{1}{2}$ miles an hour from B to A, he requires $3\frac{1}{3}$ minutes longer for the double journey what is the distance from A to B?

2 A cruiser sailing at the rate of 10 miles an hour discovers a ship 18 miles off running from her at the rate of 8 miles an hour, how many miles can the ship run before she is overtaken?

3 *B* has 5 miles start of *A*, but travels at the rate of only 3 miles an hour, while *A* travels at the rate of $4\frac{1}{2}$ miles an hour, where will *A* overtake *B*, and how long will he take to do it?

4 A boy walks to school at the rate of $3\frac{1}{2}$ miles an hour, and is one minute late, if he had walked at the rate of $3\frac{1}{3}$ miles per hour he would have been 3 minutes late, find the distance to the school

5. A boy walks to school at the rate of $3\frac{1}{3}$ miles per hour, and is 4 minutes late, the next day he increases his pace by a quarter of a mile per hour, and is 2 minutes late find the distance to the school

6 A bicyclist can ride from A to B and back in a certain time at an average rate of 10 miles an hour If he were to ride from A to B at 8 miles an hour, and return at 12 miles an hour, he would lose half an hour on the double journey Find the distance from A to B

7. A cyclist, whose average speed is 10 miles an hour, sets out to ride from A to B, at the same time his friend, whose average speed is 8 miles an hour, sets out to ride from B to A If they meet 4 miles from half-way, how far is it from A to B?

8 Divide £555 so that by investing part of it at 4 % and the remainder at 5 % the total income produced may be £25 10*s*

✓ 9. I invest £720 partly at 3 % and partly at 5 %, thereby obtaining the same income as if I had invested the whole at $3\frac{1}{3}$ % How much did I invest at each rate?

10 A person buys 20 yards of cloth and 25 yards of canvas for £1 17*s* 6*d* By selling the cloth at a gain of 15 % and the canvas at a gain of 20 % he clears 6*s* 3*d*, find the price of each per yard

11 A dealer spends £760 in buying horses at £24 each and cows at £20 each, through disease he loses 20 % of the horses and 15 % of the cows By selling the remaining animals at cost price he receives £628, find how many of each he bought

12. An income of £160 is derived partly from money invested at $3\frac{1}{2}$ % per annum, and partly from money invested at 3 % per annum, if the investments were interchanged the income would be £165 How much is invested at each rate?

13. A grocer buys 15 lbs of figs and 28 lbs of currants for £1 1*s* 8*d*; by selling the figs at a loss of 10 %, and the currants at a gain of 30 %, he clears 2*s* 6*d* on his outlay, how much per pound did he pay for each?

14. A man bought a number of eggs at three for twopence, and three times as many at two for three halfpence; if he gains half a crown by selling them all at tenpence a dozen, how many did he buy?

15. A man buys oranges at sixpence a dozen, and an equal number at ninepence a score; he sells them at ninepence a dozen, and makes a profit of 5s. 6d.: how many oranges did he buy?

16. I bought a certain number of apples at three a penny; I kept one sixth of them, and sold the rest at two a penny, and gained a penny: how many did I buy?

17. A man buys 3 horses and 7 sheep for £100; he sells the horses at a profit of 8 %, and the sheep at a profit of 12 %; his whole gain is £8. 14s. What price did he pay for a sheep?

18. On a tour a man spends £3. 10s. more on railway fares than on hotels, and the hotels cost £2. 10s. less than all other expenses. If the whole cost is £48, what did he spend on railway fares?

19. The profits of a business were £150 in the first year, and half as much in the second year as in the third. In the fourth year they were three times as much as in the first two years together. The total profit in all four years was half as much again as in the first and fourth years together. Find the total profit.

262. EXAMPLE. *A certain number of persons paid a bill; if there had been 12 fewer each would have paid 1s. 6d. more; if there had been 8 more each would have paid 6d. less: find the number of persons and what each had to pay.*

Suppose there were x persons and that each paid y shillings.

Then the total number of shillings paid is xy.

If $(x-12)$ persons paid $(y+1\frac{1}{2})$ shillings each,

$$\text{the total number of shillings paid} = (x-12)(y+1\tfrac{1}{2});$$

and if $(x+8)$ persons paid $(y-\frac{1}{2})$ shillings each,

$$\text{the total number of shillings paid} = (x+8)(y-\tfrac{1}{2}).$$

Now all these expressions for the total sum paid must be equal;

therefore $$xy=(x-12)(y+1\tfrac{1}{2}), \qquad (1)$$

and $$xy=(x+8)(y-\tfrac{1}{2}). \qquad (2)$$

From (1), $$xy=xy+1\tfrac{1}{2}x-12y-18;$$

that is, $$3x-24y=36,$$

or $$x-8y=12. \qquad (3)$$

From (2), $$xy=xy-\frac{x}{2}+8y-4;$$

that is, $$x-16y=-8. \qquad (4)$$

By combining (3) and (4) we find that $x=32$, $y=2\frac{1}{2}$.

Thus there were 32 persons and each paid 2s. 6d.

263 The following problem illustrates how a solution may be sometimes neatly effected by the introduction of an auxiliary symbol which will divide out in the course of the work

EXAMPLE *An express leaving* P *at* 3 *p m reaches* Q *at* 6 *p m*, *a slow train leaving* Q *at* 1 30 *p m arrives at* P *at* 6 *p m*, *if both trains are supposed to travel at a uniform speed, find the time when they will meet*

Let x be the number of hours after 3,

and let a be the number of miles from P to Q

The express goes a miles in 3 hours, that is, $\frac{a}{3}$ mi per hr

The slow train goes a miles in $4\frac{1}{2}$ hours, that is, $\frac{a}{4\frac{1}{2}}$ or $\frac{2a}{9}$ mi per hr

Thus in x hours the express has gone $x \times \frac{a}{3}$ miles, and the slow train, starting $1\frac{1}{2}$ hours earlier, has in $\left(x+1\frac{1}{2}\right)$ hours gone $\left(x+1\frac{1}{2}\right) \times \frac{2a}{9}$ miles

But when the trains meet the whole distance has been covered, hence

$$\frac{ax}{3}+\frac{2a}{9}\left(x+1\tfrac{1}{2}\right)=a,$$

or
$$\frac{x}{3}+\frac{2x}{9}+\frac{1}{3}=1, \text{ whence } x=1\tfrac{1}{5}$$

That is, the trains meet 1 hr 12 min after 3, or at 4 12 p m

EXAMPLES XXIII. e

1 A train travelled a certain distance at a uniform rate Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less, and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more Find the distance

2 A certain number of persons paid a bill, if there had been 10 more, each would have paid 1*s* less, if there had been 5 fewer, each would have paid 1*s* more find the number of persons, and what each had to pay

3 A certain subscription is raised in a boys' school, had each boy given a penny less, the money would have been obtained from 20 more, and if each had given 2*d* more, from 25 fewer subscribers How many contributors were there, and what did each give?

4 If the breadth of a certain rectangle were increased by 5 yds, and its length diminished by 10 yds, its area would be increased by 200 sq yds, whilst if its breadth were diminished by 5 yds, and its length increased by 15 yds, its area would be decreased by 75 sq yds What are its length and breadth?

5. A sum of money is to be divided equally among a certain number of boys If there were 3 fewer, each would receive exactly 2*s*, and if there were 2 more, each would get only 1*s* 6*d*. How much money is there for division, and how many boys are to share it?

6. At 7.40 a.m. the ordinary train starts from Norwich and reaches London at 11.40 a.m.; the express which starts from London at 9 a.m. arrives at Norwich at 11.40 a.m. If both trains travel at a uniform speed, find the time when they meet.

7. A can run 50 yards whilst B runs 45 yards. If B has 5 minutes start in a race, what time will A take to get level with B?

8. A boy starts from home and walks to school at the rate of 11 yards in 9 seconds, and is 1 minute late; if he had walked at the rate of 22 yards in 15 seconds he would have been half a minute too soon. Find the distance to the school.

9. A man has a number of coins which he tries to arrange in the form of a solid square; on the first attempt he has 116 over, and when he increases the side of the square by three coins he wants 25 to complete the square. How many coins has he?

10. A man has 81 coins, some of them crowns and the rest shillings; if he exchanged each crown for a florin and each shilling for a half-crown he would neither gain nor lose. How many crowns has he?

11. After a repulse a general found that only 5400 men more than half of his former force were fit for service, as 400 more than one fifth were wounded, and 500 more than one eighth were killed, missing, or prisoners. What was his force before the battle?

12. A box of oranges can be divided so that half the boys in a school will have 3 each, the other half 2 each, and there will be 25 over; if all the boys but 45 had been given 3 oranges each, the rest could have had 2 each, and there would have been 10 left over. How many oranges were there in the box?

13. With a capital of £415 invested partly at $2\frac{1}{2}$ per cent. and partly at 4 per cent. an income of £14. 10*s*. is secured. How is the money divided?

14. A person swimming in a stream which runs $1\frac{1}{3}$ miles per hour finds that it takes him five times as long to swim a mile up stream as it does to swim the same distance down. At what rate does he swim?

15. Three trains A, B, C travel on the railway from Bristol to Hull, a distance of 220 miles, at the rate of 25, 20, 30 miles per hour respectively; A and B leave Bristol at 7 a.m. and 8.15 a.m. respectively, and C leaves Hull at 10.30 a.m. When and where will A be equidistant from B and C?

16. A man expected to receive on a certain day 300 tons of coal at a cost of 12*s*. per ton, this he had contracted to sell at 15*s*. per ton. He only received part of the coal, and had to buy the rest at 19*s*. per ton to complete his contract. He gained £21 less than he had expected. How many tons did he get at 12*s*. per ton?

17. A man travels a certain distance, and finds that if he had gone one more mile per hour, he would have saved an hour and a half, but that, if he had gone slower than he did by half a mile an hour, he would have taken one hour longer. Find the distance and his rate.

MISCELLANEOUS EXAMPLES V.

Exercises for Revision.

A

1. Resolve into two or more factors

(i) $p^4-p^2q^2-56q^4$, (ii) $12y^2-30y+12$;

(iii) $2mn+m^2-1+n^2$, (iv) $x+3y+x^3+27y^3$

2. Find the H C F and L C M of

$$3x^2+x-10,\quad 6x^2-x-15,\quad 6x^2-19x+15$$

3. Find, by inspection, values of x which satisfy the following equations

(i) $(x+7)(x-3)=0$, (ii) $x^2+8x=0$;

(iii) $x^2-25=0$; (iv) $2x^2=3x$.

4. Simplify (i) $\left(\dfrac{a}{1+a}+\dfrac{1-a}{a}\right)\div\left(\dfrac{a}{1+a}-\dfrac{1-a}{a}\right)$;

(ii) $\dfrac{1}{2a^2(a^2+c^2)}-\dfrac{1}{4a^3(c-a)}+\dfrac{1}{4a^3(a+c)}$.

5. Solve (i) $\dfrac{1}{2}(2x+7)-\dfrac{4}{5}\left(3x-\dfrac{5}{2}\right)=9$,

(ii) $\dfrac{2x+5}{3}=\dfrac{y+4}{2}=\dfrac{2x+2y+9}{6}$

6. I bought a certain number of articles at 7 for 6*d*; if they had been 13 for 1*s* I should have spent 6*d* more how many did I buy?

7. Expand the product $(2-x^2+x^3-3x^4)(4-2x^2+3x^3)$ as far as the term which involves x^3

B

8 Simplify $5(y+1)^3-11(y+1)^2+10(y+1)-2$

9. Using Detached Coefficients,

(i) divide $x^5-7x^3-2x+12$ by x^2+x+2,

(ii) find the H C F of

$$2x^5+5x^3+3x^2-7x-3 \text{ and } 2x^4-4x^3+7x^2-11x-6$$

10. Write down the square of $2n-1$. Hence shew that the square of any odd integer when divided by 8 leaves a remainder 1

11. Simplify

$$(x+1)(x^2+x-12)(x^2-x-12)$$
$$-(x-1)(x^2+7x+12)(x^2-7x+12).$$

12. Write down the factors of $2a^2+13a+15$ Hence shew that the graph of the equation

$$2(x-2y)^2+13(x-2y)+15=0$$

consists of two parallel straight lines

13 Solve the equations

$$6x+4y-2z-5=3x-2y+4z+10=5x-2y+6z+13=0$$

14 A man buys oranges at the rate of p for a shilling, and by selling them at q pence a dozen makes a profit of r per cent Shew that

$$25pq-36r=3600$$

C

15. Draw the graphs of $3-2x$ and $3x-7$ referred to the same axes From these graphs determine the value of each expression when they have the same value

16. Find the factors of

(i) $10x(x-1)-3(x+1)$, (ii) $9b^2-6bc+c^2-16$

17 Find the square root of $a^2+4b^2+c^2-4ab-2ac+4bc$

18 If $f(x)\equiv x^3+7x^2-36$, find the value of $f(2)$, $f(-3)$, and $f(-6)$. Hence write down x^3+7x^2-36 as the product of three simple factors

19. Simplify

(i) $\dfrac{1+x+x^2}{1-x^3}+\dfrac{x-x^2}{(1-x)^3}$, (ii) $\dfrac{x^3-2x^2+2x-1}{2x^3-x^2-x}$;

(iii) $\dfrac{2x+1}{2x^2+3x+1}-\dfrac{2x-1}{2x^2-3x+1}+\dfrac{2x^2}{x^2-1}$

20. Solve the equations

(i) $\dfrac{5}{6}\left(x-\dfrac{1}{3}\right)+\dfrac{7}{6}\left(\dfrac{x}{5}-\dfrac{1}{7}\right)=4\frac{8}{9}$; (ii) $\left.\begin{array}{l}3x+5y=15\\5x-3y=8\end{array}\right\}$

21 A crew, which can row at the rate of 8 miles per hour on still water, finds that it takes twice as long to pull up a certain reach against stream as it does to come down the same reach Find how fast the river flows

D

22 Shew that $(y^2-3y)(y^2-3y+2)+1$ is a square What is its square root?

23 By means of the formula $(a+b)(a-b)=a^2-b^2$, find the value of $2117\times1883-1113\times887$

24. Resolve into factors

(i) $9(x-2)^2-4(x-1)^2$; (ii) $2x^3-x^2+8x-4$

25. Simplify (i) $\left\{\frac{p}{p-1}-\frac{p}{p+1}\right\}-\frac{1}{3}\left\{\frac{1}{p-1}-\frac{1}{p+1}+\frac{1}{1-p^2}\right\}$;

(ii) $\frac{3x^3-2x^2-x}{4x^3-2x^2-3x+1}$.

26. By means of factors, find the product of

(i) $2a^2-3ab+5b^2$ and $2a^2-3ab-5b^2$,

(ii) $1-2a^2$, $1-2a^2+4a^4$, $1+2a^2$, and $1+2a^2+4a^4$

27. A cash box contains three equal sums of money, one in sovereigns, one in shillings, and one in sixpences If the total number of coins in the box is 732, find how much money the box contains

28. Taking one inch as unit, draw with the same axes the graphs of $2x+y=2$, $2x+6=3y$, $y=1$; and find from the diagram the coordinates of the three points at which the lines intersect

E

29. Factorize, as fully as possible,

(i) $2a-18a^3$, $ax+2bx-a^2-2ab$, p^2+p-42,

(ii) $27d^3+8$, $6a^2-ab-2b^2$, $m^2-(n-r)^2$;

(iii) a^4-13a^2+36, $x^2+2x-255$, $(a^2+b^2)^2-4a^2b^2$.

30. A journey of x miles takes me n hours; a journey of y miles takes a cyclist m hours express in symbols the fact that the cyclist's pace is 6 miles per hour faster than mine If $x=20$, $n=5$, shew from the formula that on an average the cyclist covers a mile in 6 minutes

31. Divide x^6-5x^3+8 by x^2+x+2 by using Detached Coefficients; and verify the result

32. Find the H.C F. of

$$x^3+3ax^2-6a^2x-8a^3 \quad \text{and} \quad x^3-2ax^2-a^2x+2a^3$$

33. Simplify $\frac{1}{x^2-3x+2}+\frac{1}{x^2-4x+3}+\frac{2}{5x-x^2-6}$.

34. Solve the equations:

$$x+y+z=11, \quad 2y-x-z=10, \quad 6z-5y=1$$

35. With half an inch for the x-unit and one-tenth of an inch as the y-unit, draw the graph of $y=x^2+x$ for integral values of x between -5 and $+4$.

On the same scale draw the graph of $y=3x+8$, and find the coordinates of the points where the two graphs intersect.

F

36. If a men and b boys can unload a ship in c days, and A men and B boys unload it in C days, in how many days will $A+a$ men and $B+b$ boys unload it?

37. Find the L.C.M. of

$$4x^2+8x-12, \quad 9x^2-9x-54, \quad 6x^4-30x^2+24$$

38. Solve the equations

$$\text{(i)} \left.\begin{aligned} \frac{12}{x}-\frac{5}{y}&=3 \\ \frac{9}{x}+\frac{2}{y}&=3\tfrac{2}{5} \end{aligned}\right\}, \qquad \text{(ii)} \left.\begin{aligned} 9x+8y&=xy \\ \frac{12}{y}-\frac{7}{x}&=5\tfrac{3}{4} \end{aligned}\right\}$$

39. By the method of completing the square, find the factors of

(i) $x^2+40x+391$, (ii) $p^2+10p-551$

40. Divide $\dfrac{a}{a+b}-\dfrac{b}{a-b}-\dfrac{2b^2}{b^2-a^2}$ by $\left\{1-\dfrac{2b}{a+b}\right\}^2$

41. The expenses of a certain number of persons would have amounted to $6d$ per head less if there had been 15 more to share in them, and $3d$ per head more if there had been 5 fewer to share. how many persons were there, and what had each to pay?

42. Draw a graph to shew the fall of the barometer due to increase of height above sea level from the following data

Height in thousands of feet	0,	5,	10,	15,	20,	25,	30
Height of barometer in inches	30,	24.9,	20.6,	17.1,	14,	11,	9.8

Estimate the height of the barometer at 12000 feet above sea level

G

43. If I buy an article for $£(c-d)$ and sell it for $£(c+d)$, what is my gain per cent?

44. Divide $\dfrac{a^3}{2}+\dfrac{3a^2x}{2}-2x^3$ by $\dfrac{a}{2}+x$

45. Why is it obvious that

$$(c+a-2b)d^2+(a+b-2c)d+(b+c-2a)$$

is exactly divisible by $d-1$?

46. Distinguish between an *Equation* and an *Identity*. Construct a simple example of each.

Prove the identity

$$a^4+b^4+(a+b)^4\equiv 2a^2b^2+2(a^2+b^2)(a+b)^2$$

47. Solve the following equations, verifying the solutions in each case.

$$\text{(i)}\ \frac{x+a+c}{x+b+c}=\frac{b}{a}, \qquad \text{(ii)}\ \left.\begin{aligned} x+y+1&=3(x+y-1) \\ x-y+1&=2(x-y-1)\end{aligned}\right\}$$

48. Two men A and B run a race A runs at the uniform rate of 22 feet per second, B at the uniform rate of 20 feet per second If B has 20 yards start, and A wins by 3 seconds, find the length of the race

49. Draw a smooth curve lying evenly among the points given by the following corresponding values of x and y

$$x=7,\quad 11,\quad 15,\quad 19,\quad 24,\quad 29,$$
$$y=4,\quad 5\,8,\quad 8,\quad 11\,5,\quad 19,\quad 27\,5$$

Find the value of y when $x=21$, and the value of x when $y=25$

H

50 Prove that the sum of the squares of a number with two digits, and of the number with the same digits reversed, is greater than 81 times the sum of the squares of the digits by 20 times the square of the sum of the digits

51. Solve the following equations, and verify the solutions

$$\text{(i)}\ \frac{x}{x+a}+\frac{x}{x+b}=2, \qquad \text{(ii)}\ \frac{3}{x}-y=2,\ \frac{1}{3x}+7y=50$$

52. Simplify the following expressions

$$\text{(i)}\ \left(\frac{a^2}{b^3}+\frac{1}{a}\right)-\left(\frac{a}{b^2}-\frac{1}{b}+\frac{1}{a}\right), \qquad \text{(ii)}\ \frac{3+x}{1+3x}-\frac{3-x}{1-3x}-\frac{1+16x}{9x^2-1}$$

53 Find the length of the side of a square carpet if when a border, whose width is 1 foot, is put round it the area is increased by 40 square feet

54 Shew that if two expressions have a common factor it will divide their sum and difference By means of this principle and the Remainder Theorem, find the H C F of

$$3x^3-13x^2+23x-21 \quad \text{and} \quad 6x^3+x^2-44x+21$$

55. A man bicycles half the distance from one town to another at 12 miles per hour, and the other half at 8 miles per hour; a second man bicycles all the way at $11\frac{1}{4}$ miles per hour If the difference in the times they take is $5\frac{1}{2}$ minutes, what is the whole distance?

56 Draw a graph from which the equivalent of a pressure given in pounds per square inch may be read off in kilograms per square centimetre, given that 27 lbs per sq in is approximately equivalent to 1 90 Kg per sq cm

Read off the equivalents of 30 lbs and 57 lbs per sq in

Express 2 55 Kg per sq cm in lbs per sq in

K.

57. Resolve into factors

$$\text{(i)}\ 12x^2-xy-y^2; \qquad \text{(ii)}\ a^3+2a^2+2a+1$$

Prove that x^2+px+q is divisible by $x-a$ if $a^2+pa+q=0$

58. Find the L C M of

$$a^3b^3(a^3-b^3),\quad a^2b^4(a^4-b^4),\quad a^3b(a-b)^2,\quad a^2+ab+b^2$$

59. Solve the following equations, and test the solutions

$$\text{(i)}\ \frac{3(x-2)}{2x-3}=\frac{3x-13}{2(x-4)}; \qquad \text{(ii)}\ \frac{px}{x-q}+\frac{qx}{x-p}=p+q.$$

60. Find the value of

$$\text{(i)}\ \frac{a}{a+b}-\frac{a}{b-a}+\frac{2a^2}{a^2+b^2}-\frac{4a^2b^2}{b^4-a^4},$$

$$\text{(ii)}\ \frac{a^2-2ax-x^2}{2(a^2-x^2)}-\frac{2ax(a-x)}{(a-x)(a^2+2ax+x^2)}-\frac{x^2-a^2}{2(x-a)^2}$$

61. There are two mixtures of wine and water, one of which contains twice as much water as wine, and the other three times as much wine as water How much must be taken from each in order to fill a pint cup, in which the water and the wine shall be equally mixed?

62. A man rides one-third of the distance from A to B at the rate of a miles per hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles an hour, he could have ridden from A to B and back again in the same time Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$$

63. A man buys 100 eggs for 5*s* and has to pay 1*s* 8*d* for carriage He wishes to sell them so as to gain 15 per cent on his whole outlay. Draw a graph to shew to the nearest penny the selling price of any number of eggs up to 100, and read off the price of 65 From the graph find the number of eggs which could be bought for 6*s* 8*d*

CHAPTER XXIV

GRAPHS OF QUADRATIC FUNCTIONS.

264 ANY function which involves the square of the variable x, but no higher power, is called a **quadratic function of x**, or a **function of the second degree in x.**

We shall begin by tracing the graph of the simplest form of such a function, viz x^2

EXAMPLE *Draw the graph of* $y=x^2$

This is one of the most useful graphs the pupil will meet with, it is, therefore, important to plot the curve carefully on a suitable scale

Positive values of x and y may be tabulated as follows

x	0	0 5	1	1 5	2	2 5	3	3 5	4	
y	0	0 25	1	2·25	4	6·25	9	12 25	16	

These values shew that it will be convenient to take the x unit four times as great as the y unit

Now if we take the following *negative* values of x

$$-0\ 5,\ \ -1,\ \ -1\ 5,\ \ -2,\ \ -2\ 5,\ \ -3,\ \ -3\ 5,\ \ -4,$$

we shall obtain the same series of values for y as before

If the points we have now determined are plotted and connected by a continuous line drawn freehand, we shall obtain the curve shewn in Fig 16 This curve is called a parabola, and the point O is known as its vertex.

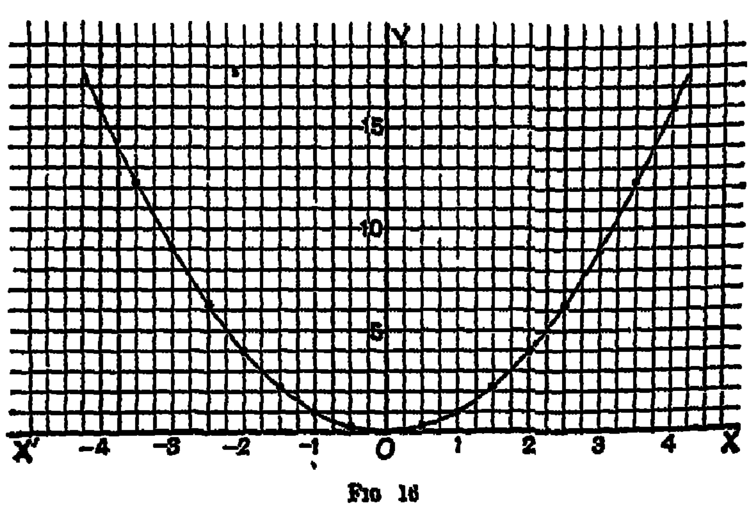


FIG 16

There are three facts to be specially noted in this example

(i) Since from the equation we have $x=\pm\sqrt{y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign* Hence the graph is symmetrical with respect to the axis of y, so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes

(ii) We observe that all the plotted points lie above the axis of x This is evident from the equation, for since x^2 must be positive for all values of x, every ordinate obtained from the equation $y=x^2$ must be positive

In like manner the pupil may shew that the graph of $y=-x^2$ is a curve similar in every respect to that in Fig 16, but lying entirely below the axis of x

(iii) As the numerical value of x increases that of y increases very rapidly Hence, as there is no limit to the values which may be selected for x, it follows that the curve extends upwards and outwards to an infinite distance in both the first and second quadrants

265 Any equation of the form $y=ax^2$, where a is constant, will represent a parabola If a is a positive integer, the curve will be as in Fig 16, but will rise more steeply in the direction of OY If a is a positive fraction, we shall have a flatter curve, extending more rapidly to right and left of OY If a is negative, the curve will lie below the x-axis, and will be steeper or flatter than the graph of $y=x^2$, according as a is numerically greater or less than unity In every case the origin is the vertex of the parabola, and the axis of x is a tangent at that point

EXAMPLES XXIV a

1. Draw the graph of $y=x^2$, taking 1 inch as unit on both axes, and using the following values of x

$$-0{\cdot}4,\quad -0{\cdot}3,\quad -0{\cdot}2,\quad -0{\cdot}1,\quad 0,\quad 0{\cdot}1,\quad 0{\cdot}2,\quad 0{\cdot}3,\quad 0{\cdot}4$$

2 Taking 1 inch as unit for x, and 0·1 inch as unit for y, draw the graphs of

(i) $y=8x^2$, (ii) $y=-8x^2$, (iii) $y=16x^2$

[*Choose values of* x *differing by* 0·25 *between* -2 *and* $+2$]

3. Draw the graphs of $y=x^2$ and $x=y^2$ on a large scale, and shew that they have only one common chord Find its equation

4. Plot the graph of $y=x^2$ for values of x between -5 and $+5$ Read off the approximate values of

(i) $(2{\cdot}7)^2$, (ii) $(\pm 3{\cdot}6)^2$, (iii) $(4{\cdot}2)^2$, (iv) $(-1{\cdot}9)^2$.

266 The most general form of a quadratic function of x is ax^2+bx+c It will be found that the graph of such a function is always a parabola, differing in shape and position according to the values of a, b, c

EXAMPLE *Find the graph of* $y=2x+\frac{x^2}{4}$

Here the following arrangement will be found convenient

x	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9
$2x$	6	4	2	0	−2	−4	−6	−8	−10	−12	−14	−16	−18
$\frac{x^2}{4}$	2·25	1	·25	0	·25	1	2·25	4	6·25	9	12·25	16	20·25
y	8·25	5	2·25	0	−1·75	−3	−3·75	−4	−3·75	−3	−1·75	0	2·25

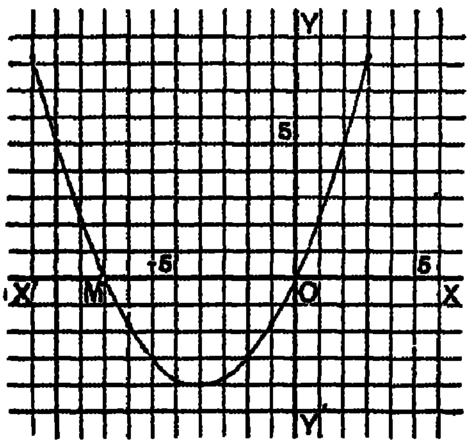


FIG 17

From the form of the equation it is evident that every positive value of x will yield a positive value of y and that as x increases y also increases Hence the portion of the curve in the first quadrant lies as in Fig 17, and can be extended indefinitely in this quadrant In the present case only two or three positive values of x and y need be plotted, but more attention must be paid to the results arising out of negative values of x It is found that the values of y are negative between $x=0$ and $x=-8$ When $x=-8$, $y=0$, and the curve crosses the x axis, after this the values of y are positive

267 In the last example, since the value of $\frac{x^2}{4}+2x$ is represented by y, the *expression* $\frac{x^2}{4}+2x$ becomes zero when the ordinate is zero Thus we can obtain the roots of the *equation* $\frac{x^2}{4}+2x=0$ by reading off the values of x at the points where the curve cuts the x-axis These are $x=0$, $x=-8$, at the points O and M

We can apply this method to an equation of any degree: thus if any function of x is represented by $f(x)$, the solution of the equation $f(x)=0$ may be obtained by plotting the graph of $y=f(x)$, and then measuring the intercepts made on the axis of x These intercepts are values of x which make y equal to zero, and are therefore roots of $f(x)=0$.

268 In the graph of $y=x^2$ (Fig 16) it will be noticed that as we pass from right to left along the curve the ordinate is constantly decreasing until it becomes zero at O, after this the ordinate begins to increase The point at which this change takes place in a graph is known as a **turning-point** Thus the origin is a turning-point of $y=x^2$, and of all curves represented by an equation of the form $y=ax^2$ Again in Fig 17 there is a turning-point at the point $(-4, -4)$ In each of these cases the algebraically least value of the ordinate is found at the turning-point

269 If a function gradually increases till it reaches a value a, which is algebraically greater than neighbouring values on either side, a is said to be a **maximum** value of the function

If a function gradually decreases till it reaches a value b, which is algebraically less than neighbouring values on either side, b is said to be a **minimum** value of the function

The following illustration will make these points clearer

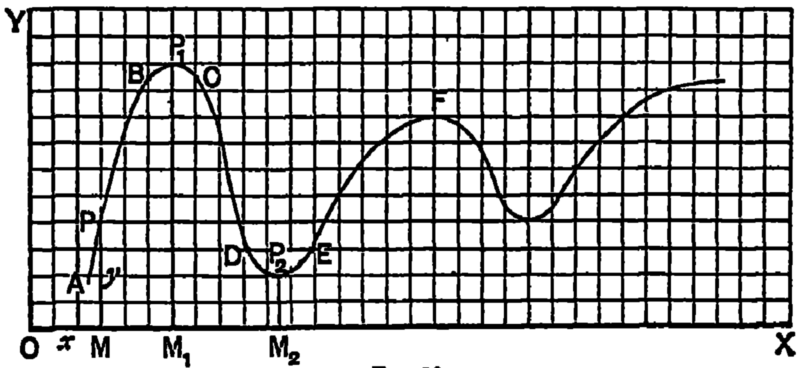


FIG 18

In this figure the continuous curve ABCDEF represents the graph of a variable quantity $f(x)$ As x increases gradually, the ordinate y travels parallel to OY, and its value at any point gives the value of $f(x)$ for the corresponding value of x At P_1 the value of y is greater than that at B or C on either side, and here $f(x)$ is a maximum Similarly at P_2 the value of y is less than that at D or E, and here $f(x)$ is a minimum

It will now be evident that maximum and minimum values occur at the turning-points where the ordinates are algebraically greatest and least respectively in the immediate vicinity of such points

The following points should also be noticed

(i) In any continuous curve maximum and minimum values occur alternately

(ii) There will always be a maximum or a minimum value between any two equal values of the ordinate

(iii) The slope of the curve at any point indicates the rate of change at that point of the function under discussion, and at each point of maximum or minimum value the tangent to the curve is parallel to the axis of x

270 The following example should be studied very carefully

EXAMPLE *Draw the graph of* $y=3-4x-4x^2$ *Thence find the roots of the quadratic equation* $4x^2+4x-3=0$ *Shew that the expression* $3-4x-4x^2$ *is positive for all real values of* x *between* 0·5 *and* −1·5, *and negative for all real values of* x *outside these limits Also find the maximum value of the expression* $3-4x-4x^2$

Take 0·4″ as unit for x, and 0·1″ as unit for y, and use the following table of values

x	2	1·5	1	0·5	0	−0·5	−1	−1·5	−2	−2·5
$-4x$	−8	−6	−4	−2	0	2	4	6	8	10
$-4x^2$	−16	−9	−4	−1	0	−1	−4	−9	−16	−25
y	−21	−12	−5	0	3	4	3	0	−5	−12

After plotting these points we have the graph given in Fig 19

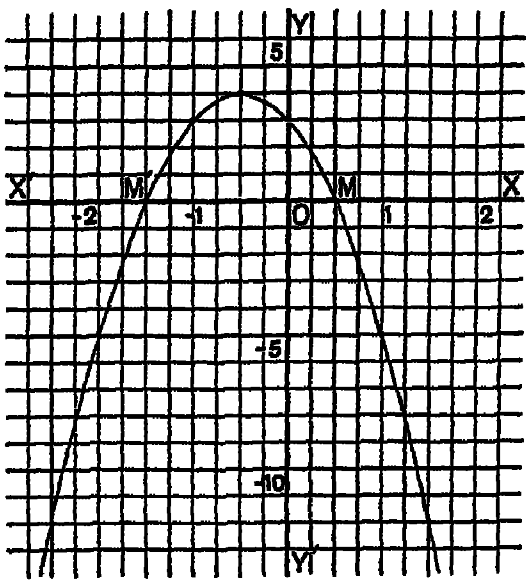


FIG 19

The roots of the equation $4x^2+4x-3=0$ are the values of x which make y equal to 0 These are found at the points M and M′ where the curve cuts the x-axis Thus the required roots are 0·5 and −1·5

Again between the points M and M′ the graph lies above the x-axis; that is, the value of y, or $3-4x-4x^2$, is positive so long as x lies between 0·5 and −1·5, and is negative for other values of x.

The maximum value of the expression $3-4x-4x^2$ is the value of the greatest ordinate in the graph, namely 4

The maximum value of the expression $3-4x-4x^2$ may also be found as follows

$$3-4x-4x^2=3-(4x^2+4x)$$

Complete the square within the bracket (Art 198), thus

$$3-4x-4x^2=3+1-(4x^2+4x+1)=4-(2x+1)^2$$

Since $(2x+1)^2$ cannot be negative for any real value of x, the expression $3-4x-4x^2$ will be greatest when $2x+1=0$, or $x=-\frac{1}{2}$ Thus the maximum value is 4

Similarly to find the *minimum* value of x^2+6x-3

We have $x^2+6x-3=(x^2+6x+9)-3-9=(x+3)^2-12$

This expression will have its least value when $x+3=0$ Hence the required minimum value is -12

This may be illustrated by drawing the graph of $y=x^2+6x-3$

271 **Infinite and Zero Values.** Consider the fraction $\frac{a}{x}$, in which the numerator a has a *certain fixed value*, and the denominator is a *quantity subject to change*, then it is clear that the smaller x becomes the larger does the value of the fraction $\frac{a}{x}$ become

Thus $\quad \frac{a}{0 \cdot 1}=10a, \quad \frac{a}{0 \cdot 01}=100a, \quad \frac{a}{0 \cdot 0001}=10000a,$ and so on

By making the denominator x sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please, that is, if x is made *less than any quantity that can be named*, the value of $\frac{a}{x}$ will become *greater than any quantity that can be named*

A quantity *less* than any assignable quantity is called **zero** and is denoted by the symbol 0 A quantity *greater* than any assignable quantity is called **infinity** and is denoted by the symbol ∞ Hence we may now say briefly

$$\textit{when } x=0, \textit{ the value of } \frac{a}{x} \textit{ is } \infty$$

Again if x is a quantity which gradually increases and finally becomes *greater than any assignable quantity* the fraction becomes *smaller than any assignable quantity* Or more briefly

$$\textit{when } x=\infty, \textit{ the value of } \frac{a}{x} \textit{ is } 0$$

When the symbols for zero and infinity are used in the sense above explained, they are subject to the rules of signs which affect other algebraical symbols Thus we shall find it convenient to use a concise statement such as "when $x=+0$, $y=+\infty$" to indicate that when a *very small and positive* value is given to x, the corresponding value of y is *very large and positive*

EXAMPLE. *Find the graph of $xy=4$ Shew that it consists of two infinite branches, one in the first and the other in the third quadrant*

The equation may be written in the form $y=\frac{4}{x}$,
from which it appears that when $x=0$, $y=\infty$ and when $x=\infty$, $y=0$ Also y is positive when x is positive, and negative when x is negative Hence the graph must lie entirely in the first and third quadrants

Take the positive and negative values of the variables separately

(1) *Positive values*

x	0	1	2	3	4	5	6		∞
y	∞	4	2	$1\frac{1}{3}$	1	8	$\frac{2}{3}$		0

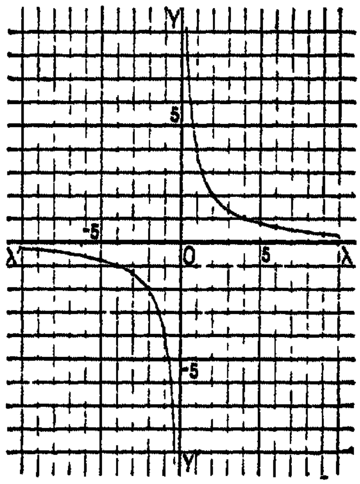


FIG 20

Graphically these values shew that as we recede further and further from the origin on the x axis in the positive direction, the values of y are positive and become smaller and smaller Hence the graph is continually approaching the x axis in such a way that by taking a sufficiently great positive value of x we obtain a point on the graph as near as we please to the x axis but never actually reaching it until $x=\infty$ Similarly, as x becomes smaller and smaller the graph approaches more and more nearly to the positive end of the y axis, never actually reaching it as long as x has any finite positive value, however small

(2) *Negative values*

x	-0	-1	-2	-3	-4	-5		$-\infty$
y	$-\infty$	-4	-2	$-1\frac{1}{3}$	-1	- 8		-0

The portion of the graph obtained from these values is in the third quadrant as shewn in Fig 20, and exactly similar to the portion already traced in the first quadrant It should be noticed that as x passes from $+0$ to -0 the value of y changes from $+\infty$ to $-\infty$ Thus the graph which in the first quadrant has run away to an infinite distance on the *positive* side of the y axis, reappears in the third quadrant coming from an infinite distance on the *negative* side of that axis Similar remarks apply to the graph in its relation to the x axis

This curve is known as a **rectangular hyperbola** Any equation of the form $xy=c$, where c is constant, will give a graph similar in form to that in Fig 20

272 When a curve continually approaches more and more nearly to a line without actually meeting it until an infinite distance is reached, such a line is said to be an **asymptote** to the curve In the above example each of the axes is an asymptote, and the curve is called a *rectangular* hyperbola because in this case the asymptotes are *at right angles*

273 The distance from the origin of any point $P(x, y)$ is given by the relation $OP^2=x^2+y^2$ Hence any equation of the form $x^2+y^2=a^2$, where a is constant, represents a circle, of radius a, whose centre is at the origin, since every point (x, y) which satisfies the equation is at a constant distance a from the origin

EXAMPLE *Solve graphically the simultaneous equations*

$$\text{(i) } x^2+y^2=41, \qquad \text{(ii) } y=2x-3$$

The graph of (i) is a circle By putting the equation in the form $y=\pm\sqrt{41-x^2}$, we easily find by trial that the equation is satisfied by $x=4$, $y=5$ This determines a point P on the graph, which can now be drawn by describing a circle with centre O and radius OP

The graph of (ii) is a straight line, which cuts the axes at the points (1 5, 0), (0, -3) This line cuts the circle at P and Q

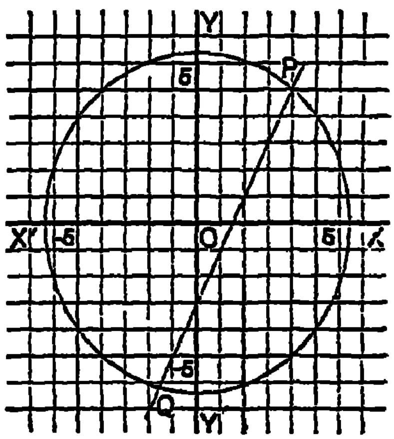


Fig 21

At the points where the two graphs meet they will be satisfied by the same values of x and y The coordinates of these points are (4, 5) and (-1 6, -6 2)

Thus the solution of the equations is given by

$$x=4,\ y=5, \text{ and } x=-1{\cdot}6,\ y=-6{\cdot}2$$

274 The graphical treatment of quadratic functions and equations will be further illustrated in the following chapter. In particular we shall explain how the roots of equations may be found graphically by a method which is sometimes shorter and more easy of application than that illustrated in Art 270 For the present it is sufficient for the pupil to remember the following general principles·

(i) The roots of an equation $f(x)=0$ may always be found by first tracing the graph of $y=f(x)$ and then reading off the abscissæ of the points where it is cut by the axis of x.

A fairly large unit for x should always be chosen If one inch is taken as the x-unit it will be possible to read accurately to tenths of the unit, and hundredths may be estimated as explained in Art 138

(ii) *Any two* simultaneous equations in two unknowns x and y may be solved by tracing the graphs of the equations and reading off the coordinates of their points of intersection

EXAMPLES XXIV. b.

[*In the following examples the scales of measurement on the two axes must be very carefully chosen Much time will be saved if the selection of scales is postponed until after the corresponding values of the variables have been tabulated*]

1. With 0 5" as unit for x and 0 1" as unit for y, plot the graphs of the following quadratic functions

(i)	x^2-3x+2	for integral values of x	from	-3 to 5,
(ii)	$(x+3)(x+2)$	,,	,,	-5 to 4;
(iii)	$6+x-x^2$	,,	,,	-4 to 5;
(iv)	$9-6x+x^2$	,,	,,	-1 to 7,
(v)	$4x^2-4x-15$	,,	,,	-2 to 3

Find the maximum value of (iii) and the minimum value of (v)

2. Draw the graphs of

$$\text{(i) } y=2x-\frac{x^2}{4}, \qquad \text{(ii) } y=x^2+2x-4$$

In each case give the coordinates of the turning point

3. Find graphically the roots of the following equations to 2 places of decimals

$$\text{(i) } \frac{x^2}{4}+x-2=0, \qquad \text{(ii) } x^2-2x=4; \qquad \text{(iii) } 4x^2-16x+9=0$$

Deduce solutions of

$$\text{(iv) } \frac{x^2}{4}+x-2=6; \qquad \text{(v) } x^2-2x=8; \qquad \text{(vi) } 4x^2-16x+9=-6$$

[*The roots of* (iv) *will be the values of* x *which satisfy the equations* $y=\frac{x^2}{4}+x-2$, *and* $y=6$ *simultaneously*]

4 On a large scale draw the graph of $x^2-7x+11$, hence find the roots of the equation $x^2-7x+11=0$, and the minimum value of the expression $x^2-7x+11$

5. Draw the graph of $4-3x-x^2$ and deduce the value of x when the function is a maximum

6 Draw the graph of $y=\frac{1}{2}(x-1)(2-x)$ from $x=-1$ to $x=5$, and find approximately the roots of the equation $x^2+1=3x$ Deduce the solution of $x^2-4=3x$

7 Plot the graph of $y=\frac{1}{4}(3+6x-x^2)$ from $x=-1$ to $x=7$

Find from the graph the approximate values of the roots of the equation $x^2-6x-3=0$

8 Find graphically, and algebraically, the maximum value of $5+4x-2x^2$, and the minimum value of x^2-2x-4 In each case give the coordinates of the turning-points

9. Draw the graph of $y=(x-1)(x-2)$ and find the minimum value of $(x-1)(x-2)$ Measure, as accurately as you can, the values of x for which $(x-1)(x-2)$ is equal to 5 and 9 respectively

10 Shew graphically that the expression x^2-4x+7 is positive for all real values of x

11 Shew graphically that the expression x^2-2x-8 is negative for all values of x between -2 and 4, and positive for all values of x outside those limits

12 Plot the graph of $y=1{\cdot}2+1{\cdot}8x-0{\cdot}6x^2$ between the values $x=-1$ and $x=5$

Find, from the graph, the greatest value of y

13. Draw the graphs of $y=1+\frac{1}{3}x$, and $2y=x(x+3)$, for values of x from -4 to $+2$, and find from the figure the values of x where the two graphs intersect

14 Draw the graph of $y=(2+x)(3-x)$, and find the maximum value of $(2+x)(3-x)$ Also find, as accurately as possible, the values of x for which $(2+x)(3-x)$ is equal to 2

15 On a large scale draw the graphs of

(i) $xy=1$, (ii) $xy=-6$

16 Draw the graphs of $x^2+y^2=53$, $y-x=5$, and find the coordinates of the points where they meet

17 Solve the following pairs of equations graphically

(i) $x+y=15$,	(ii) $x-y=3$,	(iii) $x^2+y^2=13$,
$xy=36$,	$xy=18$,	$xy=6$

Explain why (iii) has four solutions while (i) and (ii) each have only two.

CHAPTER XXV.

QUADRATIC EQUATIONS AND FUNCTIONS

275 SOME easy types of quadratic equations have been given in Chap XVII In the present chapter the solution of quadratics will be treated more fully

276 Standard Form Any equation which involves the square of one unknown quantity x, but no higher power, can by suitable reduction be written in the form

$$ax^2+bx+c=0,$$

where a, b, c are known quantities, *and the term* ax^2 *is positive*

Solution by Factorization.

277 The *expression* ax^2+bx+c is said to be the **quadratic expression or function** which corresponds to the quadratic equation $ax^2+bx+c=0$

In each case the term c, which does not involve x, is spoken of as **the constant or absolute term**

If the *expression* can be put into linear factors the roots of the *equation* can at once be found · [See Arts 202–204]

We give two further examples

EXAMPLE 1 *Solve the equation* $\dfrac{9x}{4}+\dfrac{x-9}{x}=1$

Clearing of fractions, $\quad 9x^2+4x-36=4x,$

$$9x^2-36=0,$$

or $\quad x^2-4=0, \qquad (1)$

that is, $\quad (x-2)(x+2)=0,$

whence $\quad x-2=0, \quad$ or $\quad x+2=0,$

the required roots are 2 and -2

Or thus From (1) we obtain $x^2=4$,

whence by taking the square root of each side, $x=\pm 2$

In such a case it is not necessary to write the double sign on both sides, for $\pm x=\pm 2$ gives the four cases

$$+x=+2, \quad -x=-2, \quad +x=-2, \quad -x=+2$$

The first two of these statements give the same result, viz. $x=+2$, while the last two give $x=-2$ Hence in extracting the square root of both sides of an equation it is sufficient to put the double sign on *one* side

EXAMPLE 2 *Solve the equation* $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}$

Clearing of fractions, $4x(x+2)-5x(x-1)=3(x-1)(x+2)$;

simplifying, $4x^2+8x-5x^2+5x=3x^2+3x-6$;

that is, $-4x^2+10x+6=0$

Dividing by -2, making the square term positive,

$$2x^2-5x-3=0,$$

that is, $(2x+1)(x-3)=0$,

whence $x-3=0$, or $2x+1=0$,

the required roots are 3 and $-\frac{1}{2}$

EXAMPLES XXV. a.

Solve the following equations and verify the solutions

1. $(x+3)(2x-1)=0$
2. $(x-a)(x+2a)=0$
3. $x(x-c)=0$.
4. $(3x+7)(2x-5)=0$
5. $(2x-m)(x+2m)=0$
6. $x^2-p^2=0$
7. $3x^2-10x+3=0$
8. $6x^2-13x+6=0$
9. $2x^2-15a^2=ax$
10. $2x^2-7x=39$
11. $2x(x+1)=15+x$
12. $3x^2-2ax-bx=0$.
13. $x^2-\dfrac{3x}{4}-\dfrac{1}{8}=0$
14. $9x=\dfrac{40b^2}{x+b}$
15. $\dfrac{x}{a}-\dfrac{a}{x}=\dfrac{x+5a}{x}$.
16. $\dfrac{x+10}{x-5}-\dfrac{10}{x}=\dfrac{11}{6}$.
17. $\dfrac{x+2}{x-1}+\dfrac{x-4}{2x}=\dfrac{7}{2}$
18. $\dfrac{7}{x+5}-\dfrac{1}{x-3}=1\frac{2}{3}$
19. $\dfrac{2x}{x+1}-\dfrac{1}{1-x^2}+1=0$
20. $\dfrac{2x}{x-1}+\dfrac{3x-1}{x+2}=\dfrac{5x-11}{x-2}$

Solution by Completing the Square.

278 When the factors of a quadratic expression are not easily found by trial, we resort to the process known as completion of the square [Art 198] The solution of a quadratic equation can be made to depend upon the same principle

From the identities

$$x^2+2ax+a^2=(x+a)^2, \quad x^2-2ax+a^2=(x-a)^2,$$

we see that in a quadratic function of x, which is a perfect square with $+1$ *as the coefficient of* x^2, the constant term is always *the square of half the coefficient of* x Hence if the terms of a quadratic equation are so arranged that the terms involving x^2 and x are on the left-hand side with the coefficient of x^2 *unity and positive*, we can complete the square on the left-hand side by adding the square of half the coefficient of x

EXAMPLE 1 *Solve the equation* $x^2+16x=57$.

The square of half 16 is 8^2, or 64

$$x^2+16x+8^2=57+64,$$

that is,
$$(x+8)^2=121,$$
$$x+8=\pm 11$$

Hence we now have the two simple equations

$$x+8=11, \text{ and } x+8=-11,$$
$$x=3, \text{ or } -19$$

EXAMPLE 2 *Solve the equation* $13x=x^2+42$

Transpose so as to have the terms involving x on the left hand side, and the square term positive

Thus
$$x^2-13x=-42.$$

Completing the square,

$$x^2-13x+\left(\frac{13}{2}\right)^2=-42+\frac{169}{4},$$

that is,
$$\left(x-\frac{13}{2}\right)^2=\frac{1}{4},$$
$$x-\frac{13}{2}=\pm\frac{1}{2},$$
$$x=\frac{13}{2}\pm\frac{1}{2},$$
$$x=7, \text{ or } 6$$

NOTE. We do not work out $\left(\frac{13}{2}\right)^2$ on the left-hand side

[*Examples* XXV b 1–12, *page* 254, *may conveniently be taken here*]

279. When the coefficient of x^2 is not unity we must divide the equation throughout by the coefficient of x^2, before completing the square

EXAMPLE 1 *Solve the equation* $\dfrac{3x-8}{x-2}=\dfrac{5x-2}{x+5}$

Simplifying, we have $3x^2+7x-40=5x^2-12x+4$;

that is,
$$-2x^2+19x=44$$

Dividing by -2,
$$x^2-\frac{19x}{2}=-22$$

Completing the square,

$$x^2-\frac{19x}{2}+\left(\frac{19}{4}\right)^2=-22+\frac{361}{16},$$

that is,
$$\left(x-\frac{19}{4}\right)^2=\frac{9}{16};$$
$$x-\frac{19}{4}=\pm\frac{3}{4};$$
$$\therefore x=\frac{19\pm 3}{4}, \quad \text{whence } x=\frac{11}{2}, \text{ or } 4.$$

280 Roots which cannot be expressed in an exact numerical form are called irrational quantities

Thus $\sqrt{6}$, $\sqrt{10}$ $\sqrt[3]{5}$ are irrational Any quantities which do not involve such roots are, for the sake of distinction, called rational.

NOTE. The meaning of the terms *irrational* and *unreal* (Art 184) must not be confused The irrational quantities quoted above are all real, while an unreal quantity, since it involves the square root of a negative number, must always be irrational Rational quantities are always real, while irrational quantities may be real or unreal

EXAMPLE *Solve the equation* $9x^2-12x-1=0$

We have $$x^2-\frac{4}{3}x=\frac{1}{9}$$

Completing the square, $$x^2-\frac{4}{3}x+\left(\frac{2}{3}\right)^2=\frac{4}{9}+\frac{1}{9},$$

that is, $$\left(x-\frac{2}{3}\right)^2=\frac{5}{9},$$

$$x-\frac{2}{3}=\frac{\pm\sqrt{5}}{3},$$

$$x=\frac{2\pm\sqrt{5}}{3}.$$

Since 5 is not a square number the roots of this equation are irrational. Hence in this case no numerical quantity, positive or negative, can be found which will *exactly* satisfy the given equation, but the value of $\sqrt{5}$ may be found to any required degree of accuracy

Thus $\sqrt{5}=2{\cdot}236$ to four significant figures, and to the same degree of accuracy the roots of the above equation are

$$\frac{4{\cdot}236}{3} \text{ and } -\frac{0{\cdot}236}{3}, \text{ or } 1{\cdot}412 \text{ and } -0{\cdot}079$$

NOTE The roots of the *equation* $9x^2-12x-1=0$ are *irrational* because the *expression* $9x^2-12x-1$ has no rational factors In such cases it is futile to attempt to solve the equation by the method of factorization.

281 The process of solving a quadratic by completing the square requires the following steps of work

(1) *If necessary, arrange the equation so that the terms in* x^2 *and* x *are on the left-hand side and the constant term on the other.*

(2) *Divide throughout by the coefficient of* x^2, *if that coefficient is not unity*

(3) *Complete the square on the left-hand side by adding to each side of the equation the square of half the coefficient of* x

(4) *Take the square root of each side, and solve the two resulting simple equations*

282 In all the instances considered hitherto the quadratic equations have had two unequal roots Sometimes, however, there is only one solution Thus if $x^2-6x+9=0$, then $(x-3)^2=0$, whence $x=3$ is the only solution In such cases it is convenient to say that the quadratic has *two equal roots*

EXAMPLES XXV. b.

Solve the following equations by completing the square Verify the solutions in Examples 1-12

1. $x^2-12x=85$ 2. $x^2+8x-105=0$ 3 $14x=240-x^2$

4. $x^2-x-56=0$ 5 $x^2+7x=98$ 6. $x(x+10)=299$

7. $x(22-x)=57$ 8. $x+88=x(x-2)$ 9 $x^2-341=20x$

10. $38x-357=x^2$ 11. $x^2+6x=247$ 12 $x^2-238=3x$

13 $2x^2+3x=2$ 14. $3x^2+7x-6=0$ 15. $2x^2-x=15$

16 $4x^2+11x=3$ 17. $5=3x^2+14x$ 18. $6x^2+35=31x$

19. $3+11x=4x^2$ 20. $18+5x^2=33x$ 21. $20-9x=20x^2$.

22. $12x^2-17x+6=0$ 23 $6x^2+35x=6$ 24 $28=31x+5x^2$

25. $27=5x(10x-3)$ 26 $9x(2x-3)=26$ 27 $143=3x(3x-2)$

28. $7x=3(1-2x^2)$ 29 $4x^2+126=65x$ 30 $2(x^2+20)=21x$

31. $x=\frac{12}{7}(1-x^2)$ 32. $\frac{9x}{25}+\frac{25}{9x}=2$ 33 $\frac{2+x}{8+3x}=\frac{x-5}{2+5x}$

34 $10x-\frac{6}{x}+11=0$ 35. $14x-\frac{10}{x}-31=0$ 36. $\frac{3x-5}{3}=\frac{1}{x}+\frac{4x-7}{5}$

37 $\frac{3x+4}{19}=\frac{x^2}{4x+3}$ 38. $\frac{1}{x}\left(16-\frac{9}{x}\right)=7\frac{1}{9}$ 39 $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$

40 $\frac{x+3}{x-2}-\frac{1-x}{x}=4\frac{1}{4}$ 41. $\frac{5}{5-x}+\frac{8}{8-x}=3$ 42. $\frac{7}{x+3}-\frac{1}{x-5}=1\frac{2}{3}$

[*In Examples 43-56 the values of irrational roots should be given correct to 2 decimal places A Table of square roots will be found on page 263*]

43 $x^2-6x+7=0$ 44. $x^2+11=8x$ 45 $x^2-4x+1=0$

46. $2x^2-6x+3=0$ 47. $x^2+2x=4$ 48 $x^2=5x-3$

49. $9x^2-6x=5$ 50 $7x^2-12x+3=0$ 51 $7x^2+16x+5=0$

52. $3x^2-7x=3$ 53. $5(x^2-1)=9x$ 54 $(x-2)(3+4x)=2x^2$.

55. $\frac{1}{x+1}+\frac{3}{x+2}=\frac{2}{x}$ 56. $\frac{5x-3}{2x+1}-\frac{x-13}{x+4}=2$

Solution by Formula

283 *To solve the general equation* $ax^2+bx+c=0$

Transposing, $$ax^2+bx=-c,$$

dividing by a, $$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Completing the square by adding to each side $\left(\frac{b}{2a}\right)^2$ we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a},$$

that is, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$

extracting the square root, $$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a},$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

284 Since a, b, c may have any values whatever, we see that every quadratic has two roots We may now apply this general formula to any particular case by substituting the numerical values of a, b, and c

EXAMPLE 1 *Solve the equation* $4x^2-10x+5=0$

Apply the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, by putting $a=4$, $b=-10$, $c=5$.

Thus $$x=\frac{10\pm\sqrt{(-10)^2-4\cdot 5\cdot 4}}{2\cdot 4}$$

$$=\frac{10\pm\sqrt{100-80}}{8}=\frac{10\pm\sqrt{20}}{8}$$

Now $\sqrt{20}=4{\cdot}472$ approximately,

$$x=\frac{10\pm 4{\cdot}472}{8}=\frac{14{\cdot}472}{8}, \text{ or } \frac{5{\cdot}528}{8}$$

Thus the roots are 1·81 and 0·69, correct to two decimal figures.

EXAMPLE 2 *Solve the equation* $x^2-3x+3=0$

Here $a=1$, $b=-3$, $c=3$,

$$x=\frac{3\pm\sqrt{(-3)^2-4\cdot 1\cdot 3}}{2}=\frac{3\pm\sqrt{-3}}{2}$$

Now the square root of -3 cannot be represented by any numerical quantity exactly or approximately Thus there is no real value of x which satisfies the equation, and the roots are said to be **imaginary**.

285 The results on the preceding page may be shewn graphically

(i) *To find the roots of* $4x^2-10x+5=0$ *graphically*

Draw the graph of $y=4x^2-10x+5$

x	0	1	2	3
$4x^2$	0	4	16	36
$-10x$	0	-10	-20	-30
y	5	-1	1	11

The roots of the equation are the abscissæ of the points where the graph cuts the x-axis. At these points the ordinate y *changes sign* Hence from the few integral values in the adjoining table, we infer that one root of the equation lies between 0 and 1, and the other between 1 and 2 It will therefore be convenient to plot this part of the curve more minutely, and we need not consider any value of x greater than 2

x	0	0·25	0 5	1	1·25	1 5	2
$4x^2$	0	0 25	1	4	6 25	9	16
$-10x$	0	-2 5	-5	-10	-12 5	-15	-20
y	5	2 75	1	-1	-1·25	-1	1

Take 1 0″ as unit for x, and 0 2″ as unit for y, the graph is given below

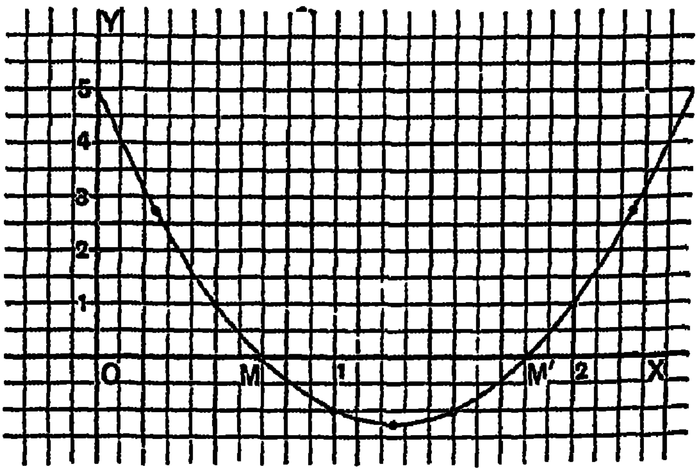


FIG 22

From this figure we find $OM=0$ 69, $OM'=1$ 81, approximately Thus the required roots are 0 69 and 1 81, correct to two decimal figures

NOTE In cases where roots are wanted with great accuracy, and it is inconvenient to use a large scale for the whole figure, it is often advisable to make a rough sketch first, to find approximately the position of points where the graph crosses the x axis A small portion of the curve can then be enlarged in the neighbourhood of such points

(ii) *To solve* $x^2-3x+3=0$ *graphically*

Draw the graph of $y=x^2-3x+3$

Corresponding values of x and y are given below

x	0	0 5	1	1 5	2	2 5	3
x^2	0	0 25	1	2 25	4	6 25	9
$-3x$	0	-1 5	-3	-4 5	-6	-7 5	-9
y	3	1 75	1	0 75	1	1 75	3

Here we may conveniently take 0 5″ as unit for x, and 0 4″ as unit for y The graph is shewn in the adjoining diagram It is evident that the curve does not meet the x-axis In other words there is no real numerical value of x which makes the expression x^2-3x+3 equal to zero

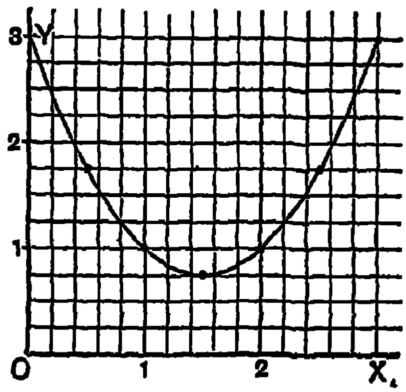


FIG 28

286 From the result $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ it appears that the square root of the compound expression b^2-4ac, *taken as a whole*, enters into every solution Hence in each case the character of the solution depends upon the value of b^2-4ac, which is known as the **discriminant** of the equation $ax^2+bx+c=0$, or of the expression ax^2+bx+c We shall denote the discriminant by the symbol Δ

(i) If Δ is a perfect square, the roots of the equation $ax^2+bx+c=0$ (or the factors of the corresponding expression) are rational and unequal

(ii) If $\Delta=0$, each root of the equation reduces to $-\dfrac{b}{2a}$ Thus the roots are rational and equal So are the factors of the expression ax^2+bx+c

(iii) If Δ is positive but not a perfect square, the roots, though real, are irrational and unequal And the expression ax^2+bx+c has no rational factors

(iv) If Δ is negative, the roots are imaginary And the expression ax^2+bx+c has no real factors

287. The different methods of solving quadratics may be summed up as follows

When the equation has been brought to standard form it may be solved (i) by factorizing the function which stands on the left side, (ii) by transposing the constant term and completing the square on the left side, (iii) by the use of the general formula When the roots are rational the first method is to be preferred if the factorization is fairly simple, in all other cases the second or third method should be adopted. In particular, quadratics with literal coefficients should be solved by factorization, if possible Otherwise the use of the general formula will usually give the readiest solution

EXAMPLE *Solve the equation* $24x^2-5cx-36c^2=0$

By the formula, $x=\dfrac{5c\pm\sqrt{(-5c)^2-4.24\ (-36c^2)}}{48}=\dfrac{5c\pm\sqrt{25c^2+3456c^2}}{48}$

$$=\frac{5c\pm\sqrt{3481c^2}}{48}=\frac{5c\pm 59c}{48};$$

$$x=\frac{4c}{3},\ \text{or}\ -\frac{9c}{8}$$

In the following set of examples the method of solution in each case is left to the pupil's discretion Irrational roots should be given correct to the second decimal figure

EXAMPLES XXV. c.

Solve the following equations

1. $2x^2-5x-12=0$	2. $23x=120+x^2$	3. $x^2-3x=2$
4. $x^2-ax=20a^2$	5 $15x^2+2cx=8c^2$	6 $x^2+3x+1=0$
7. $5x^2-15x+11=0$	8. $36x^2-35b^2=12bx$	9 $42x^2-28c^2=25cx$
10. $x^2-3x=3$	11. $4x^2=x-1$	12 $5x+2=12x^2$
13. $x^2-18x+88=0$	14. $x^2+2x=3\,2$	15 $x^2-3x-3\,51=0$
16. $x^2+x-272=0$	17. $x^2+x=1\,0956$	18 $x^2-2x=783$
19. $2x^2+9ax=180a^2$	20 $3x^2+2x=8$	21. $x^2-14x+12=0$

22 Find two values of x which will make $x(3x-1)$ equal to 0 362, giving each value to the nearest hundredth

23. Solve the equation $x^2+ax-a^2=0$ If $a=12$, give the numerical values of the roots to three decimal places

24. Solve the equation $x(a-x)=c^2$ Give the numerical values of the roots to three decimal places, when $a=16$, $c=6$

288 Combination of two graphs When the variations of a quadratic function have to be examined in detail, the best general method of procedure is that illustrated in Art 270 But the graphical solution of a quadratic equation may often be more readily obtained by combining two graphs, as we shall now shew

EXAMPLE *Solve the equation* $2x^2-x-3=0$ *graphically Between what values of* x *is the function* $2x^2-x-3$ *positive?*

We have to find values of x which will make

$$2x^2=x+3, \quad \text{or} \quad x^2=\frac{x+3}{2}$$

Put $\qquad y_1=x^2 \quad (1), \quad \text{and} \quad y_2=\frac{x+3}{2}, \qquad (2)$

and plot the graphs of these equations on the same axes

Then the required values of x are the abscissæ of the points of intersection of (1) and (2)

For at these points $y_1=y_2$, or $x^2=\frac{x+3}{2}$

For (1) we may use the following values, taking the x unit twice as great as the y unit

x	0	±0 5	±1	±1 5	±2
y	0	0 25	1	2 25	4

Thus we obtain the parabola POQ in Fig 24

The intercepts of (2) on the axes are −3, 1 5; thus the graph of (2) is the straight line PQ

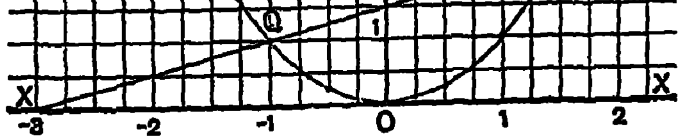


FIG 24

The roots of the equation are given by the abscissæ of P and Q Thus from the figure the roots are 1 5 and −1

Again, the expression $2x^2-x-3$ is positive or negative according as y_1 is greater or less than y_2 From the graph we see y_1 is less than y_2 between Q and P, that is for values of x between $x=-1$ and 1 5, and y_1 is greater than y_2 for all other values of x Hence $2x^2-x-3$ is positive for all values of x except such as lie between −1 and 1 5

289. The solution of the last example might have been effected equally well by drawing the graphs of $y=2x^2$ and $y=x+3$ But if a number of quadratic equations have to be solved graphically it is convenient to reduce them to the form $x^2=px+q$ as a first step. The graph of $y=x^2$ can then be plotted once for all on a suitable scale, and the line $y=px+q$ can be readily drawn on the same scale for different values of p and q

EXAMPLES XXV. d

1. From the graphs of $y=x^2$, $y=2x+8$, deduce the solution of the equation $x^2-2x-8=0$

2 By the method of Art 288, find graphically the roots of the following equations to two places of decimals

(i) $\dfrac{x^2}{4}+x-2=0$, (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$.

[*Take* 1 0″ *as unit for* x, and 0 2″ *as unit for* y]

3 On the same axes draw the graphs of

$$y=x^2, \quad y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6$$

Hence discuss the roots of the four equations

(i) $x^2-x-6=0$, (ii) $x^2-x+6=0$, (iii) $x^2+x-6=0$, (iv) $x^2+x+6=0$

4. Find the roots of $4x^2+4x-3=0$ graphically Shew that the expression $4x^2+4x-3$ is negative for all real values of x between 0 5 and -1 5, and positive for all real values of x outside these limits

5. By considering the graphs of $y=x^2$, $y=4x-7$, shew that x^2-4x+7 is positive for all real values of x

6 On a large scale plot the graphs of $y=x^2$, and $4y=10x-5$ Hence find the roots of the equation $4x^2-10x+5=0$ to two places of decimals

7. Taking 1 0″ as unit for x and 0 5″ as unit for y draw the graph of $y=x^2$, and make use of it to solve the following equations accurately to two places of decimals

(i) $x^2-3x=3$, (ii) $4x^2+4=9x$, (iii) $5x^2-5=9x$

290 There are some equations of higher degree than the second which may be solved by the methods explained in this chapter

EXAMPLE 1 *Solve* $x^4-25x^2+144=0$

By resolution into factors, $(x^2-9)(x^2-16)=0$,

$\therefore \quad x^2-9=0, \quad \text{or} \quad x^2-16=0;$

that is, $\quad x^2=9, \quad \text{or} \quad 16,$

and $\quad x=\pm 3, \quad \text{or} \quad \pm 4$

EXAMPLE 2 *Solve* $x(2x-1)+\frac{6}{2x^2-x}=7$

Write y for $2x^2-x$, then we have

$$y+\frac{6}{y}=7,$$

or

$$y^2-7y+6=0$$

From this quadratic $y=6$, or 1;

$$2x^2-x=6, \text{ or } 1$$

Thus we have *two* quadratics to solve, and finally we obtain

$$x=2, -\tfrac{3}{2}, 1, -\tfrac{1}{2}$$

EXAMPLES XXV e.

Solve the equations

1 $4=5x^2-x^4$ 2. $x^4+36=13x^2$ 3. $4(x^4+c^4)=17c^2x^2$

4 $x^6+7x^3=8$ 5 $x^6-19x^3=216$ 6 $x^6+26c^3x^3=27c^6$

7. $16\left(x^2+\frac{1}{x^2}\right)=257.$ 8. $x^2+\frac{a^2b^2}{x^2}=a^2+b^2$

9 $(x^2+x)^2+72=18(x^2+x)$ 10 $(x^2+2)^2=29(x^2+2)-198$

11. $(x^2-3x)+\frac{40}{x(x-3)}=14$ 12 $x(x-2a)=\frac{8a^4}{x^2-2ax}+7a^2$

291 The method of solution by factors is applicable to equations of higher degree than the second

For example, if $(x-2)(x+1)(x+2)=0,$

the equation must be satisfied by each of the values which satisfy the equations $x-2=0, \quad x+1=0, \quad x+2=0$

Thus the roots are $x=2, -1, -2$

EXAMPLE *Solve the equation* $3x^3+5x^2=3x+5$

Putting the equation in the form

$$3x^3+5x^2-3x-5=0,$$

we have

$$x^2(3x+5)-(3x+5)=0,*$$

or

$$(x^2-1)(3x+5)=0,$$

that is,

$$(x+1)(x-1)(3x+5)=0,$$

whence

$$x+1=0, \text{ or } x-1=0, \text{ or } 3x+5=0$$

Thus the roots are $-1, 1, -\frac{5}{3}$

NOTE At the stage marked with an asterisk we might have removed the factor $3x+5$, but in so doing the factor must be equated to zero to furnish one root of the equation

292 If one root of an equation is known, or can be obtained by trial, a corresponding factor of the first degree can be removed. When this is done we have left an equation of lower degree than the original equation.

EXAMPLE *Solve the equation* $x^3-3x^2-6x+16=0$

By trial it will be found that the left hand side vanishes when $x=2$.

Hence $x=2$ is one root of the equation and corresponding to this root we have a factor $x-2$; the equation may now be written

$$x^2(x-2)-x(x-2)-8(x-2)=0,$$

or

$$(x^2-x-8)(x-2)=0$$

Removing the factor $x-2$, we have $x^2-x-8=0$,

whence

$$x=\frac{1\pm\sqrt{33}}{2}=\frac{1\pm5\cdot745}{2}$$

Thus the three roots are 2, 3·37, −2·37, to two decimal figures

EXAMPLES XXV. f

Solve the following equations by the method of factors

1. $x^3+x^2-x-1=0$
2. $x^3-2x^2-x+2=0$
3. $x^3-4x=x^2-4$
4. $x^3+7x^2+7x-15=0$
5. $x^3-3x-2=0$
6. $x^4+2x=3x^2$
7. $x^3+30=19x$
8. $x^3+6=2x^2+5x$
9. $x^3+6a^3=7a^2x$
10. $2x^3+13x^2=36$

Solve the following equations having given one root in each case

11. $x^3-39x+70=0$ $[x=5]$
12. $x^3-37x-84=0$ $[x=-3]$
13. $x^3-12a^2x=16a^3$ $[x=4a]$
14. $x^4+432a^3x=108a^2x^2$ $[x=6a]$
15. $x^4+40x=8x^3$ $[x=-2]$
16. $4x^4-15x^2+1=0$ $[x=-\frac{1}{4}]$

(*Miscellaneous Examples on Quadratic Equations and Functions*)

17. Find to the nearest tenth the values of x which will make

(i) $2x(2-x)$ equal to 1·73, (ii) $x(5\cdot5-x)$ equal to 7·378

18. Solve the equations:

(i) $\left(x-\frac{1}{x}\right)^2-\frac{77}{12}\left(x-\frac{1}{x}\right)+10=0$, (ii) $\left(\frac{x}{p}-5+\frac{6p}{x}\right)\left(\frac{6x}{p}-5+\frac{p}{x}\right)=0$

19. By using the Discriminant (Art 286) find which of the following equations should be solved by factors, and which by the general formula

(i) $2x^2-3x-7=0$; (ii) $2x^2-5x-7=0$,

(iii) $x^2+x-552=0$, (iv) $x^2+x-550=0$

20. Draw the graphs of x^2+x-2 and of $3x+6$, and find the abscissæ of their common points What algebraical equation has been solved by the above process?

21. In each of the following quadratic expressions find, by means of the Discriminant, whether the factors are rational or irrational, real or unreal:

(i) $x^2-6x+13$, (ii) $x^2-6x-13$,
(iii) $6x^2+5ax-56a^2$, (iv) $4x^2+25x+39$.

22. Solve the equation $x^2-2x-5{\cdot}25=0$,
(i) by drawing the graph of $y=x^2-2x-5{\cdot}25$,
(ii) by combining the graphs of $y=x^2$, $y=2x+5{\cdot}25$,
(iii) " " " $y=x^2-2x$, $y=5{\cdot}25$.
[Fig. 9 *on page* 111 *may be used for* (iii).]

23. Draw the graphs of x^2 and of $3x+1$. By means of them find approximate values for the roots of $x^2-3x-1=0$.

24. Prove that if x is real, $x^2+6x+16$ cannot be less than 7.

25. If x is real prove graphically that $5-4x-x^2$ is not greater than 9, and that $4x^2-4x+3$ is not less than 2. Between what values of x is the first expression positive?

A Table of Square Roots of Numbers from 1 to 150

No	Square Root	No	Square Root.	No	Square Root.	No	Square Root.	No	Square Root.
1	1·000	31	5·568	61	7·810	91	9·539	121	11·000
2	1·414	32	5·657	62	7·874	92	9·592	122	11·045
3	1·732	33	5·745	63	7·937	93	9·644	123	11·091
4	2·000	34	5·831	64	8·000	94	9·695	124	11·136
5	2·236	35	5·916	65	8·062	95	9·747	125	11·180
6	2·449	36	6·000	66	8·124	96	9·798	126	11·225
7	2·646	37	6·083	67	8·185	97	9·849	127	11·269
8	2·828	38	6·164	68	8·246	98	9·899	128	11·314
9	3·000	39	6·245	69	8·307	99	9·950	129	11·358
10	3·162	40	6·325	70	8·367	100	10·000	130	11·402
11	3·317	41	6·403	71	8·426	101	10·050	131	11·446
12	3·464	42	6·481	72	8·485	102	10·100	132	11·489
13	3·606	43	6·557	73	8·544	103	10·149	133	11·533
14	3·742	44	6·633	74	8·602	104	10·198	134	11·576
15	3·873	45	6·708	75	8·660	105	10·247	135	11·619
16	4·000	46	6·782	76	8·718	106	10·296	136	11·662
17	4·123	47	6·856	77	8·775	107	10·344	137	11·705
18	4·243	48	6·928	78	8·832	108	10·392	138	11·747
19	4·359	49	7·000	79	8·888	109	10·440	139	11·790
20	4·472	50	7·071	80	8·944	110	10·488	140	11·832
21	4·583	51	7·141	81	9·000	111	10·536	141	11·874
22	4·690	52	7·211	82	9·055	112	10·583	142	11·916
23	4·796	53	7·280	83	9·110	113	10·630	143	11·958
24	4·899	54	7·348	84	9·165	114	10·677	144	12·000
25	5·000	55	7·416	85	9·220	115	10·724	145	12·042
26	5·099	56	7·483	86	9·274	116	10·770	146	12·083
27	5·196	57	7·550	87	9·327	117	10·817	147	12·124
28	5·292	58	7·616	88	9·381	118	10·863	148	12·166
29	5·385	59	7·681	89	9·434	119	10·909	149	12·207
30	5·477	60	7·746	90	9·487	120	10·954	150	12·247

CHAPTER XXVI.

Simultaneous Equations of the Second or Higher Degree

293 When two unknowns are connected by a pair of equations, one or both of which may be the second or higher degree, there is no method of solution universally applicable A few typical cases deserve special attention

Example *Solve the equations*

$$2x-3y=4, \qquad (1)$$

$$2x^2-3xy-2y^2=12 \qquad (2)$$

From (1), $$x=\frac{3y+4}{2}, \qquad (3)$$

Substituting this value for x in (2), we have

$$\frac{(3y+4)^2}{2}-\frac{3y(3y+4)}{2}-2y^2=12,$$

that is, $$9y^2+24y+16-9y^2-12y-4y^2-24=0.$$

On reduction, $$y^2-3y+2=0,$$

or $$(y-2)(y-1)=0,$$

$$y=2, \text{ or } 1$$

The corresponding values of x can now be found from (3)

When $y=2$, $x=5$, and when $y=1$, $x=\frac{7}{2}$

Thus the solutions are $\left.\begin{array}{l}x=5,\\ y=2,\end{array}\right\}$ $\left.\begin{array}{l}x=\frac{7}{2},\\ y=1\end{array}\right\}$

The method of this example may always be used when one of the equations contains both unknowns in the first degree only

294 Some equations which belong to the above type may be more neatly solved as shewn in the two following examples

Example I *Solve* $$x+y=11, \qquad (1)$$

$$xy=24 \qquad (2)$$

Squaring (1), we have $x^2+2xy+y^2=121$,

from (2), $4xy=96$,

by subtraction, $x^2-2xy+y^2=25$,

$$x-y=\pm 5 \qquad (3)$$

From (1) and (3) we now have two pairs of *simple* equations,

$$\left.\begin{array}{l}x+y=11,\\ x-y=5,\end{array}\right\} \text{ and } \left.\begin{array}{l}x+y=11,\\ x-y=-5\end{array}\right\}$$

By addition and subtraction, we have, after division by 2,

$$\left.\begin{array}{l}x=8,\\ y=3,\end{array}\right\} \text{ or } \left.\begin{array}{l}x=3,\\ y=8\end{array}\right\}$$

EXAMPLE 2 *Solve* $x-2y=8,$ (1)

$xy=24$. (2)

Squaring (1), $x^2-4xy+4y^2=64;$

from (2), $8xy=192,$

by addition, $x^2+4xy+4y^2=256,$

$x+2y=\pm 16$ (3)

Combining (1) and (3), we have

$$\left.\begin{array}{l} x-2y=8, \\ x+2y=16, \end{array}\right\} \text{ and } \left.\begin{array}{l} x-2y=8, \\ x+2y=-16, \end{array}\right\}$$

whence, by addition and subtraction,

$$\left.\begin{array}{l} x=12, \\ y=2, \end{array}\right\} \text{ or } \left.\begin{array}{l} x=-4, \\ y=-6 \end{array}\right\}$$

Corresponding values of the two unknowns should always be arranged accurately in pairs

In the last two examples our object has been to deduce two *simple* equations of the form

$$ax+by=c, \quad ax-by=d,$$

and the solution is then completed by addition and subtraction This method is very frequently useful

EXAMPLE 3 *Solve* $x^2+y^2=73,$ (1)

$xy=24$ (2)

Multiply (2) by 2, then by addition and subtraction we have

$$x^2+2xy+y^2=121, \quad x^2-2xy+y^2=25,$$

whence $x+y=\pm 11, \quad x-y=\pm 5$

These results furnish *four* pairs of simple equations, namely

$$\left.\begin{array}{l} x+y=11, \\ x-y=5, \end{array}\right\} \left.\begin{array}{l} x+y=11, \\ x-y=-5, \end{array}\right\} \left.\begin{array}{l} x+y=-11, \\ x-y=5, \end{array}\right\} \left.\begin{array}{l} x+y=-11, \\ x-y=-5 \end{array}\right\}$$

From these equations, by addition and subtraction, we obtain

$$x=8,\ y=3, \quad x=3,\ y=8, \quad x=-3,\ y=-8, \quad x=-8,\ y=-3$$

Here we have *four* solutions, while in Example 2 there are only *two* These results will be illustrated graphically on the following page

EXAMPLE 4 *Solve* $x^2+y^2=73,$ (1)

$x+y=11$ (2)

By subtracting (1) from the square of (2), we have

$2xy=48$, so that $xy=24$ (3)

Equations (2) and (3) have already been solved in Art 294, Ex 1

NOTE If the solution were completed by means of (1) and (3) we should get *four* solutions, as in Ex 3, but two of these do not satisfy equation (2)

295. Example *Solve the following pairs of equations graphically*

$$\left.\begin{aligned} x^2+y^2&=73, \\ xy&=24 \end{aligned}\right\} \quad (A) \qquad \left.\begin{aligned} x-2y&=8, \\ xy&=24 \end{aligned}\right\} \quad (B)$$

We shall require the graphs of

(i) $x^2+y^2=73$, which is a circle with centre at O [Art. 273]

(ii) $xy=24$, which is a rectangular hyperbola [Art 271]

(iii) $x-2y=8$, which is a straight line

Since (i) is satisfied by $x=3$, $y=8$, we have only to plot the point P(3, 8), and draw a circle with centre O and radius OP

The intercepts of (iii) on the axes are 8 and -4, whence we have the line RS

The hyperbola (ii) must be plotted fully enough to shew its intersections with the circle (i) and the line (iii) The following corresponding values of x and y may be used

$$x=\pm 2, \quad \pm 3, \quad \pm 4, \quad \pm 5, \quad \pm 6, \quad \pm 8, \quad \pm 10, \quad \pm 12,$$
$$y=\pm 12, \quad \pm 8, \quad \pm 6, \quad \pm 4.8 \quad \pm 4, \quad \pm 3, \quad \pm 2.4, \quad \pm 2$$

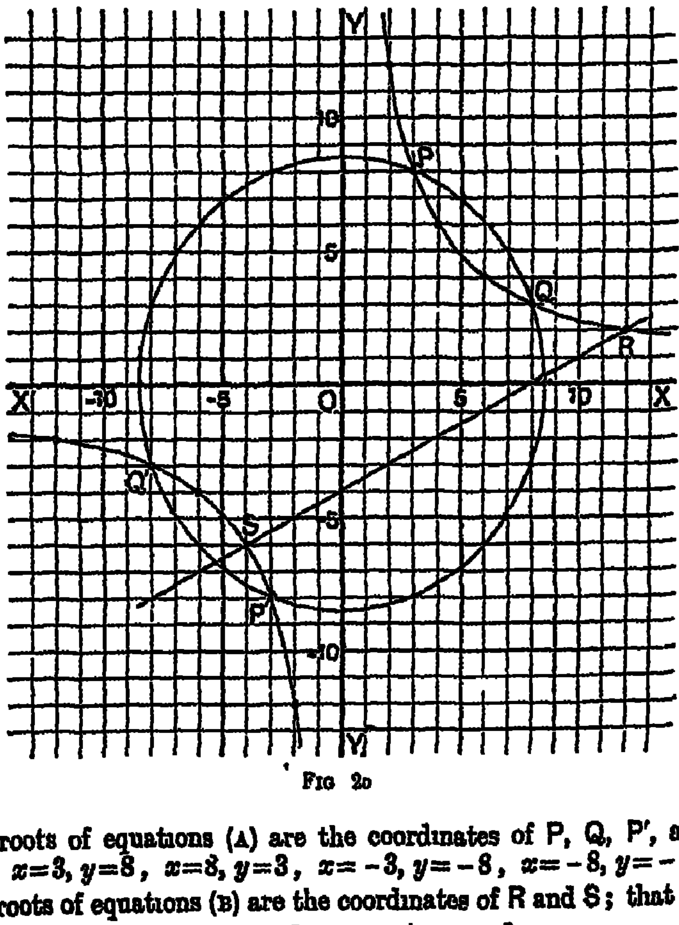


Fig 20

The roots of equations (A) are the coordinates of P, Q, P′, and Q′; that is, $x=3, y=8$, $x=8, y=3$, $x=-3, y=-8$, $x=-8, y=-3$

The roots of equations (B) are the coordinates of R and S; that is,

$$x=12,\ y=2, \quad x=-4,\ y=-6$$

The circle and the hyperbola intersect in *four* points, while the straight line cuts the hyperbola in *two* points only. This accounts for the fact that in Art. 294 there are four solutions in Example 3 and only two in Example 2.

296. Fig. 25 may be used to shew the solutions of certain other pairs of equations.

For example, the intersections of the line RS and the circle give the solutions of the equations

$$x-2y=8,\quad x^2+y^2=73.$$

These are $x=8.5$, $y=0.3$; $x=-5.4$, $y=-6.7$, approximately. These values may be verified algebraically by the method of Art. 293.

Again, if we draw the line $x+y=11$, it will be found to pass through the points P and Q. Thus the coordinates of these points are the roots of the following pairs of equations

$$\left.\begin{aligned} x+y&=11,\\ xy&=24,\end{aligned}\right\}\qquad \left.\begin{aligned} x^2-y^2&=73,\\ x+y&=11,\end{aligned}\right\}$$

each of which has two solutions only, as shewn algebraically in Art. 294, Examples 1 and 4.

EXAMPLES XXVI. a.

Solve the following equations by the method of Art. 293:

1. $x^2-3y=16,\ x=y+2.$	2. $y=x-3,\ x^2-3y^2=13.$	3. $xy=15,\ 2x-y=1.$
4. $2x+y=5,\ x^2-y^2=3.$	5. $x^2+2xy=3,\ 3x+2y=5.$	6. $xy+8=7y,\ x+3=2y.$
7. $x^2+y^2=52,\ 2x+y=8.$	8. $2x+3y=4,\ 2xy+y^2=7y-2.$	9. $x^2-2y^2=1,\ 3x-y=7.$

Solve the following equations by the method of Art. 294:

10. $x+y=11,\ xy=30.$	11. $x+y=14,\ xy=45.$	12. $x-y=2,\ xy=35.$	13. $xy=374,\ x-y=23.$
14. $x-3y=13,\ xy=12.$	15. $2x-y=11,\ xy=21.$	16. $4x-y=24,\ xy+20=0.$	17. $x+5y=19,\ xy=18.$

18. $xy=1054,\ x=y-3.$	19. $x+8=y,\ xy=1280.$	20. $x^2+y^2=74,\ xy=35.$
21. $xy=72,\ x^2+y^2=145.$	22. $x^2+y^2=365,\ xy=182.$	23. $x^2+4y^2=52,\ xy=12.$
24. $x^2+y^2=225,\ x+y=21.$	25. $x+y=27,\ x^2+y^2=369.$	26. $x^2+y^2=229,\ x-y=13.$

27. Solve Examples 12 and 20 graphically on the same diagram.

Solve the following equations

28. $x+y=11,\quad x^2+xy+y^2=91$

29. $x^2-xy+y^2=57,\quad x-y=-1$

30. $x^2-xy+y^2=75,\quad x+y=15.$

31. $\frac{1}{9}(x-y)=1,\quad x^2-3xy+y^2=29$

32. $x+y=7,\quad \frac{1}{x}+\frac{1}{y}=\frac{7}{12}$

33. $xy=35,\quad \frac{1}{x}+\frac{1}{y}=\frac{12}{35}$

[*In the following three examples solve for* $\frac{1}{x}$ *and* $\frac{1}{y}$.]

34. $\frac{1}{x^2}+\frac{1}{y^2}=178,\quad 39xy=1$

35. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6},\quad \frac{1}{x^2}+\frac{1}{y^2}=\frac{13}{36}.$

36. $\frac{1}{x^2}+\frac{1}{y^2}=4\frac{1}{4},\quad \frac{1}{x}-\frac{1}{y}=1\frac{1}{2}$

37. Draw the graph of $xy=18$, and use it to solve the following three pairs of equations on the same diagram

(i) $xy=18,\quad x-3y=-3,$ (ii) $xy=18,\quad 2x-3y=12,$ (iii) $xy=18,\quad x+2y=12$.

297 The solution of some equations of higher degree than the second can be made to depend on some of the foregoing types

EXAMPLE 1 *Solve*

$$x^3+y^3=189, \tag{1}$$
$$x+y=9 \tag{2}$$

Dividing (1) by (2), we have

$$\frac{x^3+y^3}{x+y}=21$$

Since $x+y$ is not zero, we may reduce the fraction on the left, thus

$$x^2-xy+y^2=21$$

Squaring (2), we have $x^2+2xy+y^2=81$,

whence, by subtraction, $3xy=60$,

or
$$xy=20 \tag{3}$$

Equations (2) and (3) are now of the type solved in Art 294, Example 1, and give the solutions

$$\left.\begin{array}{l}x=5, \text{ or } 4,\\ y=4, \text{ or } 5\end{array}\right\}$$

EXAMPLE 2 *Solve*

$$x^4+x^2y^2+y^4=273, \tag{1}$$
$$x^2-xy+y^2=13 \tag{2}$$

Equation (1) may be written

$$(x^2+xy+y^2)(x^2-xy+y^2)=273;$$

Dividing this by (2), $x^2+xy+y^2=21$ (3)

From (2) and (3), by addition and subtraction,

$$x^2+y^2=17, \quad xy=4$$

Solving these equations as in Art. 294, Example 3, we obtain

$$x=4,\ y=1;\quad x=1,\ y=4,\quad x=-1,\ y=-4,\quad x=-4,\ y=-1.$$

[*Examples* XXVI b. 1-15, *page* 269, *may be taken here*]

298 The following method may always be used when the terms involving the unknowns are homogeneous, and of the same degree in each equation

EXAMPLE *Solve*

$$7y^2+15xy=-68, \qquad (1)$$
$$x^2+2xy+2y^2=17 \qquad (2)$$

Dividing (1) by (2), $\dfrac{7y^2+15xy}{x^2+2xy+2y^2}=-\dfrac{68}{17}=-4$,

whence $7y^2+15xy=-4x^2-8xy-8y^2$,

that is, $15y^2+23xy+4x^2=0$,

or $(5y+x)(3y+4x)=0$;

$$y=-\tfrac{1}{5}x, \text{ or } y=-\tfrac{4}{3}x$$

(i) If $y=-\frac{1}{5}x$,

from (1), $\dfrac{7x^2}{25}-3x^2=-68$;

whence $x^2=25$,

or $x=\pm 5$,

$y=-\frac{1}{5}x=\mp 1$

(ii) If $y=-\frac{4}{3}x$,

from (1), $\dfrac{112x^2}{9}-20x^2=-68$;

whence $x^2=9$,

or $x=\pm 3$;

$y=-\frac{4}{3}x=\mp 4$

Thus the complete solution is $x=\pm 5$, $y=\mp 1$, $x=\pm 3$, $y=\mp 4$

NOTE In selecting corresponding values of x and y both upper or both lower signs must be taken together

EXAMPLES XXVI b.

Solve the following equations

1. $x^2-25y^2=156$, $x+5y=26$	2. $7x-4y=23$, $49x^2-16y^2=1081$	3. $x^3+y^3=35$, $x+y=5$
4. $x+y=11$, $x^3+y^3=341$	5. $x^3+y^3=2240$, $x+y=20$	6. $x-y=1$, $x^3-y^3=19$
7. $y^3-x^3=117$, $y-x=3$	8. $x-2y=1$, $x^3-8y^3=127$	9. $x^3+27y^3=280$, $x+3y=10$
10. $x^4+x^2y^2+y^4=21$, $x^2+xy+y^2=7$	11. $x^2+xy+y^2=19$, $x^4+x^2y^2+y^4=133$	12. $x^2-xy+y^2=19$, $x^4+x^2y^2+y^4=741$
13. $x^2-xy+y^2=37$, $x^3-y^3=37$	14. $x^3+y^3=351$, $x^2-xy+y^2=39$	15. $4x^2-2xy+y^2=31$, $8x^3+y^3=217$
16. $x^2-2xy=24$, $xy-2y^2=4$	17. $x^2-9y^2=64$, $xy+3y^2=32$	18. $3x^2-5y^2=7$, $3xy-4y^2=2$
19. $8xy-13y^2=3$, $13x^2-21xy=10$	20. $4x^2+xy=7$, $3xy+y^2=18$	21. $x^2+2xy+10y^2=146$, $xy+y^2=24$.

22. $2x^2-3xy+2y^2=2\frac{3}{4}$, $x^2-4xy+y^2+\frac{1}{2}=0$

299 The following solutions, given in brief, will furnish some useful suggestions for examples which do not fall immediately under the foregoing types

EXAMPLE 1 *Solve* $x^3+y^3=91$, $x^2y+xy^2=84$

Multiply the second equation by 3 and add to the first

thus $(x+y)^3=343$, whence $x+y=7$

But $xy(x+y)=84$, therefore $xy=12$

The equations $x+y=7$, $xy=12$ may now be solved as in Art. 294

EXAMPLE 2 *Solve* $\frac{x}{y}+\frac{3y}{x}=\frac{7}{2}$, $(x+y)(x+3y)=15$

We have $x^2+3y^2=\frac{7}{2}xy$, $x^2+3y^2+4xy=15$,

whence $\frac{7}{2}xy+4xy=15$, so that $xy=2$

The equations $x^2+3y^2=7$, $xy=2$ may now be solved as in Art 294.

EXAMPLE 3 *Solve* $x^2y^2+24=10xy$, $x^2+y^2=17$

The first equation is a quadratic in xy, which gives $xy=4$, or 6 Each of these results may now be combined with $x^2+y^2=17$

EXAMPLE 4. *Solve* $9x^2+y^2+128=21(3x+y)$, (1)

$xy=4$ (2)

From (2), we have $6xy-24=0$, adding this to (1), we have

$$(3x+y)^2+104=21(3x+y)$$

This is a quadratic in $3x+y$, of which the solution is

$$3x+y=13, \quad\text{or}\quad 3x+y=8$$

Each of these equations may now be combined with $xy=4$

NOTE The equations $3x+y=13$, $3x+y=8$ represent two parallel straight lines Thus the roots of the given equations are the coordinates of the points in which these lines meet the hyperbola $xy=4$

EXAMPLE 5 *Solve* $x^2yz=6$, $xy^2z=18$, $xyz^2=12$

Multiplying the three equations together, we have

$$x^4y^4z^4=6\times18\times12=3^4\times2^4, \quad\text{so that}\quad xyz=\pm6$$

Dividing each of the given equations by this, we have

$$x=\pm1, \quad y=\pm3, \quad z=\pm2$$

The roots must be taken either all positively or all negatively

EXAMPLE 6 *Solve* $x^2+xy+xz=42$, $xy+y^2+yz=70$, $xz+yz+z^2=84$.

These equations may be written

$$x(x+y+z)=42, \quad y(x+y+z)=70, \quad z(x+y+z)=84$$

By addition, $(x+y+z)(x+y+z)=196$, so that $x+y+z=\pm14$

Dividing each of the given equations by this, we obtain

$$x=\pm3, \quad y=\pm5, \quad z=\pm6$$

EXAMPLES XXVI. c.

Solve the following equations:

1. $4(x^2+y^2)=17xy,$ $x+y=5$.
2. $10(x^2+y^2)=29xy,$ $x-y=3$.
3. $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{74}{35},$ $y-x=2$.
4. $x^2+3xy+y^2=61,$ $x+y=7$.
5. $x^2-3xy+y^2=5,$ $x-y=3$.
6. $x^2+2xy=39,$ $2y^2-3xy=5$.
7. $x^3+y^3=152,$ $x^2y+xy^2=120$.
8. $x^3-y^3=124,$ $x^2y-xy^2=20$.
9. $x^3-y^3=243,$ $xy(y-x)=162$.
10. $(x+5)(y+2)=65,$ $xy=24$.
11. $x+3xy=35,$ $y+2xy=22$.
12. $x^2y^2+24=10xy,$ $x+y=5$.
13. $\dfrac{x-2y}{2x-y}+\dfrac{2x-y}{x-2y}=\dfrac{26}{5},$ $x^2+y^2=90$. [*Put* $u=\dfrac{x-2y}{2x-y}$, *and solve the resulting quadratic in* u.]
14. $x^2+4y^2+80=15x+30y,$ $xy=6$.
15. $x^2-3y^2+x-3y+30=0,$ $x^2+y^2+x+y=18$.
16. $2(x+y)^2+324=51(x+y),$ $xy=35$. [*Solve the first equation as a quadratic in* $x+y$. *Illustrate graphically.*]
17. $(x^2+y^2)(x+y)=15,$ $x^2+x+y^2+y=8$. [*Put* $x^2+y^2=u$, $x+y=v$, *then* $u+v=8$, $uv=15$, *whence* $u=5$, $v=3$, *or* $u=3$, $v=5$. *Illustrate graphically.*]
18. Shew that the equations $x^3+x^2y+xy^2+y^3=888$, $x^2+y^2=74$ may be replaced by the equivalent system $x^2+y^2=74$, $x+y=12$, and solve these latter equations graphically.
19. Solve the following pairs of equations graphically:

   (i) $x^2+y^2=100,$ $xy=48$; (ii) $x^2+y^2=34,$ $2x+y=11$; (iii) $x^2+y^2=25,$ $3x+4y=25$; (iv) $x^2+y^2=36,$ $4x+3y=12$.

   [*Approximate roots to be given to one place of decimals.*]

Find the values of x, y, z from the following equations:

20. $x^2yz=36,$ $xy^2z=48,$ $xyz^2=12$.
21. $xy^2z^2=36,$ $x^2yz^2=144,$ $x^2y^2z=48$.
22. $x^2yz=a^4,$ $xy^2z=b^4,$ $xyz^2=c^4$.
23. $yz+zx=16,$ $zx+xy=25,$ $xy+yz=-39$.
24. $(x+y)(x+z)=63,$ $(y+z)(z+x)=42,$ $(z+x)(z+y)=54$.
25. $x^2+xy+xz=48,$ $xy+y^2+yz=12,$ $xz+yz+z^2=84$.
26. $x+y-z=14,$ $y^2+z^2-x^2=46,$ $yz=9$.

   [*From the last two equations,* $(y-z)^2-x^2=28$. *Put* u *for* $y-z$, *and find* $u+x$ *and* $u-x$.]
27. $(x+y)^2-z^2=65,$ $x^2-(y+z)^2=13,$ $x+y-z=5$.

CHAPTER XXVII

Problems Leading to Quadratic Equations.

300 We shall now give some problems which lead to quadratic equations

Example 1 *Divide 40 into two parts such that the sum of their reciprocals is equal to $\frac{8}{75}$*

Let x be one part, then $40-x$ is the other The statement of the problem gives

$$\frac{1}{x}+\frac{1}{40-x}=\frac{8}{75},$$

whence $$40\times 75=8x(40-x);$$

that is, $$x^2-40x+375=0,$$

or $$(x-15)(x-25)=0, \text{ so that } x=15, \text{ or } 25$$

Both of these values are admissible, for if $x=15$, $40-x=25$; if $x=25$, $40-x=15$

Example 2 *A man sold a horse for £21, and lost as much per cent. as he gave for the horse, what was the cost price?*

Let the cost price be represented by x pounds; then the loss will be represented by $x-21$ pounds. And this is x per cent of x pounds,

$$x-21=x\times\frac{x}{100},$$

whence $$x^2-100x+2100=0,$$

that is, $$(x-30)(x-70)=0$$

$$x=30, \text{ or } 70$$

Thus the horse may have cost either £30, or £70

In the first case the loss is £9, which is 30 % of £30

„ second „ „ £49, „ 70 % of £70

301. In the above examples each of the two roots of the quadratic gives an intelligible answer to the problem, but it will often happen that one root of the equation is incompatible with the conditions of the case we are discussing For example, the equation may give rise to one root which is positive and integral, and another which is fractional or negative. The latter would be rejected if the conditions of the problem could only be satisfied by a positive integral solution

EXAMPLE 1 *A train runs 60 miles at a uniform rate; if the rate had been 10 miles an hour more, it would have taken half an hour less for the journey find the rate of the train*

Suppose the train runs at x miles per hour, then the time occupied is $\frac{60}{x}$ hours On the other supposition the time is $\frac{60}{x+10}$ hours,

$$\frac{60}{x+10}=\frac{60}{x}-\frac{1}{2}, \tag{1}$$

whence $$x^2+10x-1200=0,$$

or $$(x+40)(x-30)=0,$$

$$x=30, \text{ or } -40$$

Hence the train travels 30 miles an hour, the negative answer being inadmissible

NOTE The value $x=-40$ can be made applicable to a new problem by a modification of the conditions of the original problem

Since $x=-40$ satisfies equation (1), $x=+40$ satisfies the equation

$$\frac{60}{-x+10}=\frac{60}{-x}-\frac{1}{2},$$

which is obtained from (1) by writing $-x$ for x Now, by changing the signs throughout, this latter equation may be written in the form

$$\frac{60}{x-10}=\frac{60}{x}+\frac{1}{2},$$

and this is the algebraical statement of the following problem

A train runs 60 miles at a uniform rate, if the rate had been 10 miles an hour less *it would have taken half an hour* more *for the journey find the rate of the train*

The answer is 40 miles an hour

EXAMPLE 2 *A tank can be filled with water by two pipes running together in 15 minutes By the larger pipe alone the tank can be filled 16 minutes sooner than by the smaller pipe alone, find the time in which each pipe alone would fill the tank*

Suppose that the two pipes running alone would fill the tank in x and $x+16$ minutes Then running together they will fill $\left(\frac{1}{x}+\frac{1}{x+16}\right)$ of the tank in one minute

$$\frac{1}{x}+\frac{1}{x+16}=\frac{1}{15};$$

whence $$15(2x+16)=x^2+16x;$$

that is, $$x^2-14x-240=0,$$

or $$(x-24)(x+10)=0,$$

$$x=24, \text{ or } -10$$

The negative value is inadmissible

Thus the larger pipe takes 24 minutes, and the smaller 40 minutes.

EXAMPLES XXVII. a.

1. Find two numbers, differing by 7, such that the sum of their squares is 137.

2. The sum of the squares of two consecutive numbers is 145, find them.

3. Find a number which, when increased by 7, is equal to sixty times the reciprocal of the number.

4. Two numbers differ by 5, and the sum of their reciprocals is $\frac{2}{11}$; find the numbers.

5. One number is three times another number; if each is increased by 1 the sum of the reciprocals is $\frac{14}{11}$. Find the numbers.

6. The length of a rectangular field exceeds its breadth by one yard, and the area is 3 acres. Find the length of the sides.

7. A person sells goods at £31. 5s., and gains as many pounds per cent. as the goods cost: find the cost price.

8. A man sells a horse for £144, and gains as much per cent. as he gave for the horse. What did the horse cost?

9. A man rides 24 miles at a uniform rate; if he had ridden 2 miles per hour faster, he would have saved an hour. How fast did he ride?

10. A steamer goes a journey of 3240 miles; if it had gone 3 miles an hour faster it would have taken $1\frac{1}{2}$ days less: find the time the journey occupied.

11. An ordinary train, the average speed of which is 15 miles an hour less than that of the express, takes 2 hours longer to go 180 miles. What is the average speed of each train?

12. Two men start at the same time to meet each other from towns which are 54 miles apart. If one takes 3 minutes longer than the other to walk a mile, and they meet in six hours, how fast does each walk?

13. A tank can be filled by 2 pipes together in 6 hours; if the larger pipe alone takes 5 hours less than the smaller to fill the tank, find the time in which each pipe alone would fill the tank.

14. Two pipes open together can fill a cistern in $9\frac{3}{8}$ minutes: if the smaller pipe takes 10 minutes more than the larger to fill the cistern, find in what time it will be filled by each pipe singly.

15. I bought a number of cricket balls for £3; if I had bought a cheaper sort costing 2s. apiece less, I should have had 15 more for the money: what did I pay for each?

16. A man buys a number of articles for £1, and after losing 3, gets 25s. by selling the rest at 9d. apiece more than they cost; how many did he buy?

17. Sixteen guineas is divided equally among a certain number of boys; if there were 8 boys fewer each would receive 9d. more; what was the number of boys?

302 EXAMPLE 1 *A tradesman bought a number of yards of silk for £9 7s 6d, he kept 4 yards, and sold the rest at half-a-crown per yard more than he gave, thereby obtaining £1 2s 6d more than he originally spent how many yards did he buy?*

Let x be the number of yards bought

Then the *cost* price *per yard* is $\frac{75}{x}$ half crowns

But *selling price* of $(x-4)$ yards is £10 10s, or 84 half-crowns;

$$\text{the } \textit{selling} \text{ price } \textit{per yard} \text{ is } \frac{84}{x-4} \text{ half-crowns},$$

$$\text{the } \textit{gain per yard} \text{ is } \left(\frac{84}{x-4}-\frac{75}{x}\right) \text{ half crowns}$$

But from the question the gain per yard is 1 half-crown

Thus $$\frac{84}{x-4}-\frac{75}{x}=1,$$

whence $$9x+300=x^2-4x,$$

or $$x^2-13x-300=0,$$

$$(x-25)(x+12)=0,$$

$$x=25, \text{ or } -12$$

The negative value is inadmissible Thus the number of yards bought was 25

EXAMPLE 2 *What is the price of pears per gross when 120 more for a sovereign lowers the price 2d per score?*

Let x be the number of pears bought for a sovereign, then the price of each pear is $\frac{240}{x}$ pence, and that of a score is $\frac{4800}{x}$ pence

If 120 more are bought for a sovereign the price per score is $\frac{4800}{x+120}$ pence,

$$\frac{4800}{x}-\frac{4800}{x+120}=2,$$

whence $$4800\times 120=2x(x+120),$$

or $$x^2+120x-480\times 600=0,$$

$$(x-480)(x+600)=0,$$

$$x=480, \text{ or } -600$$

Thus 480 pears cost 20s, and the price per gross is $\frac{144}{480}$ of 20s, or 6s

NOTE As in Art 301, Ex 1, we may shew that the negative value -600 suggests the problem

What is the price of pears per gross when 120 fewer *for a sovereign* raises *the price 2d per score?*

EXAMPLE 3 *The small wheel of a carriage makes 22 revolutions more than the large wheel in a quarter of a mile If the circumference of each wheel were 1 ft more, the small wheel would make 6 more revolutions than the large wheel in 143 yards Find the circumference of each wheel*

Suppose the small wheel to be x feet, and the large wheel y feet in circumference

In 440 yds the two wheels make $\frac{1320}{x}$ and $\frac{1320}{y}$ revolutions respectively.

Hence $$\frac{1320}{x}-\frac{1320}{y}=22, \text{ or } \frac{1}{x}-\frac{1}{y}=\frac{1}{60} \qquad (1)$$

Similarly from the second condition we obtain

$$\frac{429}{x+1}-\frac{429}{y+1}=6, \text{ or } \frac{1}{x+1}-\frac{1}{y+1}=\frac{2}{143} \qquad (2)$$

From (1), $$x=\frac{60y}{y+60}, \text{ whence } x+1=\frac{61y+60}{y+60}$$

Substituting in (2), $$\frac{y+60}{61y+60}-\frac{1}{y+1}=\frac{2}{143},$$

that is, $$143y^2=2(y+1)(61y+60),$$

or $$21y^2-242y-120=0,$$

whence $(y-12)(21y+10)=0$, so that $y=12$, or $-\frac{10}{21}$.

The negative value is clearly inadmissible Putting $y=12$, we get $x=10$ Hence the small wheel is 10 ft and the large one 12 ft in circumference

EXAMPLES XXVII. b

1 A man bought a number of yards of silk for £6 15*s*, he kept 5 yards and sold the rest at 1*s* 6*d* per yard more than he gave, thereby gaining 15*s* on his original outlay how many yards did he buy?

2 A hawker buys a certain number of oranges for 5*s*, and retails them at 8 for 6*d*, thereby gaining as much as he paid for 25 oranges how many did he buy?

3. For £720 a man purchased some horses, 3 of them died, and he sold the remainder at £6 apiece more than he gave, thereby gaining 5 per cent on his outlay how many horses did he buy?

4. A man invests some money in $3\frac{1}{2}$ per cent stock; if the price were £15 less, he would receive $1\frac{1}{6}$ per cent more for his money what price does he pay for the stock?

5 A tradesman finds that by selling a book for 4*s* 8*d* his percentage of profit is the same as the number of pence the book cost him what did he pay for it?

6 The interest on a sum of money for one year is £31 17*s* 6*d*, if the rate of interest were less by $\frac{1}{2}$ per cent it would be necessary to invest £100 more to produce the same amount of interest Find the sum invested at first

7 A person selling a horse for £37 10*s* finds that his loss per cent is half the number of pounds that he paid for the horse what was the cost price?

8 What are eggs selling at when, if the price were raised threepence per dozen, one would get four fewer in a shilling's worth?

Find a meaning for the negative root as in Art 301

9 I bought a certain number of books for 30*s*, if each book had been subject to a discount of 4*d*, I should have had three more for the money find the cost of each

10 If 6 fewer bottles of wine can be bought for £5 when the price is raised ten shillings per dozen, what is the original price?

Alter the wording so as to state a new problem suggested by the negative solution

11 *A* and *B* distribute £5 each in charity *A* relieves 5 persons more than *B*, and *B* gives to each 1*s* more than *A* How many did each relieve?

12 How many pears are bought for 1*s*, when 6 more for the money lowers the price 2*d* a dozen?

13 The price of one kind of sugar is 1*s* 9*d* per stone more than that of another kind, and 8 lbs less of the first kind than of the second can be bought for £1 Find the price of each per stone

14 The product of two numbers added to their sum is 23, and 5 times their sum taken from the sum of their squares leaves 8 find the numbers

15 An officer forms his men into a hollow square, four deep If he has 1392 men, find how many there will be in the front

16 A rectangular plot of grass is surrounded by a gravel walk four feet wide The area of the plot is 1200 square feet, and the area of the walk is 624 square feet Find the dimensions of the plot

17 The hypotenuse of a right-angled triangle is less than the sum of the other sides by 6 ft, and the area of the triangle is 60 sq ft find the lengths of the sides

18 Two men, *A* and *B*, travel in opposite directions along a road 180 miles long, starting simultaneously from the ends of the road. *A* travels 6 miles a day more than *B*, and the number of miles travelled each day by *B* is equal to double the number of days before they meet. Find the number of miles which each travels in a day

19. Find two numbers such that their product multiplied by their sum is 330, and their product multiplied by their difference is 30

20. A certain number of pears were sold for a certain number of pence; if 5 more had been sold for the same money they would each have cost one halfpenny less; if 5 fewer had been sold for the same money they would each have cost one penny more. What was the number of pears and the price of each?

21. A man invests some money in 3 per cent. stock; if the price were £1 more, he would receive 1 per cent. less for his money: at what price did he buy the stock?

22. If $37\frac{1}{2}$ minutes would be saved in a railway journey by increasing the speed by 5 miles an hour, and 50 minutes would be lost by diminishing the speed by 5 miles an hour, find the length of the journey and the speed of the train.

23. A man buys a certain number of photographs for £1. Two get damaged, and by selling the remainder for $2d.$ more than they cost he makes one shilling profit. How many did he buy?

24. There is a number consisting of two digits such that the difference of the cubes of the digits is 109 times the difference of the digits. Also the number exceeds twice the product of its digits by the digit in the units' place. Find the number.

25. A boat's crew can row 8 miles an hour in still water: what is the speed of a river's current if it takes them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

26. How long will it take each of two pipes to fill a cistern if one of them alone takes 27 minutes longer to fill it than the other, and 75 minutes longer than the two together?

27. Find the speed of a train if when the speed is increased by 6 miles an hour 20 minutes are saved in 144 miles.

Alter the wording so as to state a new problem suggested by the negative solution.

28. If a carriage wheel $14\frac{2}{3}$ ft. in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less: how fast is the carriage travelling?

29. Find two numbers such that their sum multiplied by the sum of their squares is equal to 40, and their difference multiplied by the difference of their squares is equal to 16.

30. A man cannot afford to spend more than £18 a week in paying labourers; after a time he finds that he has to pay each labourer $6d.$ a day more, and so is obliged to dismiss four of them. Find how many labourers he employed at first.

31. A sets out from London to meet B, who starts at the same time from Maidstone, 35 miles distant. A walks $1\frac{1}{2}$ miles an hour faster than B, but after two hours stops at a friend's house on the way for $1\frac{1}{3}$ hours; he then proceeds again, and meets B half way between London and Maidstone. Find the rate at which each walks.

303 Geometrical Applications

EXAMPLE *Divide a line* AB, *whose length is* a *units, into two parts at* X *so that* $AB \cdot BX = AX^2$ *Explain both solutions*

Let $AX = x$ units, then $BX = (a - x)$ units

Hence $$a(a-x)=x^2, \text{ or } x^2+ax-a^2=0,$$

$$x=\frac{a}{2}(-1\pm\sqrt{5})=\frac{a}{2}(-1\pm 2{\cdot}236)=0{\cdot}62a, \text{ or } -1{\cdot}62a \text{ approximately.}$$

In the figure AX is the positive root The negative root is AX′, and it is measured in the direction opposite to AX

Thus $$AB \cdot BX = AX^2, \text{ and } AB \cdot BX' = AX'^2$$

EXAMPLES XXVII b (*Continued*)

32 If a straight line 6 cm in length is divided internally so that the rectangle contained by the whole and one part is equal to the square on the other part, find the segments of the line to the nearest millimetre.

33 A line AB is produced to P so that $AB \cdot AP = BP^2$ If $AB = 8$ cm, find the lengths of AP and BP to the nearest millimetre

34 If a line AB of any length is divided externally as in Ex 33, shew that (i) $AB^2 + AP^2 = 3BP^2$; (ii) $(AB+AP)^2 = 5BP^2$

35 A line AB is produced to P so that $BP^2 = 2AB^2$ If $AB = 3{\cdot}5$ cm, find AP to the nearest millimetre

36 Find a point P in a straight line AB so that $AP(AP - BP) = BP^2$ If $AB = 4{\cdot}2$ cm, find AP and BP to the nearest millimetre By substituting these values verify the truth of the given relation

37 Divide a straight line 13 cm long into two parts so that the rectangle contained by them may be equal to 36 sq cm

38 Justify the following graphical solution of Ex 37

On AB, a line 13 cm in length, describe a semicircle At A draw AP perpendicular to AB and 6 cm in length, through P draw a line PQR to cut the semicircle in Q and R, draw QX, RY perpendicular to AB Then AB is divided as required either at X or Y Verify the algebraical solution of Ex 37 by actual measurement

39 Solve the following equations graphically, taking a centimetre as unit and giving the roots to the nearest millimetre

(i) $x(7-x)=12$; (ii) $x^2-11x+30=0$,

(iii) $x^2-6x+4=0$; (iv) $x^2+13=8x$

CHAPTER XXVIII

Graphical Problems

304 When two quantities x and y are so related that a change in one produces a proportional change in the other, their variations can always be expressed by an equation of the form $y=ax$, where a is some constant quantity. Hence in all such cases the graph which exhibits their variations is *a straight line through the origin*, so that in order to draw the graph it is only necessary to know the position of one other point on it. For instance, examples which deal with work and time, distance and time (when the speed is uniform), quantity and cost of material, principal and simple interest at a given rate per cent, may all be illustrated by linear graphs through the origin.

Example. *At noon A starts to cycle from* P *to* Q, *a distance of* 40 *miles. He rides at* 6 *miles an hour, resting for an hour after riding* 12 *miles. At* 3 *p.m. B starts from* P *at* 10 *miles an hour. Find graphically*

(i) *When and where B overtakes A*,

(ii) *At what time B is* 8 *mi. behind A*,

(iii) *Their distance apart at* 5 *p.m.*

If a third cyclist C starts at noon, riding from Q *to* P, *and meets B at* 4 *p.m., at what speed does he ride?*

In Fig. 26, on the opposite page, let the position of P be chosen as origin; let time be measured horizontally from 12 o'clock (1 inch to 2 hours), and let distance be measured vertically (1 inch to 20 miles). Thus each division on the horizontal axis represents 12 minutes, and each division on the vertical axis stands for 2 miles.

In 2 hours A has ridden 12 mi., therefore if D is taken 0·6 inch (representing 12 miles) above the point which marks 2 p.m., PD is the graph of A's motion for the first 2 hours; that is to say, the ordinate of every point on this line will mark the distance travelled in the time given by the corresponding abscissa. In the next hour he makes *no advance towards* Q, therefore the corresponding portion of the graph is DE. As A now proceeds at the same rate as before, EG, drawn parallel to PD, gives the details of his motion between 3 p.m. and 7 p.m.

B starts at 3 p.m., and covers 10 mi. per hour, therefore for the graph of B's motion we use the point which marks 3 p.m. as origin, and join it to H whose ordinate is 0·5 inch (representing 10 miles), and produce the line.

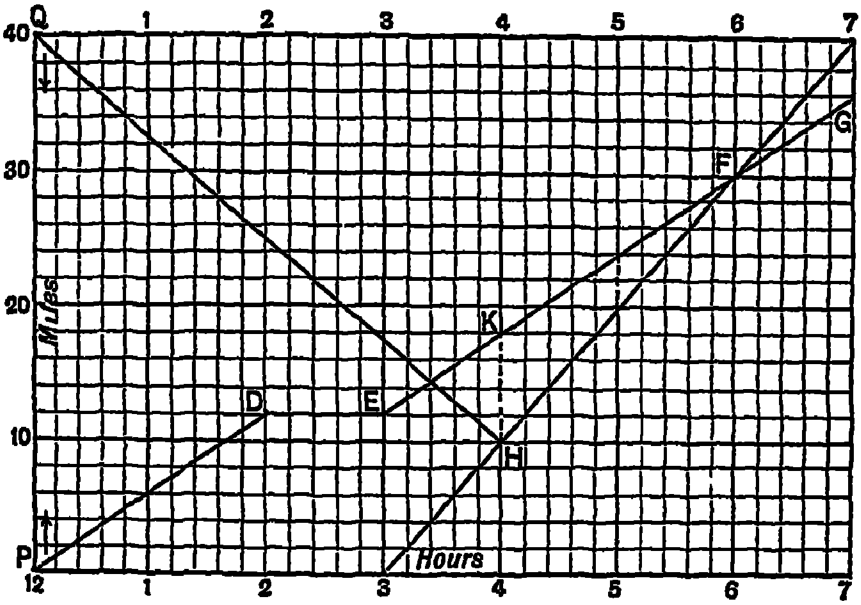


FIG 26

(i) The graphs of A's and B's motion meet at F, which is 30 mi from P And the time is 6 p m

(ii) To find when A and B are 8 mi apart slide a graduated ruler parallel to the vertical axis till the difference of the ordinates of the two graphs is found to be 8 This is shewn by KH, thus the time is 4 p m

(iii) The difference of the ordinates at 5 p m represents 4 mi, which is the distance between A and B at that time

As C walks towards P, his distances from Q at different times will be denoted by ordinates measured *downwards* A 4 p m he meets B, whose position at that time is represented by H Therefore QH is the graph of C's motion Thus C has ridden 30 mi in 4 hours, and his speed is $7\frac{1}{2}$ mi per hour

EXAMPLES XXVIII. a

1 At 10 a m A starts to ride at 8 miles an hour, and at 11 30 B follows at 12 miles an hour find graphically when B overtakes A, and at what times A and B are 4 miles apart

[*Take* 1 *inch to* 1 *hour, and* 1 *inch to* 10 *miles*]

2 Two men ride towards each other from two places 60 miles apart if they ride at 12 miles and 9 miles an hour respectively starting at noon, find when they are first 18 miles apart Also find (to the nearest minute) their time of meeting

3 At noon A starts to ride at 6 miles per hour, two hours later B follows, riding at 12 miles an hour, but resting for half an hour at the end of each hour Find when and where B overtakes A Shew also that at the end of three consecutive hours B is just 6 miles behind A.

305 The solution in Art 304 has been given very fully Solutions may usually be presented with rather less detail, as we shall now shew

EXAMPLE *A and B ride to meet each other from two towns* X *and* Y *which are 60 miles apart A starts at 1 p m, and B starts 36 minutes later If they meet at 4 p m, and A gets to* Y *at 6 p m, find, by means of a graph, the time when B gets to* X *Also find the times when they are 22 miles apart When A is half-way between* X *and* Y, *where is B?*

In Fig 27, let each horizontal division represent 12 minutes and each vertical division 2 miles

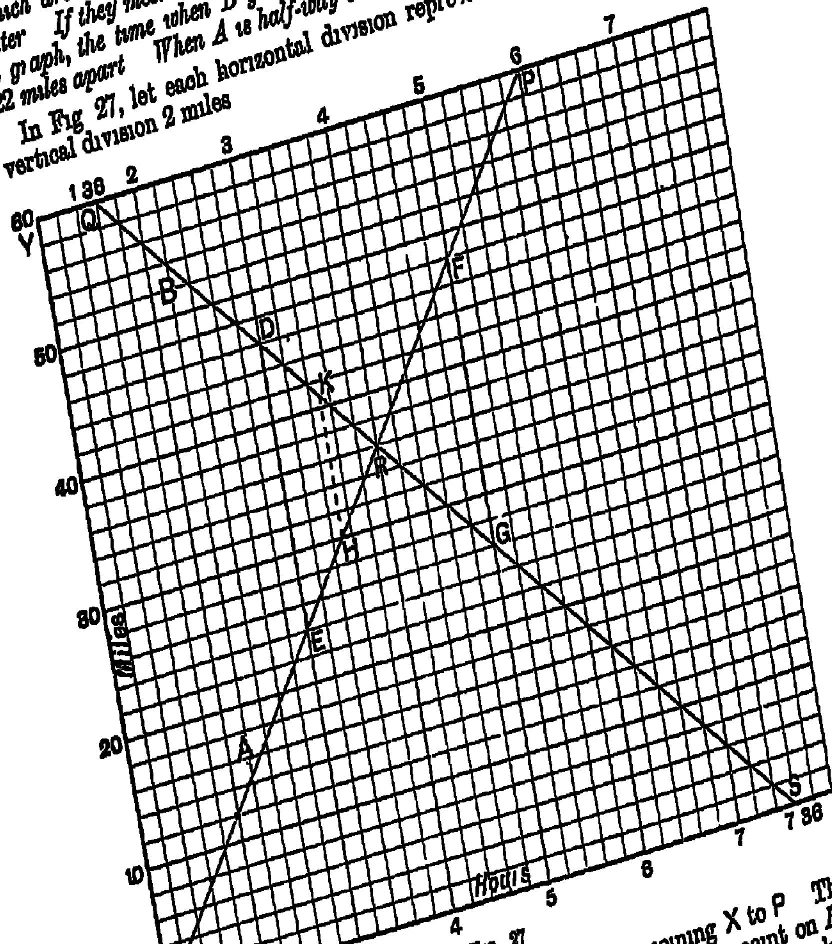


FIG 27

The graph of *A*'s motion is obtained at once by joining X to P The point R on this line corresponds to 4 p m, so that R is a point on *B*'s graph If Q be the point marking the time of *B*'s start from Y, the line QR is the graph of *B*'s motion up to 4 p m The graph is completed by producing QR to meet the horizontal through X in the point S

Thus *B* rides from Y to S in the time represented by Q'S, that is, he arrives at S at 7 36 p m

The dotted lines DE and FG each represent 22 miles, thus *A* and *B* are 22 miles apart at 3 p m and 5 p m

The point H marks *A*'s position half way between X and Y, and the point K marks *B*'s position at that time, thus *B* is 41 miles from X

EXAMPLES XXVIII a. (*Continued*)

4. Two bicyclists ride to meet each other from two places 95 miles apart *A* starts at 8 a m at 10 miles an hour, and *B* starts at 9 30 a m at 15 miles an hour Find when and where they meet, and at what times they are $37\frac{1}{2}$ miles apart

[*Take* 1 *inch to* 1 *hour, and* 1 *inch to* 25 *miles*]

5 At what distance from London, and at what time, will a train which leaves London for Rugby at 2 33 p m, and goes at the rate of 35 miles an hour, meet a train which leaves Rugby at 1 45 p m and goes at the rate of 25 miles an hour, the distance between London and Rugby being 80 miles?

Also find at what times the trains are 24 miles apart, and how far apart they are at 4 9 p m

[*Take* 1 *inch to* 2 *hours, and* 1 *inch to* 20 *miles*]

6 A man starts at noon to ride from A to B at a uniform speed of 6 miles an hour, but after riding for 1 hour he has to return to A, where he is detained half an hour. By increasing his speed to 10 miles an hour he finds he can just reach B as soon as if there had been no delay find the total length of his ride and the time of his arrival at B

7 At 8 a m *A* starts from P to ride to Q, which is 48 miles distant At the same time *B* sets out from Q to meet *A* If *A* rides at 8 miles an hour, and rests half an hour at the end of every hour, while *B* walks uniformly at 4 miles an hour, find graphically

(i) the time and place of meeting

(ii) the distance between *A* and *B* at 11 a m,

(iii) at what time they are 14 miles apart

[*Take* 1 *inch to* 1 *hour, and* 1 *inch to* 20 *miles*]

8 A cyclist has to ride 75 miles He rides for a time at 9 miles an hour and then alters his speed to 15 miles an hour, covering the distance in 7 hours At what time did he change his speed?

9 A party of tourists set out for a station 3 miles distant and go at the rate of 3 miles an hour After going half a mile one of them has to return to the starting point, at what rate must he now walk in order to reach the station at the same time as the others?

10. A motor car on its way to Bristol overtakes a cyclist at 9 a m; the car reaches Bristol at 10 30 and after waiting 1 hour returns, meeting the cyclist at noon Supposing the speeds of car and cyclist to be uniform, find when the cyclist will reach Bristol Also compare the speeds of the car and cyclist

11 Two trains start at the same time, one from Liverpool to Manchester, and the other in the opposite direction, and running steadily complete the journey in 42 min and 56 min respectively How long is it from the moment of starting before they meet?

306 Some of the ordinary processes of Arithmetic lend themselves readily to graphical illustration For example, the graph of $y=x^2$ may be used to furnish numerical square roots For since $x=\sqrt{y}$, if a series of numbers are represented by ordinates, the corresponding abscissæ will give the square roots of those numbers Similarly cube roots may be found from the graph of $y=x^3$

EXAMPLE *Draw a graph to find the cube roots of* 10 *and* 14 *correct to* 3 *places of decimals*

The cube root of 10 is a little greater than 2, hence it will be sufficient to plot the graph of $y=x^3$, taking $x=2\ 1, 2\ 2, 2\ 3, 2\ 4$, The corresponding ordinates are 9 26, 10 65, 12 17, 13 82, approximately

When $x=2$, $y=8$ Take the axes through this point, and let the units for x and y be 10 inches and 0 5 inch respectively The requisite portion of the curve is shewn in Fig 28 on the opposite page

When $y=10$, we find $x=2\ 154$ Thus the cube root of $10=2\ 154$

When $y=14$, $x=2\ 410$ Thus the cube root of $14=2\ 410$

The graph may be used to read off the cube roots of all numbers between 8 and 14 For example, the cube roots of 8 6 and 13 are found to be 2 050 and 2 350

NOTE Solutions of this kind can only be regarded as a further *illustration* of the graphical method As a *substitute* for arithmetical evolution they serve no useful purpose

EXAMPLES XXVIII b

1. Taking 1 inch as unit for x, and 0 5 as unit for y, draw the graph of $y=x^2$, and employ it to find the squares of 1 54, 1 8, 3 4, and the square roots of 7 56, 5 29, 4 81

2 Draw the graph of $y=\sqrt{x}$ taking the unit for y five times as great as that for x Use this curve to check the values of the square roots found in Ex 1

3. Draw a graph which will give the square roots of all numbers between 25 and 36 Read off $\sqrt{29}$, $\sqrt{33}$, to three places of decimals

4. From the graph of $y=x^3$ (on the scale of Fig 28) find the values of $\sqrt[3]{28\ 4}$ and $\sqrt[3]{34}$ to 4 significant figures

5. A boy who was ignorant of the rule for cube root required the value of $\sqrt[3]{14\ 71}$ He plotted the graph of $y=x^3$, using for x the values 2 2, 2 3, 2 4, 2 5, and found 2 45 as the value of the cube root Verify this process in detail From the same graph find the value of $\sqrt[3]{13\ 8}$ to two places of decimals

6 Draw a graph which will give the cube roots of all numbers between 27 and 64 correct to two places of decimals

Read off the cube roots of 44, 60; and the cubes of 3 42, 3 78

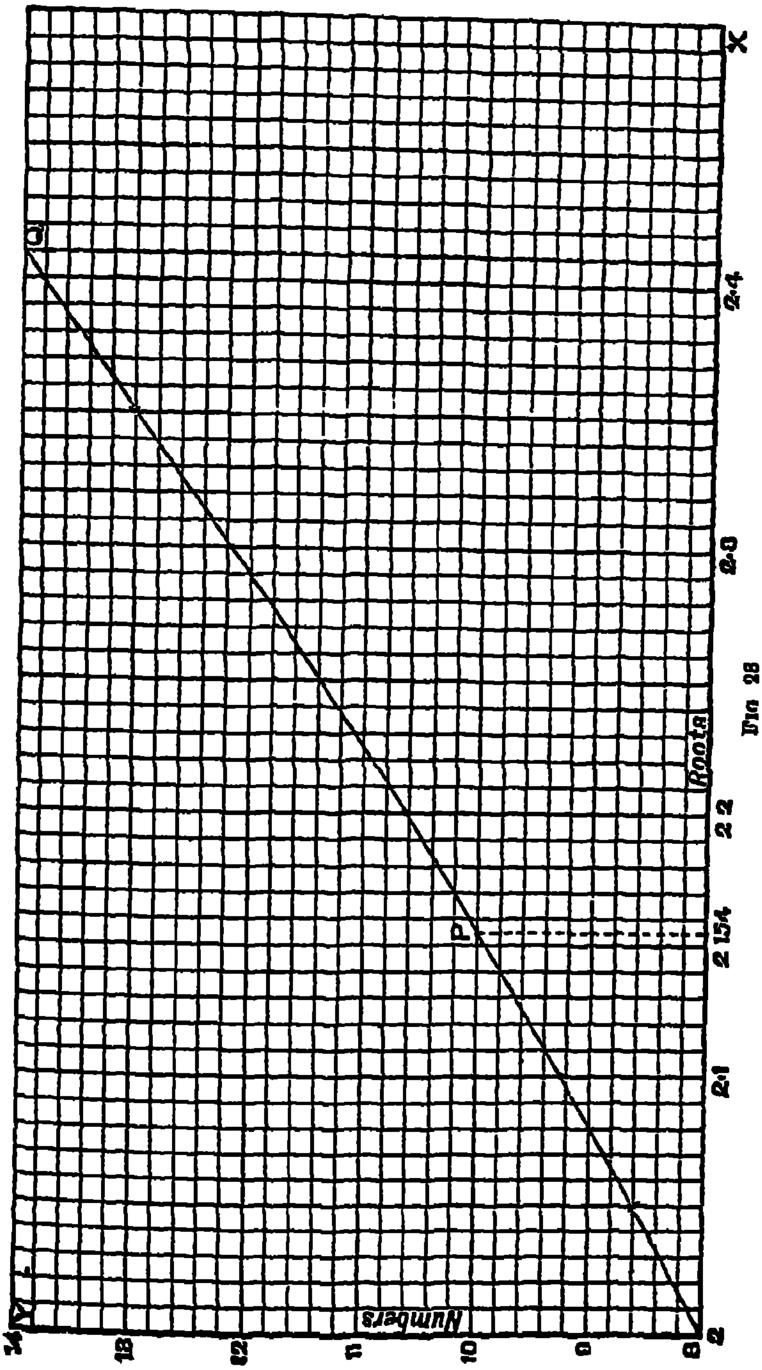


Fig 28

307. After a little practice graphical solutions can often be given very concisely.

EXAMPLE. *A, B, and C run a race of 200 yards. A gives B a start of 8 yards, and C starts some seconds after A. A runs the distance in 25 seconds and beats C by 40 yards. B beats A by 1 second, and when he has been running 15 seconds, he is 48 yards ahead of C. Find graphically how many seconds C starts after A. Shew also from the graphs that if the three runners started level they would run a dead heat.*

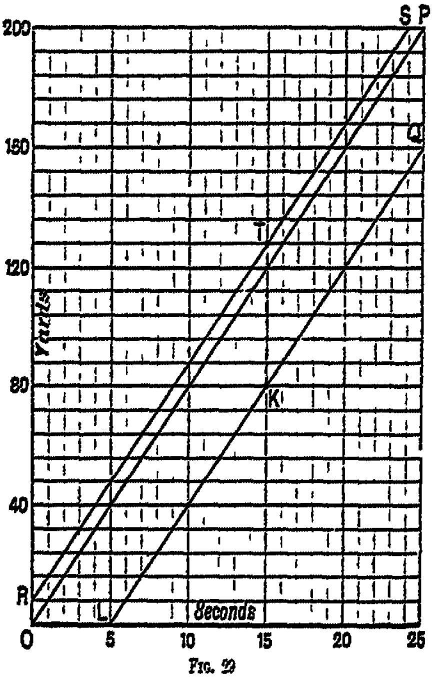


FIG. 29

A's graph is the line OP, since he runs 200 yds. in 25 seconds.

Since B has a start of 8 yds. ∴ R is a point on his graph.

Also B beats A by 1 second; ∴ S is a point on his graph.

Thus B's graph is the line RS.

A beats C by 40 yds.; ∴ Q is a point on C's graph.

Find T the point on B's graph corresponding to 15 seconds, and measure TK downwards to represent 48 yds. Then K is also a point on C's graph.

Thus C's graph is the line QK. When produced this meets the time-axis at L. Then, since OL represents 5 seconds, C must have started 5 seconds after A.

As the graphs are three *parallel* lines, the ratio of any ordinate to the corresponding abscissa is the same in each case.

Thus the speeds of the runners are equal, and if they started level they would run a dead heat.

308 When a variable quantity y is partly constant and partly proportional to a variable quantity x, the algebraical relation between x and y is of the form $y=ax+b$, where a and b are constant The corresponding graph will therefore be a straight line

EXAMPLE *The expenses of a school are partly constant and partly proportional to the number of boys The expenses were £650 for 105 boys, and £742 for 128 Draw a graph to represent the expenses for any number of boys, find the expenses for 115 boys, and the number of boys that can be maintained at a cost of £710*

If the total expenses for x boys are represented by £y, the variable part may be denoted by £ax, and the constant part by £b Hence x and y satisfy a linear equation $y=ax+b$, where a and b are constant quantities Hence the graph is a straight line, which can be drawn at once by joining the two points whose coordinates are given by the conditions of the question

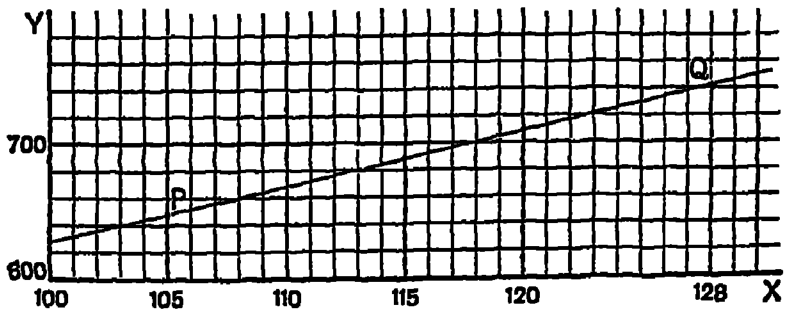


FIG 30

As the numbers are large, it will be convenient if we begin measuring ordinates at 600, and abscissæ at 100 This enables us to bring the requisite portion of the graph into a smaller compass When $x=105$, $y=650$, and when $x=128$, $y=742$ Thus two points P and Q are found, and the line PQ is the required graph

By measurement we find that when $x=115$, $y=690$, and that when $y=710$, $x=120$ Thus the required answers are £690, and 120 boys

EXAMPLES XXVIII c

1. X and Y are two towns 35 miles apart At 8 30 a m A starts to walk from X to Y at 4 miles an hour, after walking 8 miles he rests for half an hour and then completes his journey on a bicycle at 10 miles an hour At 9 48 a m B starts to walk from Y to X at 3 miles an hour, find when and where A and B meet Also find at what times they are $6\frac{1}{2}$ miles apart

2 At 8 a m A begins a ride on a motor car at 20 miles an hour, and an hour and a half later B, starting from the same point, follows on his bicycle at 10 miles an hour After riding 36 miles, A rests for 1 hr 24 min, then rides back at 9 miles an hour Find graphically when and where he meets B Also find (i) at what time the riders were 21 miles apart, (ii) how far B will have ridden by the time A gets back to his starting point

3. A and B start at the same time from London to Blisworth, A walking 4 miles an hour, B riding 9 miles an hour B reaches Blisworth in 4 hours, and immediately rides back to London After 2 hours' rest he starts again for Blisworth at the same rate How far from London will he overtake A, who has in the meantime rested $6\frac{1}{4}$ hours?

4 A, B, and C set out to walk from Bath to Bristol at 5, 6, and 4 miles an hour respectively C starts 3 minutes before, and B 7 minutes after A Draw graphs to shew (i) when and where A overtakes C, (ii) when and where B overtakes A, (iii) C's position relative to the others after he has walked 45 minutes

5. A can beat B by 20 yards in 120, and B can beat C by 10 yards in 50 Supposing their rates of running to be uniform, and that they start together, find where A and C are when B has run 80 yards

6 A, B, and C run a race of 300 yards, A and C start from scratch, and A covers the distance in 40 seconds, beating C by 60 yards B, with 12 yards' start, beats A by 4 seconds Supposing the rates of running to be uniform, find graphically the relative positions of the runners when B passes the winning post Find also by how many yards B is ahead of A when the latter has run three fourths of the course

7 In a race of 180 yards A, starting from scratch, runs a dead heat with C in 25 seconds, and beats B by 30 yards When A has run 108 yards he is 8 yards behind C and 14 yards ahead of B Find graphically how much start B and C received.

8. A cyclist started to ride 38 miles, after riding for some time at 12 miles an hour he reduced his speed to 8 miles an hour, and reached his destination in exactly 4 hours At what time did he make the change, and how far did he ride at each speed? Find graphically how much time he would have saved if the last part of his ride had been at 10 miles an hour instead of 8

9 I row against a stream flowing $1\frac{1}{2}$ miles an hour to a certain point, and then turn back, stopping two miles short of the place whence I originally started If the whole time occupied in rowing is 2 hrs 10 mins and my uniform speed in still water is $4\frac{1}{2}$ miles an hour, find graphically how far upstream I went

10 At 7 40 a m the ordinary train starts from Norwich and reaches London at 11 40 a m, the express starting from London at 9 a m arrives at Norwich at 11 40 a m if both trains travel uniformly, find when they meet Shew that the time is independent of the distance between London and Norwich, and verify this conclusion by solving an algebraical equation [Compare Art 263]

11. A boy starts from home and walks to school at the rate of 10 yards in 3 seconds, and is 20 seconds too soon The next day he walks at the rate of 40 yards in 17 seconds, and is half a minute late Find graphically the distance to the school, and shew that he would have been just in time if he had walked at the rate of 20 yards in 7 seconds

12 The annual expenses of a Convalescent Home are partly constant and partly proportional to the number of inmates The expenses were £384 for 12 patients and £432 for 16. Draw a graph to shew the expenses for any number of patients, and find from it the cost of maintaining 15

In a rival establishment the expenses were £375 for 5, and £445 for 15 patients Find graphically for what number of patients the cost would be the same in the two cases

13 A body is moving in a straight line with varying velocity The velocity at any instant is made up of the constant velocity with which it was projected (measured in feet per second) diminished by a retardation of a constant number of feet per second in every second. After 4 seconds the velocity was 320, and after 13 seconds it was 140 Draw a graph to shew the velocity at any time while the body is in motion

A second body projected at the same time under similar conditions has a velocity of 450 after 5 seconds, and a velocity of 150 after 15 seconds Shew graphically that they will both come to rest at the same time Also find at what time the second body is moving 100 feet per second faster than the first, and determine *from* the graphs the velocity of projection in each case

14. The table below shews the distances from London of certain stations, and the times of two trains, one up and one down Supposing each run to be made at a constant speed, shew by a graph the distance of each train from London at any time, using 1 inch to represent 20 miles, and 3 inches to represent an hour

Distance in miles			↓	↑
	London,		4 30 p m	7 0 p m
5½	Willesden,	arrive	4 38	
		depart	4 42	(No intermediate
66	Northampton,	arrive	5 50	stop)
		depart	5 54	
113	Birmingham,		7 0	5 0 p m

At what point do they pass one another, and how far is each from London at 5 30? Which of the three runs by the stopping train is the fastest, and which is the slowest?

MISCELLANEOUS EXAMPLES VI.

[*The following Examples are arranged in four sets I may be taken after* Chap x, II *after* Chap xv, III *after* Chap xxi, IV *after* Chap xxviii]

I (*Including* Chapters i-x.)

1 Find the value of $\dfrac{2a+b+c}{ab-c}+(2a+b-c)^2$.
when $a=2$, $b=3$, $c=-4$

2. Remove the brackets from

$$2\{a-5(b+c)\}-3\{b-2(2a-c)\},$$

and simplify the result

3 Write down the values of

(i) $(x-7)(x+13)$, (ii) $(2y-3)(2y+3)$, (iii) $(2a+3)(3a+2)$;
(iv) $(3p-8)(5p-4)$, (v) $(4m+3n)(4m-3n)$; (vi) $(5x-3)(8x-9)$

4 If the product of $x+7$ and $x-2$ is equal to the product of $x+3$ and $x+4$, what is the value of x?

5 Shew that x^3+20x is equal to $9x^2$ when $x=0$, 4, or 5 Which of the expressions is the greater when $x=3$?

6 A journey of x miles takes me n hours, a journey of y miles takes a cyclist m hours Express algebraically the fact that the cyclist's pace is 6 miles an hour faster than mine

7 With as little work as possible find the value of $ma+mb+mc$ when $m=3\,45$, $a=27\,32$, $b=13\,71$, $c=58\,97$

8 A boy's age is one-third of his father's, six years ago his age was two-ninths of his father's age at that time, what are their ages?

---

9 If $\mathrm{A}=2x-3$, $\mathrm{B}=3x-4$, $\mathrm{C}=x-2$, find the values of

(i) $\mathrm{CB}-\mathrm{CA}$; (ii) $\mathrm{AB}-6\mathrm{C}^2$

10. Simplify $\dfrac{2x-1}{6}-\dfrac{x-2}{3}+\dfrac{4x+5}{2}$ If the expression is equal to 17, what is the value of x?

11 Illustrate the following identities graphically, as in Art 66

(i) $(x+y)^2\equiv x^2+2xy+y^2$, (ii) $(x+3)(x+5)\equiv x^2+8x+15$

12 From a plank, x yards long, y feet are cut off, and the remainder is c inches longer than the part cut off Express this by an equation

13 A library contains 452 volumes, of these $12(m+4)$ are English, $7m-36$ are French, and $4(m-5)$ are Latin How many were there in each language?

14 Tabulate the values of $\frac{x^3}{100}+\frac{x^2}{10}+3$ when x has the values 1, 2, 6, 10

15. Solve the equations:

(i) $(x-3)^2-(x+9)(x-1)=5(2-x)-13x$;

(ii) $\frac{4x+1}{15}-\frac{5x-1}{3}=\frac{7x-12}{5}$

16 Divide £117 between A, B, and C, so that C may have twice as much as A, and A's share may be three fourths of B's

---

17. Add together

$$-x^3-2ax^2+a^2x,\quad 2x^3-ax^2,\quad 4x^3-a^3,\quad 5ax^2-a^2x-4a^3$$

Test your answer by putting $x=2$, $a=-1$ in the four expressions and in the sum

18 Simplify $7x^3-(3-4ax)-3x(4x+1)(ax+5)$, and then bracket the result according to powers of x

19. What is the value of $a(x^3-ay^3)+xy(x-a^2y)$ when $x=ay$?

20. If $\qquad \mathrm{M}=3m(x-1)^2-m(x-1)-4$,

and $\qquad \mathrm{N}=16+n(x-1)-3n(x-1)^2$,

find the value of $n\mathrm{M}+m\mathrm{N}$

21. Express

(i) p miles per hour in feet per second,

(ii) the price in pence per dozen of articles which cost x shillings per score,

(iii) the price in shillings per cwt of sugar which cost y pence per pound.

22 A rectangular solid of length l, breadth b, and thickness t, is made up of rectangular blocks each of which has a volume V Find the formula for the number (N) of such blocks contained in the solid From the formula find

(i) N when $l=6$ ft, $b=2$ ft, $t=1\frac{1}{2}$ ft, and $\mathrm{V}=24$ cu in,

(ii) V when $\mathrm{N}=25$, $l=15$ cm, $b=6$ cm, $t=4$ cm

23 Solve the equation $x-\frac{x-1}{2}-\frac{x}{3}=\frac{x-1}{3}-\frac{x-2}{4}+\frac{x-3}{5}$

24. How much coffee at 1*s* 6*d* per pound must be mixed with coffee at 1*s* 8*d* per pound to make 140 pounds worth £11?

II. (*Including* Chapters I–XV.)

25. If $a=2$, $b=3$, $c=5$, prove that $3a^2-ab+b^2$ is greater than $bc+2ca-c^2$ by the value of b^2-a^2

26. Subtract the sum of $5a-(7b-c)$ and $3b-(9a+c)$ from $c-4b$

27. If x lbs of tea cost y shillings, how many shillings will y lbs of the same tea cost?

28. What value of x will make the sum of $(x-1)(x-3)$ and $(x-2)^2$ equal to twice the product of $x-2$ and $x-3$?

29. Find the factors of

(i) $p^2-8p-65$, (ii) $4x^2y+2xy^2-6y^3$, (iii) $12x^4-27a^2x^2$

30. Solve the simultaneous equations

$$2x-3y=5, \qquad 3x-\frac{2y-3}{5}=4$$

31. A is twice as old as B, and B is twice as old as C. The sum of their ages will be trebled in 28 years; how old is each now?

32. Draw a graph to shew the relation between x and y from the following corresponding values

$x=5$, 10, 15, 20, 26, 33, 40, 45;
$y=7$, 11 9, 15, 16 8, 18, 18 4, 18, 17 6

From the graph read off, as accurately as possible, the values of y corresponding to $x=12$ and $x=23$

---

33. When $a=-3$, $b=2$, $c=-2$, find the value of

$$\frac{a}{2}(b^2-2c^2)^2\times\{b-3(c-ab)\}$$

34. If a men do as much work as b boys, and c men take d days to finish a job, how long would e boys take?

35. Divide x^4-13x^2+36 by x^2-5x+6

36. If $f(x)\equiv x^3+2x^2-9x-18$, find the values of $f(-2)$, $f(4)$, and $f(-5)$. Separate $f(x)$ into simple factors

37. Find the first four terms of the product

$$(1-2x+3x^3+x^4)(1+3x^2-x^4+2x^5).$$

38. Solve the equations

(i) $\dfrac{17-3x}{5}-\dfrac{4x+2}{3}=5-6x+\dfrac{7x+14}{3}$;

(ii) $3(x-4)-4(y+3)=1, \quad 5(y+3)-4(x-4)=1.$

39. A horse costs £7 more than a wagon, and a wagon with three horses costs £177. What is the value of a wagon?

40. Draw the graphs of $x=1+y$ and $2x+4y=17$ in the same diagram. Hence solve the two equations simultaneously.

---

41. If $V=5a+4b-6c$, $X=7c-3a-9b$, $Y=20a+7b-5c$, and $Z=13a-5b+9c$, calculate the value of $V-(X+Y)+Z$.

42. (i) A rectangular lawn x feet wide and y feet broad is surrounded by a path z feet wide. What is the area of the path?

(ii) Apples cost x shillings a score. How many will be obtained for a sovereign? How many more will be obtained if the price is lowered a shilling a score?

43. What value of c makes $(x-2)^2-(x-1)(x-3)=c$ an identity?

44. Write down the values of the following products:

(i) $(2x-1)(x-2)$, (ii) $(3x-5)(2x+3)$; (iii) $(6x+8)(x-2)$.

If the values of (ii) and (iii) are equal, what is the value of x?

45. Find the factors of the following expressions:

(i) x^3+6x^2-91x, (ii) $27+8x^3$, (iii) $x^2y-a^2y+x^2a-a^3$.

46. Find x, y, z from the equations

$$4x+5y+z=6, \quad x+7y+2z=10, \quad 5x-3y-6z=16.$$

47. A man gave $3d$ to each of a number of boys and had $1s\ 9d$ left; if he had had $1s\ 6d$ more he could have distributed all his money by giving $4\frac{1}{2}d$ to each boy. How many boys were there?

48. Shew, by drawing graphs, that the values of x and y which satisfy the equations $x+2y=1$, $3x-y=2$, also satisfy the equation $2x-3y=1$.

III (*Including* Chapters I.-XXI.)

49. Divide $8a^3-b^3+c^3+6abc$ by $2a-b+c$.

50. Resolve into factors:

(i) $2x^2-3xy-2y^2$, (ii) $(x-2)^3-(x-2)$, (iii) $a(a+1)-b(b+1)$.

51. A train travels x miles in p hours, how many minutes will it take to travel q yards?

52. Reduce to their simplest form:

(i) $\dfrac{28x^2-41x+15}{4x^3-7x^2+7x-3}$, (ii) $\dfrac{2b-a}{ab+b^2}-\dfrac{2a+b}{a^2-ab}+\dfrac{a-b}{ab}$.

53. Prove that if $x+y=1$,

$$x^3(y+1)-y^3(x+1)-x+y=0.$$

54. Solve the equations

$$\text{(i)}\ \frac{x-1}{3}-\frac{x-9}{2}+\frac{3x-2(x-2)}{7}=4\tfrac{1}{2};$$

$$\text{(ii)}\ 5x+11y=146, \quad 11x+5y=110$$

55 The expression $ax-3b$ is equal to 30 when $x=3$, and to 42 when $x=7$, what is its value when $x=4.3$, and for what value of x is it equal to zero?

56 An officer forms his men into a hollow square, four deep If he has 1424 men, find how many there will be in the front

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57. Resolve into factors

(i) $a^3b^3-9a^2b^2+20ab$, (ii) $(x^2-1)(x+2)+(x^2+2x)(x+1)$.

58. Simplify

$$\text{(i)}\ \left(\frac{y+x}{y-x}+\frac{y-x}{y+x}\right)-\left(\frac{y+x}{y-x}-\frac{y-x}{y+x}\right), \quad \text{(ii)}\ \frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{x^2-16x}{4-x^2}$$

59 By the use of Detached Coefficients, find the product of

$$1+x+x^2-x^4-x^5 \text{ and } 1-x+x^2$$

60. Express the square root of

$$(2x^2+5xy-3y^2)(6x^2-5xy+y^2)(3x^2+8xy-3y^2)$$

in the form of a product of three simple factors

61. If $x+5y$ exactly divides $x^3+3x^2y+5by^3$, find the value of b

62 Solve the equations

$$\text{(i)}\ \frac{x^2-x+4}{3}=\frac{(2x-3)(3x-2)}{18}+\frac{5}{12},$$

$$\text{(ii)}\ 3(x-2)-2(y+3)=1, \quad 2(x-3)+3(y+2)=0$$

63 A man has a sum of money, consisting of shillings and half-crowns After paying a bill of 15s entirely with half-crowns he finds that he has left six times as many shillings as half crowns, but, if he had paid the bill entirely with shillings, he would have had left three times as many half crowns as shillings How much money had he originally?

64. Plot the graphs of

$$y=\frac{x}{2}+\frac{3}{2}, \quad y=x+\frac{5}{4}, \text{ from } x=-5 \text{ to } x=5.$$

Find from the graphs the values of x and y which satisfy both equations, and verify the solution algebraically

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65. Assuming $v=29$, $u=45$, $g=32$, $t=2$, find s from the following formulæ.

$$\text{(i)}\ v^2=u^2-2gs; \quad \text{(ii)}\ s=ut-\tfrac{1}{2}gt^2$$

66 Find by inspection values of x which satisfy the equations

(i) $2(x+8)=5(x+8)$, (ii) $x(2x+1)=x(x-5)$;

(iii) $(2x-3)(x-7)=0$; (iv) $6(2x-5)=x(2x-5)$

67 Resolve into their simplest factors

$$a^3-a^2-6a,\quad a^3-9a,\quad a^4+2a^3,$$

and write down their lowest common multiple

68 If $f(x)\equiv x^3-13x-12$, find the values of

(i) $f(4)$, (ii) $f(-2)$, (iii) $f(x-1)+f(x+1)$

Express $f(x)$ in simple factors

69 Solve (i) $\frac{13}{25}\left(2x-\frac{3}{4}\right)-\frac{7}{10}\left(x+\frac{2}{3}\right)=\frac{5}{12}(x-5)$,

(ii) $x+2y+z+7=0,\quad 2x+y-z=1,\quad 3x-y+2z=2$

70 Simplify $$\frac{2}{a+x}-\frac{1}{a-x}+\frac{3x}{a^2-x^2}+\frac{ax}{a^3+x^3}$$

71. If $2b=a+c$, shew that $(a-b)^3+2b^3+(b-c)^3=a^3+c^3$

72 A certain train is ten minutes late when it performs its usual journey at the rate of $26\frac{1}{4}$ miles per hour, but it is only 1 min 20 secs late when it travels 27 miles an hour, find the length of its journey

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IV (*Including* Chapters I.-XXVIII.)

73 A man buys oranges at x pence per hundred and sells them at the rate of y for a shilling, what profit per score does he make? If he gains $1d$ per score when $x=75$, find the value of y

74 When the expression $x^4+x^3-ax^2+17x+b$ is divisible by x^2-3x+2, find the values of a and b by means of the Remainder Theorem

75 Express in their simplest form

(i) $\frac{2(b-c)}{a-b}-\frac{3a(2c-b)}{b^2-a^2}+\frac{b-4c}{a+b}$, (ii) $\frac{x^2+x-6}{x^3-1}\times\frac{x^2-1}{x+3}\times\frac{x^4+x^2+1}{x^3-x-2}$

76 Solve the equations

(i) $\frac{5}{x-3}+\frac{3}{x+3}=2$, (ii) $5(x-1)^2=(x-2)^2$

77 Using Detached Coefficients, find the square of

$$2+x+x^3-2x^4-x^6,$$

(i) in ascending powers as far as the term containing x^4;

(ii) in descending powers „ „ „ x^8.

78. Prove that

$$(3a-b)^3+3(3a-b)^2(b-a)+3(3a-b)(b-a)^2+(b-a)^3\equiv 8a^3.$$

79. What is the speed of a train, if an increase of speed of 3 miles per hour saves 10 minutes in 120 miles?

80. Plot the graph of $y=\frac{1}{3}(x-2)(3-x)$ from $x=-1$ to $x=5$ Find from the graph the approximate values of the roots of the equation $x^2-5x+3=0$

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81. A rectangular room a yards long and b yards wide is carpeted so as to leave a margin c inches in width all round Find the area (i) of the carpet, (ii) of the margin, each in square feet

82. Shew that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$

83. Simplify (i) $\dfrac{2x^2-13x+18}{2x^3-11x^2+3x+27}$,

(ii) $\dfrac{2}{1-a}+\dfrac{1}{1+a}+\dfrac{1}{a^2-1}+\dfrac{3}{a^2+1}-\dfrac{5}{1-a^4}$

84. With as little work as possible, find the coefficient of x^4 in the product

$$(3-2x+x^3-x^4+5x^5)(2+x^2+x^3-3x^4)$$

85 A railway embankment H feet high, A feet wide at the top, and B feet wide at the bottom, contains $\dfrac{A+B}{2}\times H\times L$ cubic feet of earth in a length of L feet If 18 cubic feet of earth weigh a ton, find the weight of an embankment 100 feet long, 27 feet high, 15 feet wide at the top, and 35 feet wide at the bottom Also find L, to the nearest foot, when the weight is 2260 tons, and A=13, B=27, H=15

86 Find the square root of $\dfrac{3x}{y}\left(2+\dfrac{3x}{y}\right)+\dfrac{y}{x}\left(1+\dfrac{y}{4x}\right)+4$

87 Of the candidates in a certain examination 64 per cent passed If there had been 11 more candidates, 9 of whom failed, the successes would have been 62 5 per cent how many candidates were there?

88. A pressure of 10 lbs per square inch is approximately equivalent to 0 7 Kg per square centimetre Draw a graph to convert pressures in pounds per square inch into kilograms per square centimetre Read off the equivalents of (i) 66 lbs per sq in, (ii) 6 4 Kg per sq cm

Shew that 56 lbs per sq in = 3·9 Kg per sq cm

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89. Resolve into factors

(i) x^4-10x^2+9, (ii) $x^2+2xy+y^2-x-y$

90. Solve the equations

(i) $\dfrac{1}{x-1}-\dfrac{2}{x-2}+\dfrac{1}{x-3}+\dfrac{x}{(x-1)(x-2)(x-3)}=0$;

(ii) $ax-by=2(a^2-b^2)$, $x-2y=-3b$.

91. Prove that if a, b, c are three consecutive numbers,

$$a^3+c^3=2b(b^2+3)$$

92 Simplify (i) $\dfrac{x+3}{x^2-5x+6}-\dfrac{x+2}{x^2-9x+14}+\dfrac{4}{x^2-10x+21}$;

(ii) $\dfrac{(a+b)^2-c^2}{a^2+ab-ac}\times\dfrac{a}{(a+c)^2-b^2}\times\dfrac{(a-b)^2-c^2}{ab-b^2-bc}$

93 Shew that the equations

$$5x-3y+z=8, \quad x+3y-3z=10, \quad 8x=3y+17,$$

are not independent

94. A man walks $15\frac{3}{4}$ miles at a uniform rate If he walked one mile per hour faster he would arrive at his destination one hour sooner Find the rate at which he walks

95 Plot the graph of $y=1\ 7+1\ 8x-0\ 6x^2$ between the values $x=-1$ and $x=5$ Find, from the graph, the greatest value of y

96 If $f(m)\equiv 3m^2-m+1$, prove that

$$f(m+1)-f(m)-2f(0)=6m$$

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97. By selling eggs at p pence per score a man makes a profit of r per cent If he bought them at the rate of q for a shilling, shew that $5pq-12r=1200$

98 Solve the following pairs of equations

$$\text{(i) } 9x^2-4y^2=576, \quad 3x-2y=12, \qquad \text{(ii) } x^2+y^2=85, \quad xy=42$$

Verify the solution of (ii) graphically

99. Plot the graph of $y=x^2$ between the values $x=10$ and $x=14$, and determine from the graph the square root of 150 to two places of decimals

100 Factorize (i) $x(2+x)-y(2+y)$, (ii) $a^4-b^4+2ab(a^2-b^2)$

101 If the value of $\dfrac{\mathsf{P}bc}{a}+\dfrac{\mathsf{Q}ac}{b}+\dfrac{\mathsf{R}ab}{c}$, where P, Q, R are coefficients independent of a, b, c, remains unaltered in value when b and c are interchanged, shew that Q$=$R

102 A man took away £15 for his holiday expenses He found that by reducing his expenses to the extent of 3 shillings a day he could have extended his holiday 5 days. How long a holiday did he take?

103 Find graphically the values of y for which the expression y^2-2y-9 vanishes Shew that for values of y between these limits the expression is negative, and for all other values positive Also find the least value of the expression

104. In 4 hours 40 minutes a man travels a distance of $25\frac{1}{2}$ miles. For part of the time he walks at $3\frac{1}{2}$ miles an hour, and for the remainder he rides a bicycle at 9 miles an hour. Find graphically how many miles he walks and rides respectively, and at what time he began to ride. Verify the results by solving an equation.

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105. If the square of half the difference between a given number and its square be subtracted from the square of half the sum of the same number and its square, shew that the result is the cube of the given number.

106. Find the difference between

$$(x+1)^3-3x^2(x+1) \quad \text{and} \quad x^3-3x(x+1)^2.$$

107. Simplify the expressions

$$\text{(i)} \quad \frac{4}{2x+x^2}+\frac{1}{4-x^2}-\frac{4}{2x-x^2},$$

$$\text{(ii)} \quad \frac{a^2}{(a-b)(a-c)}-\frac{b^2}{(b-c)(b-a)}-\frac{c^2}{(c-a)(c-b)}.$$

108. If the expressions x^3-7x-a and x^3+x^2-32 leave the same remainder when divided by $x-3$, find the value of a.

109. Express the square root of

$$(2c^2-cd-3d^2)(3c^2-cd-2d^2)(6c^2+13cd-6d^2)$$

as the product of three simple factors.

110. Solve the equations:

$$\text{(i)} \quad bx-ay=b^2-a^2, \qquad \text{(ii)} \quad x-3y=3,$$
$$x-y=\tfrac{1}{2}(a-b); \qquad 2y^2-3xy-2x^2=8.$$

111. A number consists of three digits of which the middle one is equal to the sum of the other two; the square of the middle digit exceeds four times the product of the other two by 1, and 4 times the first digit equals 3 times the third. Find the number.

112. Draw the graphs of $y=x-4$ and of $y=2x^2-2x-4$ for values of x from -2 to -2.

Find from the figure the solutions of the simultaneous equations

$$x-y=4, \quad 2x^2-2x-y=4.$$

---

113. Shew that any factor which divides two algebraical expressions A and B will also divide any multiples of A and B, and also the sum and difference of any such multiples.

Hence find a root common to the two equations

$$3x^3-11x+2=0, \quad 2x^3-5x+6=0.$$

114. Simplify

(i) $\frac{2x-1}{x-1}-\frac{4x-6}{x-2}+\frac{2x-5}{x-3}$;

(ii) $\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{yz}{(z-x)(z-y)}$

115. Expand the product

$$(2-x^2+3x^3-x^4)(3-x-2x^3+5x^4)$$

as far as the term involving x^5

116. The income tax paid on an income of £A was £T, the earned part of the income being taxed at 9*d* in the £, and the remainder at 1*s* in the £. Find how much was earned income, and how much unearned.

Find the respective amounts from your formula, when A=1150, and T=47½.

117. If m is the difference between any quantity and its reciprocal, and n the difference between the square of the same quantity and the square of its reciprocal, shew that

$$m^2(m^2+4)=n^2$$

118. If 4 is taken from a certain number the result is the square of another number, but if 21 is added to the first number the result is the square of a number greater by 1 than the second number. Find the three numbers.

119. Plot the graph of $y=\frac{1}{3}(2x^2-5)$ for values of x from -1 to 4.

Find by the use of graphs approximate solutions of the equations

$$2x^2-3y-5=0, \qquad 2y^2-3x+4=0$$

120. P and Q are two towns 30 miles apart. At 1 p.m. X starts to walk from Q to P at 3 miles an hour, and after walking two hours finds it necessary to run back to Q. This he does at $6\frac{2}{3}$ miles an hour, and after a delay of 6 minutes he again starts from Q, at 4 miles an hour. Meanwhile Y starting from P at 1 p.m. sets out for Q, at 4 miles an hour; after walking for two hours, he spends half an hour with a friend from whom he borrows a bicycle on which he continues his journey at 12 miles an hour. Draw graphs to shew the position of each man relative to P and Q at any time between 1 p.m. and 5.30 p.m. Also from the graphs find

(i) when and where X and Y meet;

(ii) at what times respectively they were 18 miles and 8 miles apart.

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121. If $x=\frac{a-b}{z}$, and $y=\frac{a-b}{z}$, shew that

$$\frac{x^3+y^3}{2}=\frac{a}{z}\left\{\left(\frac{a}{z}\right)^2+3\left(\frac{b}{z}\right)^2\right\}.$$

122. Find a numerical value of k which will make the expressions

$$2(k^3+k^2)x^3+(11k^2-2k)x^2+(k^2+5k)x+5k-1$$

and
$$2(k^2+k)x^2+(11k-2)x+4,$$

have a common factor other than unity

123 Find for what values of λ the equations

$$5x^2+(9+4\lambda)x+2\lambda^2=0, \qquad 5x+9=0,$$

are satisfied by the same value of x

124. Multiply $3x+4y+\dfrac{11xy}{x-\frac{3}{2}y}$ by $10x-3y-\dfrac{11xy}{\frac{x}{4}+y}$

125. Shew that the lowest common multiple of

$$a(a-b)^2-ac^2, \quad a^2b-b(b-c)^2, \quad (a+c)^2c-b^2c$$

is
$$abc(a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2)$$

126. Prove the identity

$$(al+bm+cn)^2+(bn-cm)^2+(cl-an)^2+(am-bl)^2$$
$$\equiv(a^2+b^2+c^2)(l^2+m^2+n^2)$$

127. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in 120 yards but only 4 revolutions more when its own circumference is increased by one-fourth, and that of the hind-wheel by one-fifth Find the circumference of each wheel

128 The keeper of a restaurant finds that when he has G guests a day his total daily expenditure is E pounds, and his total daily receipts amount to R pounds The following numbers are averages obtained by comparison of his books on many days

G	210	270	320	360
E	16 7	19 4	21 6	23 4
R	15 8	21 2	26 4	29 8

By plotting these values find E and R when he has 340 guests What number of guests per day gives him (i) no profit, (ii) £6 profit? Find simple algebraical relations between E and G, R and G, P and G, where £P is the daily profit

PART II.

CHAPTER XXIX

Arithmetic, Harmonic, and Geometric Progression

309 Series A set of quantities each of which is formed from one or more of the preceding according to some fixed law is called a series. The successive quantities are called **terms** of the series

Thus (i) 5, 8, 11, 14, 17, , A.P
(ii) 1, 3, 9, 27, 81, , G P
(iii) 1, 3, 4, 7, 11, 18, H P

are examples of series In (i) each term is formed by adding 3 to the preceding term, in (ii) each term is 3 times the preceding, and in (iii) each term after the second is the sum of the two preceding terms

From these cases it is evident that when the law of formation of successive terms is recognised, we can write down as many terms of the series as we please

Arithmetic Progression

310 Definition A series in which each term is formed from the preceding by adding to it a constant quantity is called an Arithmetic Progression The constant quantity is called the common difference, and it is found by subtracting *any* term from the term which *follows* it

The abbreviation A P is used for the words *arithmetic progression*

Thus the following series of terms are in A P

$$8,\ 19,\ 30,\ 41,\ \ldots,\ \text{common difference}=11,$$

$$25c,\ 15c,\ 5c,\ -5c,\ \ldots,\ \text{common difference}=-10c$$

311 The standard form of an A P is

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,$$

in which the first term is a, and the common difference is d In this series we notice that the coefficient of d in any term is one less than the number which indicates the place of that term in the series.

Thus the 3^{rd} term is $a+2d$, or $a+(3-1)d$,
the 5^{th} term is $a+4d$, or $a+(5-1)d$

More generally, the n^{th} term is $a+(n-1)d$

EXAMPLE 1 *Find the 7^{th} and 24^{th} terms of the series* 21, 18, 15, ...

Here the common difference $= 18 - 21 = -3$

$$\text{the } 7^{th} \text{ term} = 21 + 6(-3) = 3,$$

$$\text{and the } 24^{th} \text{ term} = 21 + 23(-3) = -48$$

EXAMPLE 2 *The 5^{th} and 22^{nd} terms of an A.P. are 39 and -114 respectively, find the series*

Let a denote the first term, and d the common difference,

then $\qquad 39 = \text{the } 5^{th} \text{ term} = a + 4d, \qquad (1)$

and $\qquad -114 = \text{the } 22^{nd} \text{ term} = a + 21d \qquad (2)$

By subtraction, $\qquad 153 = -17d$, whence $d = -9$

From (1), $\qquad a = 39 - 4d = 39 + 36 = 75$

Thus the series is 75, 66, 57, ...

The last example shews that an A.P. is completely determined when any two terms are known, for these data supply two independent equations from which the first term and common difference can be found.

EXAMPLES XXIX. a.

(Some of the following Examples may be taken orally.)

Find the 8^{th} and 20^{th} terms of the following series

1. $4, 9, 14, \dots$ 2. $21, 18, 15, \dots$ 3. $10, 4, -2, \dots$

4. $-3\frac{1}{2}, 0, 3\frac{1}{2}, \dots$ 5. $a, 4a, 7a, \dots$ 6. $x, -2x, -5x, \dots$

7. $-4.2, -3, -1.8, \dots$ 8. $5p, p, -3p, \dots$ 9. $2, 7.2, 12.4, \dots$

Find the common difference and the 7^{th} term of the following series

10. $c, c-2d, c-4d, \dots$ 11. $x-2y, x-y, x, \dots$

12. $2a+3b, 3a+2b, 4a+b, \dots$ 13. $3a-b, 4a, 5a+b, \dots$

Find the series in Examples 14–18, given two terms in each

14. The 7^{th} term is 10, and the 13^{th} is -2

15. The 6^{th} term is 25, and the 20^{th} is 81

16. The 6^{th} term is 50, and the 41^{st} is 155

17. The 11^{th} term is $37\frac{1}{2}$, and the 16^{th} is 25

18. The 4^{th} term is 21, and the 51^{st} is -355

19. The 7^{th} and 11^{th} terms of an A.P. are $7b+5c$, and $11b+9c$, find the first term and common difference

20. Find the series in which the 5^{th} and 21^{st} terms are $7x-8y$ and $23x-40y$

21. Write down and express in the simplest form

(i) the n^{th} term of $3, 5, 7, \dots$; (ii) the m^{th} term of $8, 6, 4, \dots$;

(iii) the p^{th} term of $x, 5x, 9x, \dots$; (iv) the n^{th} term of $-y, 6y, 13y, \dots$

22 The 1st and 3rd terms of an A P are 60 and 32, find the n^{th} term

23 The n^{th} term of the series $3b+2c$, $5b+c$, $7b$, ... is $17b-5c$; find n

24 If p, $5p$, $6p+9$ are in A P, find p, and continue the series for 4 terms

312 Arithmetic Mean. When three quantities are in A P the middle term is called the **arithmetic mean** of the other two

Thus $8c$ is the arithmetic mean of $3c$ and $13c$

313 *To find the arithmetic mean of two given quantities* a *and* b

Let A be the required mean, then since a, A, b are in A P,

$$\text{the common difference} = b - A = A - a,$$

whence
$$A = \frac{a+b}{2}$$

Thus *the arithmetic mean of two quantities is half their sum*

314 When any number of quantities are in A P, the terms intermediate between the first and last are called the *arithmetic means* between those two terms

Between two given quantities it is always possible to insert any required number of arithmetic means

EXAMPLE *Insert 11 arithmetic means between 25 and* -11

Including the two given numbers there will be 13 terms, so that we have to find a series of 13 terms in A P of which 25 is the first and -11 the last

Let d be the common difference,

then
$$-11 = \text{the } 13^{th} \text{ term}$$
$$= 25 + 12d,$$

whence
$$d = -3,$$

and the series is $25 [22, 19, 16, \ldots, -2, -5, -8,] -11$,

the required means being the terms within brackets

315 *To insert* n *arithmetic means between* a *and* b

Including the two given terms at the beginning and end there will be $n+2$ terms in all, so that we have to find $n+2$ terms in A P of which a is the first and b the last

Let d be the common difference,

then
$$b = \text{the } (n+2)^{th} \text{ term}$$
$$= a + (n+1)d$$

whence
$$d = \frac{b-a}{n+1},$$

and the required means are

$$a + \frac{b-a}{n+1}, \quad a + \frac{2(b-a)}{n+1}, \quad \ldots \quad a + \frac{n(b-a)}{n+1}.$$

316. *To find a formula for the sum of* n *terms of the series*

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$$

Let S denote the sum of n terms, and let l denote the last term. Then the last term but one is $l-d$, the last term but two is $l-2d$, and so on

Hence $S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l$

By writing the series in the reverse order, beginning with l,

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a$$

From these results, by addition of corresponding terms, we have

$$2S=(a+l)+(a+l)+(a+l)+\ldots$$

the bracket being repeated n times,

that is,
$$S=\frac{n}{2}(a+l) \qquad (1)$$

But
$$l=a+(n-1)d, \qquad (2)$$

$$S=\frac{n}{2}\{2a+(n-1)d\} \qquad (3)$$

EXAMPLE 1 *Find the last term, and the sum of 25 terms of the series*

$$12, \; 9, \; 6, \ldots$$

Here $d=-3$, hence from formula (2),

$$l=12+24(-3)=-60$$

Again, from formula (1),

$$S=\tfrac{25}{2}(12-60)=-600$$

EXAMPLE 2. *Find the sum of* 4·9, 5·6, 6·3, ... *to 20 terms*

Here $a=4\cdot9$, $d=0\cdot7$, $n=20$, hence from (3),

$$S=\tfrac{20}{2}(2\times4\cdot9+19\times0\cdot7)$$
$$=10(9\cdot8+13\cdot3)=231$$

317 In the last article we have three standard formulæ In these we notice that *five* symbols a, d, l, n S are involved, and each formula contains *four* of them Hence when *any three* are given, a fourth can be found from the suitable formula, and one of the other formulæ will give the fifth

EXAMPLE 1 *Given* $l=-88$, $n=16$, $S=-448$, *find* a *and* d

We have $S=\frac{n}{2}(a+l)$, hence substituting for l, n, and S,

$$-448=8(a-88), \text{ whence } a=32.$$

We have $l=a+(n-1)d$, hence substituting for a, l, and n,

$$-88=32+15d; \text{ whence } d=-8$$

EXAMPLE 2. *How many terms of the series 42, 39, 36, ... must be taken that the sum may be 312?*

Here $S=312$, $a=42$, $d=-3$, and we require n.

Substituting in $S=\frac{n}{2}\{2a+(n-1)d\}$, we have

$$312=\frac{n}{2}\{2\times42+(n-1)(-3)\},$$

or
$$624=84n-3n(n-1),$$

whence
$$n^2-29n+208=0,$$

that is,
$$(n-13)(n-16)=0;$$
$$\therefore n=13, \text{ or } 16.$$

Both these values satisfy the conditions of the question. For if we write down the 14^{th}, 15^{th}, and 16^{th} terms we find that they are $3, 0, -3$, the sum of which is 0. Thus the sum of 16 terms is the same as that of 13 terms.

NOTE. It is obvious that when S, a, and d are given, we shall always have a quadratic from which to find n.

In the above example both roots of the quadratic in n give an intelligible answer to the problem. But the equation may give rise to one root which is positive and integral, and another which is fractional or negative. The latter would be rejected as incompatible with the conditions of the case.

318. The statement of the conditions of a problem in connection with series may sometimes be conveniently shortened by using the following notation. The successive terms may be denoted by

$$T_1,\ T_2,\ T_3,\ \ldots\ldots\ T_{n-2},\ T_{n-1},\ T_n,$$

where the suffix indicates the *number* of the term in the series. Similarly the sum of any assigned number of terms may be denoted by the letter S with a suitable suffix number. Thus in any example the symbols S_{25}, S_n may be used instead of the words "sum to 25 terms," "sum to n terms" respectively.

EXAMPLE. *In an A. P., if* $T_3+T_7=13$, $S_{13}=104$, *find* T_1, T_2, *and* T_3.

Let a be the first term and d the common difference, then

$$T_3+T_7=(a+2d)+(a+6d);$$
$$\therefore 2a+8d=13. \qquad (1)$$

Also
$$S_{13}=\frac{13}{2}(2a+12d)$$
$$=13a+78d;$$
$$\therefore 13a+78d=104. \qquad (2)$$

Equations (1) and (2) give $a=3\frac{1}{2}$, $d=\frac{3}{4}$.

Thus $T_1=a=3\frac{1}{2}$, $T_2=3\frac{1}{2}+\frac{3}{4}=4\frac{1}{4}$, $T_3=4\frac{1}{4}+\frac{3}{4}=5$.

319 The results of the following example should be noted and remembered for future use

EXAMPLE *Find the sum of*

(i) *the first* n *integers*, (ii) *the first* n odd *integers*

(i) The first term $=1$, the last $=n$, and the number of terms is n,

$$S=\frac{n}{2}(n+1)$$

(ii) The first term $=1$, the common difference $=2$, and the number of terms is n,

$$S=\frac{n}{2}\{2 \cdot 1+(n-1)2\}$$
$$=n^2$$

Thus the sum of any number of consecutive odd numbers beginning with unity is a perfect square

NOTE The numbers 1, 2, 3, are sometimes referred to as *the natural numbers*

EXAMPLES XXIX. b.

1. Write down the arithmetic mean between
   (i) 13 and 29, (ii) 56 and 12, (iii) $\frac{1}{3}$ and $\frac{1}{16}$,
   (iv) $5a$ and $-15a$, (v) 6 7 and 5 3, (vi) $a-x$ and $a+x$
2. Insert 4 arithmetic means between 130 and 55
3. Insert 11 arithmetic means between 25 and -11
4. Insert 5 arithmetic means between 7 4 and 20
5. Insert 6 arithmetic means between 19 and $36\frac{1}{2}$
6. Insert 7 arithmetic means between $11a$ and $-13a$
7. Insert 8 arithmetic means between 42 6 and 17 4

Find the last three terms of the following series

8. 3, 5, 7, to 20 terms
9. 15, 11, 7, to 35 terms
10. p, $7p$, $13p$, to 12 terms
11. x, $-3x$, $-7x$, to 16 terms.

Find the last term and the sum of each of the following series

12. 8, 17, 26, to 21 terms
13. 1, 3, 5, to 100 terms
14. 6, $5\frac{3}{5}$, $5\frac{1}{5}$, to 31 terms
15. 20 5, 18, 15 5, to 15 terms

Find the sum of each of the following series

16. 11, 15, 19, to 18 terms
17. 1, $2\frac{1}{4}$, $3\frac{1}{2}$, to 32 terms
18. 18, 15, 12, to 23 terms
19. 55, 66, 77, to 17 terms.

Find the sum of each of the following series:

20. $2, 3\frac{1}{3}, 4\frac{2}{3}, \ldots$ to 60 terms.

21. $25\frac{1}{5}, 16\frac{4}{5}, 8\frac{2}{5}, \ldots$ to 17 terms.

22. $4\cdot 9, 5\cdot 6, 6\cdot 3, \ldots$ to 20 terms.

23. $5\cdot 1, 4\cdot 7, 4\cdot 3, \ldots$ to 30 terms.

24. $x, -x, -3x, \ldots$ to x terms.

25. $-5p, 0, 5p, \ldots$ to p terms.

26. $4a-b, 3a-2b, 2a-3b, \ldots$ to 9 terms.

27. $5m-n, 3m-2n, m-3n, \ldots$ to 20 terms.

28. Find the sum of 50 arithmetic means between 20 and 120.

29. How many numbers between 100 and 500 are divisible by 9? Find their sum. Also find the sum of all the numbers from 100 to 500 inclusive which are *not* divisible by 9.

30. How many numbers between 65 and 200 are multiples of 6, and how many are not multiples? Find the sum of the multiples.

31. Find the sum of x arithmetic means between x and $3x$.

32. In a pile of timber each horizontal layer contains 3 beams more than the one above it. If on the top there are 70 beams, and on the ground 376, how many beams, and how many layers are there?

How many terms must be taken of the series

33. $39+33+27+\ldots$ to make 144?

34. $6\frac{1}{2}+5\frac{1}{2}+4\frac{1}{2}+\ldots$ to make $-137\frac{1}{2}$?

35. $20+18\frac{3}{4}+17\frac{1}{2}+\ldots$ to make $162\frac{1}{2}$?

36. $5a+7a+9a+\ldots$ to make $621a$?

37. In an A. P., given $S_n=45$, $a=18$, $d=-3$, find n.

38. A man has to travel 162 miles; he goes 30 miles the first day, 27 the second, 24 the third, and so on. How many days does he take for the journey?

39. Given $a=15b$, $d=-3b$, $S_n=-270b$, find n.

40. Two particles are projected in opposite directions (towards each other) from the ends of a straight tube, 268 inches long. One passes over 20 inches in the first second, 18 inches in the second, 16 inches in the third, and so on; the other passes over 24 inches in the first second, 23 inches in the second, 22 inches in the third, and so on. In what time will they meet and what distance will each have gone over?

320. Example 1. *In an A. P., if $S_4=28$, $S_8=48$, find S_{12}.*

Let a be the first term and d the common difference;

then
$$S_4=2(2a+3d)=28,$$
$$S_8=4(2a+7d)=48.$$

These equations give $a=\frac{31}{4}$, $d=-\frac{1}{2}$.

$$S_{12}=6(2a+11d)=6\left(\frac{31}{2}-\frac{11}{2}\right)=60.$$

EXAMPLE 2. *The sum of* 5 *numbers in* A P *is* 30, *and the sum of their squares is* 220, *find the numbers*

Let a denote the *middle* number, and d the common difference; then the numbers are

$$a-2d, \quad a-d, \quad a, \quad a+d, \quad a+2d$$

The sum of these is $5a$, whence $5a=30$, and $a=6$

If we take the terms in pairs, first and last, and so on, the sum of their squares is

$$2(a^2+4d^2)+2(a^2+d^2)+a^2,$$

$$5a^2+10d^2=220$$

Combining this with $a=6$, we obtain $d=\pm 2$

Thus the required numbers are 2, 4, 6, 8, 10

NOTE In examples of this kind when the number of terms to be found is *odd*, it is convenient to take a for the *middle* term, and d for the common difference When we have to find an *even* number of terms it is best to take $a-d$ and $a+d$ for the *two middle* terms, so that $2d$ is the common difference Thus four such terms may be denoted by

$$a-3d, \quad a-d, \quad a+d, \quad a+3d$$

EXAMPLES XXIX. c.

(*Miscellaneous*)

1. How many terms are there in the series 205, 192, 179, … -107?

2. To how many terms must the series 126, 117, 108, … be continued to make the sum equal to 945? Explain the double answer

3 Sum each of the following series to 80 terms

(i) $2+3+6+7+10+11+\dots$, (ii) $-2+3-6+7-10+11-\dots$.

4 In an A P, if $S_3=24$, $S_6=54$, find S_9

5 Find the A.P in which $S_{10}=465$, and $9S_3=4S_6$

6. Sum the following series

(i) $(a-2b+c)+(a-b)+(a-c)+\dots$ to 12 terms,

(ii) $(p+2q-r)+(p+q)+(p+r)+\dots$ to 10 terms

[*The terms are separated by the signs in deeper type*]

7. In a series of 10 terms the sum of the first 5 is 7, and the sum of the last 5 is 12 Find the 1st term and the common difference

8 If the 8th term of an A P is double the 13th term, shew that the 4th term is double the 11th

9. If a, x, y, b are in A P, find x and y in terms of a and b

10 Find 5 numbers in A P such that their sum is 315, and the difference between the last and first is 28

11. The sum of 4 numbers in A.P is 38, if the greatest is 22, what are the others?

12 The sum of 3 numbers in A P is 111, and the difference of the squares of the greatest and least is 1776 Find the numbers

13 The sum of 4 numbers in A P is 28, and the sum of their squares is 216 Find the numbers

14 An A P consists of 21 terms, the sum of the three terms in the middle is 129, and the sum of the last three terms is 237, find the series

15 Between x and y there are 4 arithmetic means and also 3 arithmetic means The sum of the four exceeds the sum of the three by 10, and the first of the three exceeds the first of the four by $\frac{1}{2}$ Find x and y

16 Four numbers are in A P The product of the second and third exceeds the product of the other two by 32, and the product of the second and fourth exceeds the product of the other two by 72 Find the numbers

17 How long will it take to pay a debt of £10 by weekly payments increasing by 6*d* per week and beginning with 2*s*?

18 Two men set out to meet each other from two places 165 miles apart One travels 15 miles the first day, 14 the second, 13 the third, and so on The other travels 10 miles the first day, 12 the second, 14 the third, and so on When will they meet?

19 A person saves each year £10 more than he saved in the preceding year, and he saves £20 the first year How many years would it take for his savings, not including interest, to amount to £10,000?

Harmonic Progression

321 Definition A series of quantities is said to be in **Harmonic Progression**, when their reciprocals are in Arithmetic Progression

Hence a, b c are in H P when $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A P

Thus $\frac{1}{3}, \frac{1}{5}, \frac{1}{7},$ are in H P, because 3, 5, 7, are in A P,

and $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d},$ are in H P, because a, $a+d$, $a+2d$, are in A P

Examples in H P are usually solved by inverting the terms and using the properties of the corresponding A P There is no general formula for the sum of a number of terms in H.P

322. Harmonic Mean. If a, H, b are in harmonic progression, H is said to be the harmonic mean between a and b

Since $\dfrac{1}{a}$, $\dfrac{1}{\mathsf{H}}$, $\dfrac{1}{b}$ are in A.P,

$$\frac{1}{\mathsf{H}}=\text{half the sum of } \frac{1}{a} \text{ and } \frac{1}{b} \qquad \text{[Art 313]}$$

$$=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{a+b}{2ab}$$

$$\mathsf{H}=\frac{2ab}{a+b}$$

Thus *the harmonic mean between two quantities is twice their product divided by their sum*

NOTE. This result may also be conveniently remembered in the form

$$\frac{2}{\mathsf{H}}=\frac{1}{a}+\frac{1}{b}$$

EXAMPLE 1 *Insert 3 harmonic means between* $2\frac{2}{5}$ *and* 12

We first find *five* terms in A.P of which $\frac{5}{12}$ is the first and $\frac{1}{12}$ the last

Let d be the common difference;

then $\qquad \frac{1}{12}=\frac{5}{12}-4d$, whence $d=-\frac{1}{12}$

the arithmetic series is $\frac{5}{12}$, $\left[\frac{4}{12}, \frac{3}{12}, \frac{2}{12},\right]$ $\frac{1}{12}$

The required means are the reciprocals of the terms in brackets, viz., 3, 4, 6

EXAMPLE 2 *Find the* H.P *in which the* 15^{th} *term is* $\frac{1}{25}$, *and the* 23^{rd} *term is* $\frac{1}{41}$ *Find the* 40^{th} *term*

Let a be the first term, and d the common difference of the corresponding A.P,

then $\qquad 25=\text{the } 15^{\text{th}} \text{ term}=a+14d, \qquad (1)$

and $\qquad 41=\text{the } 23^{\text{rd}} \text{ term}=a+22d \qquad (2)$

From (1) and (2), $\qquad a=-3,\ d=2$

Hence the A.P. is $\qquad -3, -1, 1, 3, \ldots;$

and the H.P is $\qquad -\frac{1}{3}, -1, 1, \frac{1}{3}, \ldots.$

Again, the 40^{th} term of the A.P $=-3+39 \cdot 2=75$,

$\therefore$ the 40^{th} term of the H P $=\frac{1}{75}$

EXAMPLES XXIX. d

Find the harmonic mean between

1. 3 and 5 2. -4 and -7 3. $\frac{1}{x}$ and $\frac{1}{y}$ 4. $\frac{p}{q}$ and $\frac{q}{p}$

5. Insert 3 harmonic means between $-\frac{1}{3}$ and $\frac{1}{7}$

6. Insert 4 harmonic means between 1 and 6

7. Insert 3 harmonic means between -6 and 6

Find the 5th and 8th terms of the following series in H P

8. $\frac{1}{2}, \frac{4}{9}, \frac{2}{5},$ 9. $\frac{1}{3}, \frac{3}{7}, \frac{3}{5},$ 10. $1\frac{1}{3}, 1\frac{11}{17}, 2\frac{2}{13},$

11. The 12th term of an H P is $\frac{1}{5}$, and the 19th term is $\frac{3}{22}$ Find the 4th term

12. Find two numbers such that their arithmetic mean is 7 and their harmonic mean $6\frac{6}{7}$

13. If a, x, y, b are in H P, find x and y in terms of a and b

14. If a, b, c are in A P and b, c, d in H P, prove that $ad=bc$

Geometric Progression

323 DEFINITION A series in which each term is formed from the preceding by multiplying it by a *constant factor* is called a Geometric Progression The constant factor is more often called the common ratio, and is found by dividing *any* term by the term which *precedes* it

Thus the following series of terms are in G P

$$1, \quad 4, \quad 16, \quad 64, \quad \ldots \qquad \text{common ratio}=4,$$

$$5, \ -\tfrac{10}{3}, \ \tfrac{20}{9}, \ -\tfrac{40}{27}, \ \ldots, \qquad \text{common ratio}=-\tfrac{2}{3}$$

NOTE Ratio, which has not yet been formally defined, will be fully treated in Chap XXXIII It is sufficient here for the pupil to remember that (as in Arithmetic) *the ratio of one quantity to a second is the quotient obtained by dividing the first by the second*

324 The standard form of a G P is

$$a, \ ar, \ ar^2, \ ar^3, \ ar^4, \ \ldots,$$

in which the first term is a, and the common ratio is r In this series we notice that the index of r in any term is one less than the number which indicates the place of that term in the series

Thus the 3rd term is ar^2, or ar^{3-1},

the 5th term is ar^4, or ar^{5-1}

More generally, the pth term is ar^{p-1},

and the nth term is ar^{n-1}.

EXAMPLE *Find the* 8^{th} *term of the series* $-\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \ldots$

The common ratio is $\frac{1}{2} \div \left(-\frac{1}{3}\right)$, or $-\frac{3}{2}$,

$$\therefore \text{ the } 8^{th} \text{ term} = -\frac{1}{3} \times \left(-\frac{3}{2}\right)^7 = -\frac{1}{3} \times \left(-\frac{2187}{128}\right) = \frac{729}{128}$$

325 Geometric Mean When three quantities are in G P, the middle one is called the **geometric mean** between the other two

326 *To find the geometric mean between two given quantities* a *and* b

Let G be the required mean, then since a, G, b are in G P,

$$\frac{b}{G} = \frac{G}{a}, \text{ each being equal to the common ratio},$$

$$\therefore G^2 = ab, \text{ or } G = \sqrt{ab}$$

Thus *the geometric mean between two quantities is the square root of their product* [It is usual to take the positive square root]

327 If A, G, H are the arithmetic, geometric, and harmonic means between a and b, we have proved that

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}$$

Hence $$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$$

Therefore G is the geometric mean between A and H That is, *the geometric mean between any two quantities is the geometric mean between their arithmetic and harmonic means*

328 When any number of quantities are in G P, the terms intermediate between the first and last are called the geometric means between those two terms

EXAMPLE 1 *Insert* 4 *geometric means between* $\frac{1}{8}$ *and* 128

We have to find *six* terms in G P of which $\frac{1}{8}$ is the 1^{st} and 128 the 6^{th}.

Let r be the common ratio,

then $$128 = \text{the } 6^{th} \text{ term} = \frac{1}{8} \times r^5,$$

$$\therefore r^5 = 8 \times 128 = 4 \times 256 = 4^5$$

$$\therefore r = 4, \text{ and the means are } \frac{1}{2}, 2, 8, 32$$

EXAMPLE 2 *Find three numbers in* G P *such that their sum is* 42, *and their product* 512

Let the three numbers be represented by a, ar, ar^2

The product $= a^3r^3 = 512$, whence $ar = 8$, or $a = \frac{8}{r}$

The sum $= a(1+r+r^2) = 42$, whence $\frac{8}{r}(1+r+r^2) = 42$,

that is, $$4r^2 - 17r + 4 = 0, \text{ or } (4r-1)(r-4) = 0$$

$$\therefore r = 4, \text{ or } \frac{1}{4}, \text{ and the numbers are } 2, 8, 32.$$

EXAMPLES XXIX. e.

Find the 5^{th} and 8^{th} terms of the series

1 3, 6, 12, ... 2 64, -32, 16, ... 3. $4\frac{1}{5}$, $8\frac{2}{5}$, $16\frac{4}{5}$, ...

Find the 7^{th} and 10^{th} terms of the series

4 $-\frac{1}{27}$, $\frac{1}{9}$, $-\frac{1}{3}$, ... 5. $\frac{1}{256}$, $\frac{1}{64}$, $\frac{1}{16}$, ... 6 512, -256, 128, ...

7 Find the 6^{th} term of the series -12.15, 8.1, -5.4, ...

8 Continue the series $9\frac{18}{25}$, $-16\frac{1}{5}$, 27, ... to 3 terms

9 Write down the geometric mean between
(i) 8 and 18, (ii) -3 and -27, (iii) $2x$ and $8x^3$

10 Find the geometric mean between

$$x^2-6ax+9a^2 \text{ and } 9x^2+6ax+a^2$$

11. Insert 3 geometric means between
(i) $\frac{1}{4}$ and $\frac{4}{81}$, (ii) $1\frac{4}{5}$ and $\frac{1}{45}$

12. Insert 4 geometric means between 3 and 96

13 Insert 6 geometric means between 56 and $-\frac{7}{16}$

14 Find two numbers such that their arithmetic mean is 25, and their geometric mean 24

15 The arithmetic mean between two numbers is $242\frac{1}{2}$, and the geometric mean is 180, find the harmonic mean

16 Find three numbers in G P whose sum is $4\frac{2}{3}$, and whose product is -8

329 *To find a formula for the sum of* n *terms of the series*

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots$$

Let S_n denote the sum of n terms,

then $$S_n=a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}$$

Multiplying every term by r, we have

$$rS_n= \quad ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}+ar^n$$

Hence, by subtraction,

$$S_n-rS_n=a-ar^n,$$

$$(1-r)S_n=a(1-r^n),$$

$$S_n=\frac{a(1-r^n)}{1-r} \qquad (1)$$

Changing the signs in numerator and denominator,

$$S_n=\frac{a(r^n-1)}{r-1} \qquad (2)$$

It will be found convenient to remember both of the above forms for S_n, using (1) in all cases *except when* r *is positive and greater than* 1

EXAMPLE *Sum the following geometric progressions*

(i) 4, 12, 36, ... *to* 8 *terms*, (ii) $\frac{16}{27}$, $-\frac{4}{9}$, $\frac{1}{3}$, ... *to* 7 *terms*

(i) Here $r=3$, and we use formula (2).

$$S_8=\frac{4(3^8-1)}{3-1}=2(6561-1)=13120$$

(ii) Here $r=\frac{1}{3}\div\left(-\frac{4}{9}\right)=-\frac{3}{4}$, and we use formula (1)

$$S_7=\frac{\frac{16}{27}\left\{1-\left(-\frac{3}{4}\right)^7\right\}}{1-\left(-\frac{3}{4}\right)}=\frac{16}{27}\cdot\frac{1+\frac{3^7}{4^7}}{1+\frac{3}{4}}, \text{ for } \left(-\frac{3}{4}\right)^7=-\frac{3^7}{4^7}$$

$$=\frac{16}{27}\times\frac{4}{7}\times\frac{4^7+3^7}{4^7}$$

$$=\frac{1}{27\times7}\times\frac{18571}{4^4}=\frac{2653}{6912}$$

$4^7=16384$
$3^7=\ \ 2187$
$\quad\ \ 18571$

EXAMPLES XXIX f

Find the sum of the following geometric series

1. $\frac{1}{2}$, 2, 8, ... to 6 terms
2. 3, -1, $\frac{1}{3}$, ... to 6 terms.
3. $\frac{1}{24}$, $\frac{1}{12}$, $\frac{1}{6}$, ... to 10 terms
4. $-\frac{2}{5}$, $\frac{1}{2}$, $-\frac{5}{8}$, ... to 6 terms
5. 1, $-\frac{1}{2}$, $\frac{1}{4}$, ... to 12 terms
6. $1\frac{11}{16}$, $-1\frac{1}{8}$, $\frac{3}{4}$, ... to 7 terms
7. 108, 72, 48, ... to 8 terms
8. $\frac{3}{4}$, $-\frac{1}{2}$, $\frac{1}{3}$, ... to 8 terms
9. $15\frac{3}{16}$, $-20\frac{1}{4}$, 27, ... to 8 terms
10. 4, 3, $2\frac{1}{4}$, ... to 7 terms
11. 2, 4, 8, ... to m terms.
12. 3, -9, 27, ... to $2p$ terms

13. If l is the last term of the series $a+ar+ar^2+\dots$ to n terms, shew that $S_n=\frac{rl-a}{r-1}$.

14. Find the last term and the sum of the following series
(i) 64, 32, 16, ... to 10 terms, (ii) 6, -18, 54, ... to 6 terms

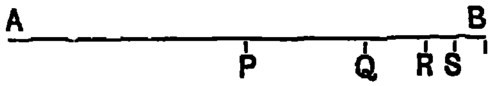
15. Find the sum of $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{9}+\frac{1}{8}-\frac{1}{27}+\dots$ to $2n$ terms

330 Consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots$

$$S_n=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}$$

Thus it appears that however many terms we take of this series *the sum is always less than* 2 And by taking n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please In other words, by taking a sufficient number of terms the sum can be made to differ from 2 by a quantity as small as we please

This result may also be illustrated as follows.



Let unity be represented by a line one inch in length. Take AB = 2 in.; bisect AB at P, PB at Q, QB at R, RB at S, and so on.

Then $\qquad AP=1, \quad PQ=\frac{1}{2}, \quad QR=\frac{1}{4}, \quad RS=\frac{1}{8}, \ldots$

Thus the sum of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is represented graphically by

$$AP+PQ+QR+RS+\ldots$$

Now by continuing the process of bisection, this latter sum can be made as nearly equal to the whole length AB as we please, but the sum of the parts can never exceed AB. That is, the sum of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ to n terms can never exceed 2, but tends to this value as its limit when the number of terms is indefinitely increased.

331. In the standard series $a+ar+ar^2+\ldots+ar^{n-1}$, we have

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now if r is numerically less than 1, r^n diminishes as n increases; thus the greater the value of n the smaller the value of r^n, and consequently of $\frac{ar^n}{1-r}$. Hence we can make S_n differ from $\frac{a}{1-r}$ by as small a quantity as we please by making n sufficiently great.

The expression $\frac{a}{1-r}$ is called the **limit of the sum** or the **sum to infinity**, and may be denoted by the symbol S_∞, replacing n by the symbol for *infinity*.

EXAMPLE 1. *Sum the following series to infinity*:

$$\text{(i)}\ 3,\ -\frac{9}{4},\ \frac{27}{16},\ \ldots; \qquad \text{(ii)}\ 2,\ \frac{1}{y},\ \frac{1}{2y^2},\ \ldots$$

(i) Here $a=3$, $r=-\frac{3}{4}$. $\qquad S_\infty=\dfrac{3}{1+\frac{3}{4}}=\dfrac{12}{7}=1\frac{5}{7}$.

(ii) The series can only be summed to infinity if $r<1$.

Now $r=\frac{1}{2y}$; hence provided that $1<2y$, or $y>\frac{1}{2}$,

$$S_\infty=\frac{2}{1-\frac{1}{2y}}=\frac{4y}{2y-1}.$$

EXAMPLE 2. *Each term of an infinite G.P. is equal to three times the sum of all the terms which follow. If the* 5^{th} *term is* $\dfrac{3}{64}$, *find the series.*

Let the series be denoted by $a+ar+ar^2+\ldots$;
then since *each* term is three times the sum of all that follow,

$$a=3(ar+ar^2+ar^3+\ldots \text{ to inf.});$$

that is,
$$a=\frac{3ar}{1-r}, \text{ whence } r=\frac{1}{4}.$$

Again,
$$\text{the } 5^{\text{th}} \text{ term}=ar^4=\frac{a}{4^4},$$

$$\therefore \frac{a}{4^4}=\frac{3}{64}=\frac{3}{4^3}, \text{ whence } a=12.$$

Thus the terms of the series are $12,\ 3,\ \dfrac{3}{4},\ \dfrac{3}{16},\ \dfrac{3}{64},\ \ldots$

332. The evaluation of a recurring decimal illustrates the use of infinite geometric progressions.

EXAMPLE. *Express* $0.2\dot{3}\dot{5}$ *as a common fraction.*

$$\begin{aligned}
0.2\dot{3}\dot{5} &= 0.2353535\ldots \\
&= \frac{2}{10}+\frac{35}{1000}+\frac{35}{100000}+\ldots \\
&= \frac{2}{10}+\frac{35}{10^3}\left(1+\frac{1}{10^2}+\frac{1}{10^4}+\ldots\right) \\
&= \frac{2}{10}+\frac{35}{10^3}\cdot\frac{1}{1-\frac{1}{10^2}}=\frac{2}{10}+\frac{35}{10^3}\cdot\frac{10^2}{99} \\
&= \frac{2}{10}+\frac{35}{990}=\frac{233}{990}.
\end{aligned}$$

EXAMPLES XXIX. g.

Sum the following series to infinity:

1. $9,\ 3,\ 1,\ \ldots$ 2. $\frac{1}{4},\ -\frac{3}{16},\ \frac{9}{64},\ \ldots$ 3. $5,\ \frac{3}{2},\ \frac{9}{20},\ \ldots$

4. $\frac{3}{5},\ -\frac{1}{3},\ \frac{5}{27},\ \ldots$ 5. $0.9,\ 0.03,\ 0.001,\ \ldots$ 6. $3.2,\ -1.6,\ 0.8,\ \ldots$

Which of the following series can be summed to infinity? Give the sum when possible, stating any necessary condition.

7. $3,\ -3^2,\ 3^3,\ \ldots$ 8. $x,\ -x^2,\ x^3,\ \ldots$ 9. $p,\ 1,\ \frac{1}{p},\ \ldots$

10. $a,\ \frac{a}{r},\ \frac{a}{r^2},\ \ldots$ 11. $1,\ 0.5,\ 0.25,\ \ldots$ 12. $a^3b,\ -a^2b^2,\ ab^3,\ \ldots$

Express in the form of fractions, by the method of Art. 332:

13. $0.\dot{7}$ 14. $0.6\dot{8}$ 15. $0.\dot{3}\dot{6}$ 16. $0.\dot{4}7\dot{9}$ 17. $3.1\dot{6}$

18 Find the G P in which $S_\infty = 24$, and the second term is 6

19 The first two terms of an infinite G P are together equal to 1 and every term is twice the sum of all that follow Find the series

20 Find the G P in which the first term is 12, and $S_\infty = 8$

21. Find the G P in which $S_4 = \frac{65}{54}$, and $S_\infty = 1\frac{1}{2}$

22 Find an infinite G P in which the first term is n, and each term is n times the sum of all that follow it

23 Sum to infinity the geometric progression

$$\frac{x+1}{x^2} - \frac{1}{x} + \frac{1}{x+1} - \ldots,$$

where x is any positive quantity.

24. Prove that the sums of the two progressions

$$1,\ \tfrac{4}{5},\ (\tfrac{4}{5})^2,\ (\tfrac{4}{5})^3, \ \ldots, \qquad \tfrac{5}{2},\ \tfrac{5}{4},\ \tfrac{5}{8},\ \tfrac{5}{16}, \ \ldots,$$

approach each other without limit as the number of terms is increased

25 A person is entitled to an annual payment which in each year is less by one-tenth than it was the year before Shew that however long it goes on, he cannot receive more than a certain sum in all

26 The middle points of the sides of an equilateral triangle are joined, forming a second triangle, a third triangle is formed by joining the middle points of the second, and the process is continued indefinitely If the perimeter and area of the original triangle are p and A respectively, find (i) the sum of the perimeters, (ii) the sum of the areas of all the triangles

333 Miscellaneous Applications Hitherto each series given for summation has been known to be either arithmetic or geometric. In some cases the first step is to discover the law of the series

EXAMPLE 1 *To sum the following series to n terms*

(i) $2,\ 2\frac{1}{5},\ 2\frac{21}{50}, \ldots$; (ii) $1,\ 2\frac{1}{5},\ 3\frac{2}{5}, \ldots,$ (iii) $-\frac{1}{7},\ -\frac{1}{2},\ \frac{1}{3}, \ldots$

(i) The terms are clearly not in A P

Since $2\frac{21}{50} - 2\frac{1}{5} = \frac{11}{10} = 2\frac{1}{5} - 2$, the terms are in G P. with common ratio $\frac{11}{10}$ Hence S_n can be found

(ii) This is evidently an A P, common difference $1\frac{1}{5}$, S_n can be found

(iii) By writing the series in the form $\frac{1}{-7}, \frac{1}{-2}, \frac{1}{3}, \ldots$ the denominators are in A P Hence the series is harmonic and cannot be summed

EXAMPLE 2 *Find the sum to* n *terms of the series*

(i) $(a+p)+(2a+px)+(3a+px^2)+ \dots$,

(ii) $3+6+10+16+ \dots$

(i) is neither arithmetic nor geometric, but by separating the terms in a from those in x, the series may be written

$$(a+2a+3a+ \dots +na)+(p+px+px^2+ \dots +px^{n-1})$$

Now the a-series is arithmetic and the x-series geometric;

$$S_n=\frac{n(n+1)}{2}a+\frac{p(x^n-1)}{x-1}.$$

(ii) may be written $(2+1)+(4+2)+(6+4)+(8+8)+ \dots$,

or $$(2+4+6+8+ \dots +2n)+(1+2+4+8+ \dots +2^{n-1}),$$

$$S_n=2\left[\frac{n(n+1)}{2}\right]+\frac{2^n-1}{2-1}=n(n+1)+2^n-1$$

EXAMPLE 3 *Sum to* n *terms the series whose* n[th] *terms are* $4n-5$ *and* $3 \cdot 2^n-4n$ *respectively*

Put $n=1, 2, 3, \dots$ successively in $4n-5$.

Then $T_1=4-5=-1$, $T_2=4 \cdot 2-5=3$, $T_3=4 \cdot 3-5=7$, $\dots$.

Thus the series is the A P $-1+3+7+ \dots$,

$$S_n=\frac{n}{2}\{-2+(n-1)\times 4\}=n(2n-3)$$

Again, putting $n=1, 2, 3, \dots$ successively in $3 \cdot 2^n-4n$

$$T_1=3 \cdot 2-4 \cdot 1,\quad T_2=3 \cdot 2^2-4 \cdot 2,\quad T_3=3 \cdot 2^3-4 \cdot 3,\quad \dots$$

Thus the series $=3(2+2^2+2^3+ \dots +2^n)-4(1+2+3+ \dots +n)$.

$$S_n=3\left\{\frac{2(2^n-1)}{2-1}\right\}-4\left\{\frac{n(n+1)}{2}\right\}=6(2^n-1)-2n(n+1).$$

EXAMPLE 4 *If* a, b, c *are in* H P, *prove that*

(i) $(b+c-a)^2$, $(c+a-b)^2$, $(a+b-c)^2$ *are in* A.P

(ii) $(2a-b)$, b, $(2c-b)$ *are in* G P

(i) $(b+c-a)^2$, $(c+a-b)^2$, $(a+b-c)^2$ will be in A.P.

if $$(a+b-c)^2-(c+a-b)^2=(c+a-b)^2-(b+c-a)^2,$$

that is, if $$2a(2b-2c)=2c(2a-2b),$$

that is, if $$\frac{a(b-c)}{abc}=\frac{c(a-b)}{abc}, \text{ or } \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$$

and this is the condition that a, b, c are in H P

(ii) $2a-b$, b, $2c-b$ will be in G P

if $$b^2=(2a-b)(2c-b),$$

that is, if $$b^2=4ac-2b(a+c)+b^2,$$

that is, if $$2ac=b(a+c), \text{ or } b=\frac{2ac}{a+c},$$

that is, if b is the harmonic mean between a and c

EXAMPLES XXIX h.

(*Miscellaneous.*)

1. Find the 6^{th} term and the sum to 6 terms of each of the following series

(i) $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots$; (ii) $2, 2\frac{1}{2}, 3\frac{1}{8}, \ldots$; (iii) $2, 2\frac{1}{2}, 3, \ldots$

2. Sum the following series

(i) $1+1\frac{3}{4}+3\frac{1}{16}+\ldots$ to 6 terms, (ii) $1+1\frac{3}{4}+2\frac{1}{2}+\ldots$ to 6 terms

3. Sum the following series

(i) $1\frac{4}{5}-1\frac{1}{5}+\frac{4}{5}-\ldots$ to 8 terms, (ii) $1\frac{4}{5}+1\frac{1}{5}+\frac{3}{5}+\ldots$ to 12 terms

4. The n^{th} term of a series is $\frac{n}{5}+2$, find the sum of 49 terms

5. Find the law of the following series, and write down the sum of n terms in each case

(i) $2+4+10+28+\ldots$, [*subtract 1 from each term.*]

(ii) $1+5+13+29+\ldots$, [*add 3 to each term.*]

(iii) $2+8+26+80+\ldots$

6. The n^{th} term of a series is $3n-2$, find S_n

7. The n^{th} term of a series is 3^n-2, find S_n

Sum the following series, each to 20 terms

8. $3(2a+3b)+3(3a+2b)+3(4a+b)+\ldots$

9. $2(3a-4b)+2(4a+3b)+2(5a+2b)+\ldots$

10. Sum the series $(t+1)+\frac{3}{2}(t+1)+2(t+1)+\ldots$ to t terms.

11. In an A P the sum of the first 7 terms is 10, and the sum of the next 7 terms 17, find the series

12. I propose to take 30 consecutive terms of the series

$$100+99+98+97+\ldots;$$

at which term must I begin that their sum may be 1155?

13. Find the sum of n terms of the series

$$1 \cdot 1+2 \cdot 01+3 \cdot 001+4 \cdot 0001+\ldots$$

14. The difference between two numbers is 3, and the difference between their arithmetic and harmonic means is $\frac{3}{14}$ Find the numbers

15. The sum of an infinite G P is 3, and the sum of its first two terms is $2\frac{2}{3}$, show that there are two such series, and find them

16. In an infinite G P in which all the terms are positive, shew that the sum cannot be less than four times the second term of the series

17 If the sum of n terms of an A P is $3n^2+4n$ for all values of n, find the series

18 If a, b, c are in G P, prove that

(i) $\dfrac{a}{c}=\dfrac{(a+b)^2}{(b+c)^2}$, (ii) $a+b$, $2b$, $b+c$ are in H P

19 If $b+c$, $c+a$, $a+b$ are in H P, then a^2, b^2, c^2 are in A P

20. If $pn+qn^2$ is the sum of n terms of an A P, find the common difference, and the r^{th} term of the series

21. Sum the following series

(i) $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+$ to n terms,

(ii) $\left(a+\frac{1}{3}\right)+\left(3a-\frac{1}{6}\right)+\left(5a+\frac{1}{12}\right)+$ to $2p$ terms

22 If the m^{th} term of an H P is equal to n, and the n^{th} term is equal to m, prove that the $(m+n)^{th}$ term is equal to $\dfrac{mn}{m+n}$.

23 A boy arranges rows of marbles one against the other so that each row contains one marble less than the preceding The last row consists of one marble only, which forms the apex of a triangle If the boy has 153 marbles, how many marbles are there in the base of the biggest triangle he can construct?

24 A person pledges his services for a year of 313 working days at the remuneration of 1d for the first day, 2d for the second, 3d for the third, 4d for the fourth, and so on. What sum, to the nearest penny, will he receive in all?

25. If xy, y^2, z^2 are in A P, then y, z, $2y-x$ are in G P

26. If a, b, c are in A P, and a, $b-a$, $c-a$ in G P, prove that

$$a=\frac{b}{3}=\frac{c}{5}$$

27 A man puts by for his son on every birthday a half-crown for every year of his age How old will the son be when the total sum put by amounts to £17?

28 The yearly output of a gold mine decreases every year 13 per cent of its amount during the previous year Given that the first year's output is £260,000, and that $(0\ 87)^{10}=0\ 24842$ approximately, find (i) the total output for the first ten years, (ii) the total output for all time

29 Shew that if a and b are such that the sum of the squares of the three arithmetic means inserted between them is equal to $(a+b)^2$, then the sum of the cubes of these means is equal to $\frac{9}{4}(a+b)^3$.

CHAPTER XXX.

THE THEORY OF INDICES

334 Positive Integral Indices Up to the present all the rules relating to indices have been based on the definition that *a^m stands for the product of m factors each equal to a*, where m is necessarily a positive whole number The laws of positive integral indices have been exemplified in special cases, thus we have seen that

(i) $a^7 \times a^3 = a^{7+3} = a^{10}$, (ii) $a^7 \div a^3 = a^{7-3} = a^4$, (iii) $(a^7)^3 = a^{7\times 3} = a^{21}$.

Each of these statements is an illustration of a general theorem relating to positive integral indices We proceed to give formal proofs of these theorems

335 THEOREM I *To prove that* $a^m \times a^n = a^{m+n}$ *when* m *and* n *are positive integers*

By definition, $a^m = a\ a\ a \ldots$ to m factors,

$a^n = a\ a\ a \ldots$ to n factors,

$$a^m \times a^n = (a\ a\ a \ldots \text{ to } m \text{ factors}) \times (a\ a\ a \ldots \text{ to } n \text{ factors})$$
$$= a\ a\ a \ldots \text{ to } (m+n) \text{ factors}$$
$$= a^{m+n}, \text{ by definition}$$

If p is also a positive integer, then

$$a^m \times a^n \times a^p = a^{m+n+p},$$

and so for any number of factors

The result $a^m \times a^n = a^{m+n}$ is usually known as the **fundamental Index Law.**

336 THEOREM II *To prove that when* m *and* n *are positive integers,*

$$a^m \div a^n = a^{m-n}, \text{ when } m > n,$$
$$= \frac{1}{a^{n-m}}, \text{ when } n > m$$

By definition, $$a^m \div a^n = \frac{a^m}{a^n} = \frac{a\ a\ a \ldots \text{ to } m \text{ factors}}{a\ a\ a \ldots \text{ to } n \text{ factors}}$$

If $m > n$, all the n factors of the denominator cancel with n factors in the numerator, leaving $m-n$ factors,

$$a^m \div a^n = a\ a\ a \ldots \text{ to } (m-n) \text{ factors}$$
$$= a^{m-n}$$

Similarly, $$a^m \div a^n = \frac{1}{a^{n-m}}, \text{ if } n > m$$

337 THEOREM III *To prove that* $(a^m)^n = a^{mn}$ *when* m *and* n *are positive integers*

$$(a^m)^n = a^m \cdot a^m \cdot a^m \ldots \text{ to } n \text{ factors}$$
$$= (a \cdot a \cdot a \ldots \text{ to } m \text{ factors})(a \cdot a \cdot a \ldots \text{ to } m \text{ factors}) \ldots$$

the bracket being repeated n times,

$$= a \cdot a \cdot a \ldots \text{ to } mn \text{ factors}$$
$$= a^{mn}$$

338 These results are derived from a definition which is intelligible only on the supposition that the indices are *positive* and *integral* But it is found convenient to use fractional and negative indices, such as $a^{\frac{4}{5}}$, a^{-7}, or, more generally, $a^{\frac{p}{q}}$, a^{-n}, and these have at present no intelligible meaning For the definition of a^m, upon which we based the three theorems just proved, is no longer *applicable when m is fractional, or negative*

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws We therefore determine meanings for symbols such as $a^{\frac{p}{q}}$, a^{-n}, in the following way we assume that they conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads us It will be found that the symbols so interpreted will also obey the other laws enunciated in Theorems II and III

339 *To find a meaning for* $a^{\frac{p}{q}}$, *when* p *and* q *are positive integers*

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing each of the indices m and n by $\frac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}$$

Similarly, $$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}$$

Proceeding in this way for 4, 5, ... q factors, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots \text{ to } q \text{ factors} = a^{\frac{qp}{q}},$$

that is, $$\left(a^{\frac{p}{q}}\right)^q = a^p$$

Hence $$a^{\frac{p}{q}} = \sqrt[q]{a^p}, \text{ by taking the } q^{\text{th}} \text{ root}$$

Or, in other words, $a^{\frac{p}{q}}$ is equal to *the* q^{th} *root of* a^p

EXAMPLES (i) $x^{\frac{5}{7}} = \sqrt[7]{x^5}$, (ii) $a^{\frac{1}{3}} = \sqrt[3]{a}$, (iii) $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

(iv) $a^{\frac{2}{3}} \times a^{\frac{5}{6}} = a^{\frac{2}{3}+\frac{5}{6}} = a^{\frac{3}{2}}$, (v) $k^{\frac{n}{2}} \times k^{\frac{2}{3}} = k^{\frac{n}{2}+\frac{2}{3}} = k^{\frac{3n+4}{6}}$.

(vi) $3a^{\frac{2}{3}}b^{\frac{1}{2}} \times 4a^{\frac{1}{6}}b^{\frac{5}{6}} = 3 \cdot 4a^{\frac{2}{3}+\frac{1}{6}}b^{\frac{1}{2}+\frac{5}{6}} = 12a^{\frac{5}{6}}b^{\frac{4}{3}}$.

340 Note on the equivalence of $a^{\frac{1}{n}}$ and $\sqrt[n]{a}$

Consider the values of $\sqrt{9}$, $\sqrt{81}$, and $\sqrt[4]{81}$

We have seen that $\sqrt{9}=9^{\frac{1}{2}}$, $\sqrt{81}=81^{\frac{1}{2}}$, $\sqrt[4]{81}=81^{\frac{1}{4}}$

Now $\qquad \sqrt{9}=+3$ or -3, and $\sqrt{81}=+9$ or -9,

$$\text{the } \textit{fourth} \text{ root of } 81 = \text{the square root of } +9, \text{ or of } -9$$
$$= \pm\sqrt{+9}, \text{ or } \pm\sqrt{-9}$$

Thus there are four values of $\sqrt[4]{81}$, two of which are real and two imaginary The *real positive* root is called the principal root In using fractional index notation we consider the principal root only

Thus $\qquad 9^{\frac{1}{2}}$ is the real positive value of $\sqrt{9}$, or 3,

$\qquad\qquad 81^{\frac{1}{4}}$ is the real positive value of $\sqrt[4]{81}$, or 3

In like manner (though the proof cannot be given here) every n^{th} root has n algebraic values when n is a positive integer In using $a^{\frac{1}{n}}$ as equivalent to $\sqrt[n]{a}$, we consider only the positive real root

Thus $\qquad 16^{\frac{1}{4}}=4,\quad 27^{\frac{1}{3}}=3,\quad 32^{\frac{1}{5}}=2,\quad 729^{\frac{1}{6}}=3$

341. *To find a meaning for* a^0

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by 0, we have

$$a^0 \times a^n = a^{0+n} = a^n;$$

hence, *if* a *is not zero,* $\qquad a^0 = \dfrac{a^n}{a^n} = 1.$

Thus a number or expression with zero index is equivalent to 1

EXAMPLE $\quad x^{b-c} \times x^{c-b} = x^{b-c+c-b} = x^0 = 1$

342 *To find a meaning for* a^{-n}

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by $-n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0$$

But $\qquad\qquad a^0 = 1,$

hence, *if* a *is not zero,* $a^{-n} = \dfrac{1}{a^n}$, and $a^n = \dfrac{1}{a^{-n}}$

From this it follows that any *factor* may be transferred from the numerator to the denominator of an expression, or vice versa, by merely changing the sign of the index

EXAMPLES. (i) $x^{-3} = \dfrac{1}{x^3}$, $\qquad$ (ii) $\dfrac{1}{y^{-\frac{1}{2}}} = y^{\frac{1}{2}} = \sqrt{y}$;

(iii) $27^{-\frac{2}{3}} = \dfrac{1}{27^{\frac{2}{3}}} = \dfrac{1}{\sqrt[3]{(27)^2}} = \dfrac{1}{\sqrt[3]{3^6}} = \dfrac{1}{3^2} = \dfrac{1}{9}.$

343 *To prove that* $a^m \div a^n = a^{m-n}$ *for all values of* m *and* n

$$a^m \div a^n = a^m \times \frac{1}{a^n}$$
$$= a^m \times a^{-n}$$
$$= a^{m-n}, \text{ by the fundamental law}$$

EXAMPLES (i) $a^3 \div a^5 = a^{3-5} = a^{-2} = \frac{1}{a^2}$,

(ii) $c \div c^{-\frac{8}{5}} = c^{1+\frac{8}{5}} = c^{\frac{13}{5}}$;

(iii) $x^{a-b} \div x^{a-c} = x^{a-b-(a-c)} = x^{c-b}$

344 We have not yet proved that $(a^m)^n = a^{mn}$ is true except when m and n are positive integers [Art 337]

The proof of this law for all values of m and n is not easy and will be postponed until the pupil has had some simple practice in the use of fractional indices Meanwhile the truth of the law may be assumed in simple cases

Thus $(a^3)^{-4} = a^{-12}$, $\left(a^{\frac{2}{3}}\right)^{\frac{3}{5}} = a^{\frac{2}{3}\times\frac{3}{5}} = a^{\frac{2}{5}}$, and so on

Again $$\sqrt{25^3} = 25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3 = 5^3 = 125,$$

and $$\sqrt[q]{x^p} = x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = \{\sqrt[q]{x}\}^p$$

345 In working examples the rules embodied in the following statements may now be used without placing any restrictions on the value of the indices

$$a^m \times a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad (a^m)^n = a^{mn} = a^{nm}$$

Also $$a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

The following examples will illustrate these principles In each case the result is expressed in a form free of negative indices and radical signs

EXAMPLES (i) $\frac{3a^{-2}}{5x^{-1}y} = \frac{3x}{5a^2y}$,

(ii) $\frac{9a^{\frac{4}{3}} \times a^{-\frac{1}{2}}}{2a^{\frac{3}{2}} \times 3a^{\frac{1}{3}}} = \frac{3}{2}a^{\frac{4}{3}-\frac{1}{2}-\frac{3}{2}-\frac{1}{3}} = \frac{3}{2}a^{-1} = \frac{3}{2a}$,

(iii) $\frac{\sqrt{x^3} \times \sqrt[3]{y^2}}{\sqrt[6]{y^{-2}} \times \sqrt[4]{x^6}} = \frac{x^{\frac{3}{2}} \times y^{\frac{2}{3}}}{y^{-\frac{1}{3}} \times x^{\frac{3}{2}}} = x^{\frac{3}{2}-\frac{3}{2}}y^{\frac{2}{3}+\frac{1}{3}} = x^0y = y$,

(iv) $2\sqrt{a} + \frac{3}{a^{-\frac{1}{2}}} + a^{\frac{5}{2}} = 2a^{\frac{1}{2}} + 3a^{\frac{1}{2}} + a^{\frac{5}{2}} = 5a^{\frac{1}{2}} + a^{\frac{5}{2}} = a^{\frac{1}{2}}(5+a^2)$

EXAMPLES XXX. a.

(*Examples 1–8 should be taken orally*)

1. Read off the values of

$$a^{\frac{1}{2}}\times a^{\frac{1}{3}},\quad a^{\frac{2}{3}}\times a^{\frac{1}{3}},\quad b^{\frac{2}{3}}\times b,\quad x^{-\frac{3}{4}}\times x,\quad y^{-\frac{2}{3}}\times y^{\frac{2}{3}},$$
$$m^{\frac{3}{4}}-m^{-\frac{1}{4}},\quad p-p^{-3},\quad k^{-3}-k^{-5},\quad c^{a}-c^{b},\quad c^{a}-c^{2a}$$

2. Express in words

$$\sqrt[m]{a^n},\quad \sqrt{a^p},\quad \sqrt[3]{b^2},\quad \sqrt[4]{x^3},\quad \sqrt[5]{y^{10}},\quad \sqrt[2]{p^2}.$$

3. Read off in integral form the values of

$$36^{\frac{1}{2}},\quad 100^{\frac{1}{2}},\quad 8^{\frac{1}{3}},\quad 64^{\frac{1}{2}},\quad 64^{\frac{1}{3}},\quad 64^{\frac{1}{6}},$$
$$32^{\frac{1}{5}},\quad 81^{\frac{1}{2}},\quad 81^{\frac{1}{4}},\quad (-8)^{\frac{1}{3}},\quad (-64)^{\frac{1}{3}},\quad 128^{\frac{1}{7}}.$$

4. Express with a single index

$$(a^2)^3,\quad (a^3)^2,\quad \left(b^{\frac{1}{2}}\right)^2,\quad (c^3)^{\frac{1}{3}}\quad \left(x^{\frac{1}{5}}\right)^{\frac{5}{6}},$$
$$\left(y^{-\frac{1}{2}}\right)^3,\quad \left(z^{\frac{2}{3}}\right)^{-3},\quad \left(p^{\frac{3}{4}}\right)^{-\frac{4}{3}},\quad \left(k^{\frac{3}{5}}\right)^{-10},\quad (m^{-n})^{\frac{2}{n}}.$$

5. Read off with indices

$$\sqrt[3]{a};\quad \sqrt[4]{a^3},\quad \sqrt[5]{b^2},\quad \sqrt[4]{c},\quad \sqrt[3]{x^{-2}},\quad \sqrt[4]{x^{-5}},\quad \sqrt[6]{p^7},\quad \sqrt[7]{m^{-5}}$$

6. Express with radical signs

$$a^{\frac{2}{3}},\quad b^{\frac{3}{4}},\quad c^{-\frac{3}{4}},\quad x^{\frac{2}{5}},\quad y^{-\frac{1}{2}},\quad p^{-\frac{3}{5}},\quad m^{\frac{4}{5}},\quad p^{-\frac{7}{5}}$$

7. Find the numerical value of

$$9^{\frac{3}{2}},\quad 4^{\frac{5}{2}},\quad 8^{\frac{2}{3}},\quad 27^{\frac{2}{3}},\quad 16^{\frac{3}{4}},\quad 32^{\frac{2}{5}}$$

8. Express with positive indices

$$a^{-2},\quad x^{-3},\quad \frac{1}{x^{-2}},\quad \frac{1}{x^{-\frac{1}{2}}},\quad \frac{a^{\frac{1}{2}}}{b^{-\frac{1}{2}}},\quad \frac{a^{-\frac{1}{3}}}{b^{\frac{1}{2}}};\quad \frac{a^{-3}}{a^2}.$$

9. Justify each step in the following example

To prove that $\sqrt[3]{8^2}\times\sqrt[4]{16^3}=32$

$$\sqrt[3]{8^2}\times\sqrt[4]{16^3}=8^{\frac{2}{3}}\times 16^{\frac{3}{4}}=(2^3)^{\frac{2}{3}}\times(2^4)^{\frac{3}{4}}=2^2\times 2^3=2^5=32.$$

10. Shew that

(i) $27^{\frac{2}{3}}=243^{\frac{2}{5}}$, (ii) $8\times 81^{\frac{3}{4}}=27\times 16^{\frac{3}{4}}$, (iii) $\sqrt[6]{27^4}\times\sqrt[4]{81^3}=243$

Express with positive indices:

11. $2x^{-\frac{1}{4}}$	12. $3a^{-\frac{2}{3}}$	13. $4x^{-2}a^3$	14. $3-a^{-2}$
15. $\frac{1}{4a^{-2}}$	16. $\frac{1}{5x^{-\frac{1}{2}}}$	17. $\frac{3a^{-3}x^2}{5y^2c^{-4}}$	18. $\frac{x^a y^{-b}}{b^{-a}}$

Express with positive indices

19. $2x^{\frac{1}{2}}/3x^{-1}$ 20. $1-2a^{-\frac{1}{2}}$ 21. $xy^{2}\times x^{-1}$ 22. $a^{-2}x^{-1}-3x$.

23. $\dfrac{1}{\sqrt{x^{3}}}$ 24. $\dfrac{1}{4\sqrt[5]{x^{-3}}}$ 25. $\dfrac{2}{\sqrt{y^{-3}}}$ 26. $\dfrac{\sqrt[4]{x^{3}}}{\sqrt{x^{-1}}}$

27. $a^{-2}x^{-\frac{1}{2}}-a^{-3}$ 28. $\sqrt[3]{a^{-1}}-\sqrt[3]{a}$ 29. $\sqrt[5]{a^{-3}}\div\sqrt[5]{a^{7}}$

Express with radical signs and positive indices under them

30 $x^{\frac{2}{5}}$ 31. $a^{-\frac{1}{2}}$ 32. $5x^{-\frac{1}{2}}$ 33 $2a^{-\frac{1}{x}}$

34 $\dfrac{1}{2a^{\frac{1}{3}}}$ 35 $\dfrac{2}{b^{-\frac{1}{4}}}$ 36. $\dfrac{c^{-\frac{1}{3}}}{2}$ 37. $\dfrac{1}{x^{-\frac{1}{x}}}$

38. $a^{-\frac{1}{3}}\times 2a^{-\frac{1}{2}}$ 39. $x^{-\frac{2}{3}}-2a^{-\frac{1}{2}}$ 40. $7a^{-\frac{1}{2}}\times 3a^{-1}$

41. $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}}$ 42. $\dfrac{a^{-\frac{1}{2}}}{3a}$ 43. $\dfrac{4x^{-1}}{x^{-\frac{1}{3}}}$ 44. $\sqrt[3]{a^{2}}\times\sqrt[2]{a^{4}}$

346 Since the index-laws are universally true, all the operations of multiplication, division, involution and evolution are applicable to expressions which contain fractional and negative indices

347 In Art 190, we pointed out that the descending powers of x are

$$\ldots\ x^{3},\ x^{2},\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^{2}},\ \frac{1}{x^{3}},$$

A reason for this may be seen if we write these terms in the form

$$\ldots\ x^{3},\ x^{2},\ x^{1},\ x^{0},\ x^{-1},\ x^{-2},\ x^{-3},\ \ldots$$

When expressions involve radical signs as well as indices, the former should be replaced by fractional indices

Thus $\qquad 4x^{2}+\sqrt{x^{3}}-2x+\frac{1}{4}+x^{3}-4\sqrt{x^{5}}$

may be written $\qquad 4x^{2}+x^{\frac{3}{2}}-2x+\frac{1}{4}+x^{3}-4x^{\frac{5}{2}}$,

or $\qquad x^{3}-4x^{\frac{5}{2}}+4x^{2}+x^{\frac{3}{2}}-2x+\frac{1}{4}$, in descending powers.

EXAMPLE 1. *Multiply* $3x^{-\frac{1}{3}}+x+2x^{\frac{2}{3}}$ *by* $x^{\frac{1}{3}}-2$

Arrange in descending powers of x

$$\begin{array}{l}
x+2x^{\frac{2}{3}}+3x^{-\frac{1}{3}} \\
\underline{x^{\frac{1}{3}}-2} \\
x^{\frac{4}{3}}+2x+3 \\
\underline{\quad -2x-4x^{\frac{2}{3}}-6x^{-\frac{1}{3}}} \\
x^{\frac{4}{3}}-4x^{\frac{2}{3}}+3-6x^{-\frac{1}{3}}.
\end{array}$$

EXAMPLE 2 *Divide* $16a^{-3}-6a^{-2}+5a^{-1}+6$ *by* $1+2a^{-1}$

$$
\begin{array}{r|l}
2a^{-1}+1 & 16a^{-3}-6a^{-2}+5a^{-1}+6\;(8a^{-2}-7a^{-1}+6 \\
 & \underline{16a^{-3}+8a^{-2}} \\
 & \quad -14a^{-2}+5a^{-1} \\
 & \quad \underline{-14a^{-2}-7a^{-1}} \\
 & \qquad\qquad 12a^{-1}+6 \\
 & \qquad\qquad \underline{12a^{-1}+6}
\end{array}
$$

EXAMPLE 3 *Write down* (i) *the product of* $x^{\frac{1}{3}}+2$ *and* $x^{\frac{1}{3}}-5$;
(ii) *the square of* $2\sqrt[4]{x^3}-3$

(i) The product $=\left(x^{\frac{1}{3}}\right)^2-3x^{\frac{1}{3}}-10=x^{\frac{2}{3}}-3x^{\frac{1}{3}}-10$;

(ii) Using a fractional index in place of the radical sign,

$$\text{the square}=\left(2x^{\frac{3}{4}}-3\right)^2=\left(2x^{\frac{3}{4}}\right)^2-12x^{\frac{3}{4}}+9=4x^{\frac{3}{2}}-12x^{\frac{3}{4}}+9$$

EXAMPLES XXX b.

(*In these examples the results are to be given free from radical signs*)

Find the value of

1. $\left(a^{\frac{2}{3}}+1\right)\left(a^{\frac{2}{3}}-3\right)$ 2. $(x^{-2}+4)(x^{-2}-4)$ 3. $\left(3c^{\frac{1}{2}}-1\right)^2$

4. $\left(x^{\frac{1}{2}}+7\right)\left(x^{\frac{1}{2}}-2\right)$ 5. $(\sqrt[3]{x^2}-2)^2$ 6. $(\sqrt[3]{a}-5)(\sqrt[3]{a}+2)$

7. $\left(a^{\frac{4}{3}}-9\right)\div\left(a^{\frac{2}{3}}+3\right)$ 8. $\left(6a^{\frac{2}{5}}-5a^{\frac{1}{5}}-6\right)\div\left(3a^{\frac{1}{5}}+2\right)$

9. Multiply $3x^{\frac{1}{3}}-5+8x^{-\frac{1}{3}}$ by $4x^{\frac{1}{3}}+3x^{-\frac{1}{3}}$

10. Divide $21a+a^{\frac{2}{3}}+a^{\frac{1}{3}}+1$ by $3a^{\frac{1}{3}}+1$

11. Multiply $3m^{\frac{1}{3}}-3m^{-\frac{1}{3}}+2m^{-1}$ by $5m^{\frac{2}{3}}+4$

12. Find the product of $c^x+2c^{-x}-7$ and $5-3c^{-x}+2c^x$

13. Find the value of $(p^{2n}-1+p^{-2n})(5p^n-3p^{-n})$

14. Divide $x^{\frac{3}{2}}+x^{\frac{1}{2}}y^{\frac{2}{3}}-2y$ by $x^{\frac{1}{2}}-y^{\frac{1}{3}}$

15. Find the square root of $4x^{\frac{1}{2}}-4x^{\frac{1}{4}}+5-2x^{-\frac{1}{4}}+x^{-\frac{1}{2}}$.

16. Divide $16a^{-3}+\frac{6}{a^2}+\frac{5}{a}-6$ by $2a^{-1}-1$

17. Multiply $1-2\sqrt[3]{x}-2x^{\frac{1}{2}}$ by $1-\sqrt[6]{x}$

Find the square root of

18. $9x-12x^{\frac{1}{2}}+10-\frac{4}{\sqrt{x}}+\frac{1}{x}$ 19. $12a^x+4-6a^{3x}+a^{4x}+5a^{2x}$

EXAMPLES XXX. c.

1. Find the values of $a^{\frac12}+b^{\frac12}$, $(a+b)^{\frac12}$, $\left(a^{\frac12}+b^{\frac12}\right)^2$,

(i) when $a=16$, $b=9$, (ii) when $a=225$, $b=64$.

In each case shew that $\sqrt{a^2+b^2}$ is not equal to $a+b$

Simplify and express with positive indices

2. $\left[(a^{-3})^{\frac23}\right]^{\frac12}$ 3. $\left[(x^{-\frac14})^6\right]^{\frac23}$ 4. $\left[(\sqrt{a^3})^{-2}\right]^{-\frac13}$.

5. $\left(x^{\frac12}\right)^3\times\left(x^{-\frac13}\right)^2$ 6. $(\sqrt[4]{x^3})^{\frac23}\times(\sqrt[5]{x^6})^{\frac{5}{12}}$. 7. $(\sqrt[3]{a^5})^{\frac12}\times\sqrt[6]{a^{-5}}$.

8. $\left(\frac{a^2}{b^3}\right)^{\frac13}\times\left(\frac{b^2}{a^3}\right)^{\frac12}$ 9. $\left(\frac{a^3}{b^2}\right)^{\frac12}\div\left(\frac{b^3}{a^2}\right)^{-\frac12}$ 10. $\left(\frac{x^3}{y^4}\right)^3\div\left(\frac{y^{-3}}{x^2}\right)^{-4}$.

11. $(c^3)^{\frac23}\times(c^{-\frac12})^{\frac45}\times\sqrt{c^{\frac45}}$ 12. $(\sqrt[4]{b^3})^{\frac16}\times\sqrt[9]{b^{-6}}\div(\sqrt{b^{-7}})^{\frac13}$

13. $(\sqrt{a^2b^3})^6$ 14. $(\sqrt[9]{x^{-4}y^3})^{-3}$ 15. $(x^ay^{-b})^3\times(x^3y^2)^{-a}$.

16. $\left(\frac{16x^2}{y^{-2}}\right)^{-\frac14}$ 17. $\left(\frac{27x^3}{8a^{-3}}\right)^{-\frac23}$. 18. $\left(\frac{a^{-\frac12}}{4c^2}\right)^{-2}$

19. $\left\{\sqrt[4]{\left(x^{-\frac23}y^{\frac12}\right)^3}\right\}^{-\frac23}$ 20. $\sqrt[4]{x\sqrt[3]{x^{-1}}}$ 21. $(4a^{-2}\div 9x^2)^{-\frac12}$

22. $(x\div\sqrt[n]{x})^n$ 23. $(\sqrt[b]{x^b}\div\sqrt[a]{x})^{\frac{1}{1-a}}$ 24. $\sqrt{a^{-2}b}\times\sqrt[3]{ab^{-3}}$.

25. $\sqrt[9]{a^{4b}x^6}\times\left(a^{\frac23}x^{-1}\right)^{-6}$. 26. $\sqrt[3]{x^{-1}\sqrt{y^3}}\div\sqrt{y\sqrt[3]{x}}$

27. $\left(\frac{a^3}{b^2}\right)^{\frac14}\times\left(\frac{c^2}{b^{-1}}\right)^{\frac13}\div\left(\frac{a^{-\frac12}}{bc^{-6}}\right)^{\frac14}$ 28. $\sqrt[3]{x^6y^{\frac23}z^{\frac12}}\times y^{\frac79}\times\left(2^{-6}x^4y^2z^{\frac13}\right)^{-\frac12}$.

29. $\sqrt[3]{(a+b)^5}\times(a+b)^{-\frac23}$ 30. $\{(x-y)^{-3}\}^n\div\{(x+y)^n\}^3$

31. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3}\div\left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$ 32. $\left\{\frac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}}\left(\frac{b^{\frac14}}{a^{\frac13}}\right)^2\div\frac{a^{-\frac13}}{b^{-\frac12}}\right\}^6$

33. $\left(a^{-\frac12}x^{\frac13}\sqrt{ax^{-\frac13}\sqrt[4]{x^{\frac43}}}\right)^{\frac13}$ 34. $\sqrt[4]{(a+b)^6}\times(a^2-b^2)^{-\frac12}$

35. $\left(\frac{a^{-3}}{b^{-\frac23}c}\right)^{-\frac12}\div\left(\frac{\sqrt{a^{-\frac12}}\ \sqrt[6]{b^3}}{a^2c^{-1}}\right)^{-2}$ 36. $\left(\frac{a^{-\frac23}x^{\frac12}}{x^{-1}a}\right)^2\div\sqrt[3]{\frac{a^{-1}}{x^{-3}}}$

37. $(a^2-b^2)^{\frac12}\times(a+b)^{-\frac12}\times(a-b)^{\frac32}$ 38. $\frac{\sqrt[3]{(a^3b^3+a^6)}}{\sqrt[3]{(b^6-a^3b^3)^{-1}}}$

39. $\frac{2^n\times(2^{n-1})^n}{2^{n+1}\times2^{n-1}}\times\frac{1}{4^{-n}}$. 40. $\frac{2^{n+1}}{(2^n)^{n-1}}\div\frac{4^{n+1}}{(2^{n-1})^{n+1}}$.

41. $\frac{3\cdot2^n-4\cdot2^{n-2}}{2^n-2^{n-1}}$. 42. $\frac{2^{n+3}}{15^{-n-1}}\times\frac{6^{-n+2}}{5^{n+1}}$. 43. $\frac{3^{n+4}-6\cdot3^{n+1}}{3^{n+2}\times7}$.

351. Miscellaneous Examples

EXAMPLE 1 $(a^{\frac{h}{k}} - b^{\frac{p}{q}})(a^{-\frac{h}{k}} + b^{-\frac{p}{q}}) = a^{\frac{h}{k}-\frac{h}{k}} - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - b^{\frac{p}{q}-\frac{p}{q}}$

$$= 1 - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - 1$$

$$= a^{\frac{h}{k}}b^{-\frac{p}{q}} - a^{-\frac{h}{k}}b^{\frac{p}{q}}.$$

EXAMPLE 2 *Express* (i) $4x^m - 9a^n$, (ii) $x - 2\sqrt{x} - 15$ *in binomial factors*

(i) $4x^m - 9a^n$ is of the form $A^2 - B^2$ where $A = 2x^{\frac{m}{2}}$, $B = 3a^{\frac{n}{2}}$

$$4x^m - 9a^n = (2x^{\frac{m}{2}} + 3a^{\frac{n}{2}})(2x^{\frac{m}{2}} - 3a^{\frac{n}{2}})$$

(ii) The expression $= \{(\sqrt{x})^2 - 2\sqrt{x} - 15\} = (\sqrt{x}+3)(\sqrt{x}-5)$

EXAMPLE 3 *Divide* $a^{\frac{3}{2}} - 8$ *by* $a^{\frac{1}{2}} - 2$

$a^{\frac{3}{2}} - 8$ is of the form $A^3 - B^3$ where $A = a^{\frac{1}{2}}$, $B = 2$

$$a^{\frac{3}{2}} - 8 = (a^{\frac{1}{2}} - 2)\{(a^{\frac{1}{2}})^2 + 2a^{\frac{1}{2}} + 4\},$$

and the required quotient $= a + 2a^{\frac{1}{2}} + 4$

EXAMPLES XXX. d.

Express in factors

1 $a + 7\sqrt{a} + 12$ 2 $x - 4x^{\frac{1}{2}} - 12$ 3 $x^n - 49$

4 $3a^{\frac{2}{3}} + 5a^{\frac{1}{3}} + 2$ 5 $x^{-2c} + x^{-c} - 20$ 6 $x^{3m} + 27$

Write down the value of

7 $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 6)$ 8 $(2x^{\frac{n}{2}} - 5)(2x^{\frac{n}{2}} + 5)$ 9 $(2\sqrt{a} - 3)(\sqrt{a} + 2)$

10 $(a^x - 2a^{-x})^2$ 11 $(a^x + a^{\frac{1}{x}})^2$ 12 $(x^{\frac{3}{2}} - 27) \div (x^{\frac{1}{2}} - 3)$

13 $\{(a+b)^{\frac{1}{2}} + (a-b)^{\frac{1}{2}}\}^2$ 14. $(x + x^{\frac{1}{2}} - 4)(x - x^{\frac{1}{2}} + 4)$

15 $(a^x - b^y)(a^{-x} + b^{-y})$ 16 $(5x^ay^b - 3x^{-a}y^{-b})(4x^ay^b + 5x^{-a}y^{-b})$

17. $(1 - 8a^{-3}) \div (1 - 2a^{-1})$ 18. $(27m^n + 1) \div (3\sqrt[3]{m^n} + 1)$

Express in simplest form, free from radical signs

19. $\dfrac{a^{\frac{4}{3}} - 8a^{\frac{1}{3}}b}{a^{\frac{2}{3}} + 2\sqrt[3]{ab} + 4b^{\frac{2}{3}}}$ 20. $\dfrac{x - 7x^{\frac{1}{2}}}{x - 5\sqrt{x} - 14} \div \left(1 + \dfrac{2}{\sqrt{x}}\right)^{-1}$.

21. $\dfrac{x^{\frac{4}{3}} - 4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2} + 4 + 4x^{-\frac{2}{3}}}$ 22. $\dfrac{a^{\frac{3}{2}} + ab}{ab - b^3} - \dfrac{\sqrt{a}}{\sqrt{a} - b}$

CHAPTER XXXI

SURDS AND IRRATIONAL QUANTITIES

352 IT is not always possible to express two quantities of the same kind in terms of a common unit For instance, if the side of a square is 1 inch, the diagonal is $\sqrt{2}$ inches Now the numerical value of $\sqrt{2}$ can be found to any required number of decimal places, but it can never be *exactly* expressed as a multiple or fraction of unity Such a quantity is said to be **incommensurable.**

353 DEFINITION When the root of a number or expression cannot be exactly determined, the root is called a **surd**

Thus $\sqrt{2}$, $\sqrt[3]{6}$, and $\sqrt{a^2+x^2}$ are surds But $\sqrt{9}$, $\sqrt[3]{8}$, and $\sqrt{a^2+2ax+x^2}$, though surd in form, are not really surds For they are capable of being expressed without root signs in the equivalent forms 3, 2, and $a+x$

354 An expression which necessarily contains one or more root signs is called **irrational** In a **rational** number or expression no root sign is necessarily involved

355 The **order** of a surd is indicated by the root symbol

Thus $\sqrt[3]{5}$, $\sqrt[n]{x}$ are respectively surds of the third and n^{th} orders

The surds which occur most frequently are those of the second order, usually called **quadratic surds**

Thus $\sqrt{6}$, $\sqrt{x}$, $\sqrt{a+b}$ are quadratic surds

NOTE Since every square root has the double sign every quadratic surd has two values, one positive and one negative Unless anything to the contrary is expressly stated, we shall only consider the positive root. Thus $\sqrt{5} = +2\,236$

356 A rational quantity may be expressed in the form of a surd of *any required order* by raising it to the power whose root the surd expresses, and prefixing the radical sign

Thus
$$3=\sqrt{9}=\sqrt[3]{27}=\sqrt[4]{81}=\sqrt[n]{3^n},$$
$$a+b=\sqrt{(a+b)^2}=\sqrt[5]{(a+b)^5}=\sqrt[m]{(a+b)^m}$$

357 A surd of any order may be transformed into a surd of a different order

EXAMPLES (i) $\sqrt[5]{3}=3^{\frac{1}{5}}=3^{\frac{2}{10}}=\sqrt[10]{3^2}$, (ii) $\sqrt[n]{a}=a^{\frac{1}{n}}=a^{\frac{m}{mn}}=\sqrt[mn]{a^m}$.

358 To compare surds of different orders they must first be transformed to surds of the same order The most convenient order is the L C M of the given orders

EXAMPLE 1 *Express $\sqrt[9]{a^4}$ and $\sqrt[6]{x}$ as surds of the same lowest order*

The L C M of 9 and 6 is 18, and expressing each surd as one of the 18^{th} order, we have $\sqrt[18]{a^8}$ and $\sqrt[18]{x^3}$

EXAMPLE 2 *Compare the surds $\sqrt[4]{9}$, $\sqrt[6]{26}$, $\sqrt[3]{5}$*

The L C M of 4, 6, and 3 is 12 Hence expressing the given surds as surds of the 12^{th} order we have

$$\sqrt[4]{9}=\sqrt[12]{9^3}=\sqrt[12]{729}, \quad \sqrt[6]{26}=\sqrt[12]{26^2}=\sqrt[12]{676}, \quad \sqrt[3]{5}=\sqrt[12]{5^4}=\sqrt[12]{625}$$

Thus $\sqrt[4]{9}$, $\sqrt[6]{26}$, $\sqrt[3]{5}$ are in descending order of magnitude

EXAMPLES XXXI. a.

1. Which of the following quantities or expressions are essentially irrational?

   (i) $\sqrt{16}$, (ii) $\sqrt{32}$, (iii) $\sqrt{1000}$, (iv) $\sqrt[3]{1000}$,

   (v) $\sqrt[3]{a^6x^3}$, (vi) $\sqrt[3]{(a+b)^6}$, (vii) $\sqrt[6]{64}$, (viii) $\sqrt[4]{729}$

2. As in Art 358, prove that

   (i) $\sqrt{5}=\sqrt[6]{125}$, (ii) $\sqrt[3]{3}=\sqrt[15]{243}$, (iii) $\sqrt[p]{x^{2p}}=\sqrt[q]{x^{2q}}$

3. Express the rational quantities 4, $3x^2$, $a-2x$ in the form of quadratic surds

4. Express as surds of the 12^{th} order:

   (i) $\sqrt{2}$, (ii) $\sqrt[3]{3}$, (iii) $\sqrt[6]{6}$, (iv) $\sqrt[4]{a^3}$, (v) $\sqrt[3]{3c^2d}$

5. Arrange $\sqrt[3]{6}$, $\sqrt[4]{10}$, $\sqrt{3}$ in ascending order

Express as surds of the same lowest order

6. $\sqrt{a}$, $\sqrt[9]{a^5}$ 7. $\sqrt[4]{a^3}$, $\sqrt{a}$ 8. $\sqrt[15]{x^4}$, $\sqrt[12]{x^{10}}$

9. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$ 10. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[6]{6}$ 11. $\sqrt[3]{2}$, $\sqrt[9]{8}$, $\sqrt[6]{4}$

359 Though the approximate values of surds which occur in arithmetical calculations can usually be determined to any required degree of accuracy, the work is often simplified and shortened by keeping quantities in surd form as far as possible, and only substituting numerical values as a last step Hence it is necessary to discuss the properties of surd quantities and their laws of combination These follow from the laws of indices since a surd can always be expressed as a quantity with a fractional index

Thus $\sqrt{2}=2^{\frac{1}{2}}$, $\sqrt[3]{6}=6^{\frac{1}{3}}$, $\sqrt[4]{a^2+x^2}=(a^2+x^2)^{\frac{1}{4}}$

360 *The n^{th} root of any expression is equal to the product of the n^{th} roots of the factors of the expression.*

For $$\sqrt[n]{ab}=(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}, \qquad \text{[Art 349]}$$
$$=\sqrt[n]{a}\cdot\sqrt[n]{b}$$

Similarly, $$\sqrt[n]{abc}=\sqrt[n]{a}\cdot\sqrt[n]{b}\cdot\sqrt[n]{c},$$

and so for any number of factors

EXAMPLES (i) $\sqrt[3]{12}=\sqrt[3]{4}\cdot\sqrt[3]{3}$, (ii) $\sqrt[5]{ab^5}=\sqrt[5]{a}\cdot\sqrt[5]{b^5}=b\sqrt[5]{a}$;
(iii) $\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$, (iv) $\sqrt[3]{16}=\sqrt[3]{8}\cdot\sqrt[3]{2}=2\sqrt[3]{2}$

Examples (iii) and (iv) shew that a surd may sometimes be expressed as the product of a rational quantity and a surd. A surd so reduced is said to be in its *simplest form*.

Conversely, a rational coefficient of a surd may be brought under the radical sign.

EXAMPLES (i) $3\sqrt{6}=\sqrt{9}\cdot\sqrt{6}=\sqrt{54}$, (ii) $a\sqrt[4]{x}=\sqrt[4]{a^4}\cdot\sqrt[4]{x}=\sqrt[4]{a^4x}$

A surd so reduced is called an **entire surd.**

361 When surds can be expressed with the same irrational factor they are said to be **like**; otherwise, they are said to be **unlike**.

Thus $3\sqrt{5}$, $2\sqrt{5}$ are like, and $\sqrt{3}$, $2\sqrt{5}$ are unlike surds.

Again, $\sqrt{12}$, $5\sqrt{3}$, and $\sqrt{\frac{1}{3}}$ are like surds;

for $\sqrt{12}=\sqrt{4}\cdot\sqrt{3}=2\sqrt{3}$, and $\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{9}}=\frac{1}{3}\sqrt{3}$.

362 The sum of a number of like surds can be found when they have been expressed in their simplest form.

EXAMPLE 1 *Find the sum of* $3\sqrt{12}$, $10\sqrt{3}$, *and* $\sqrt{\frac{1}{3}}$

The required sum $=3\cdot2\sqrt{3}+10\sqrt{3}+\frac{1}{3}\sqrt{3}=\frac{49}{3}\sqrt{3}$,
by collecting the coefficients of $\sqrt{3}$.

EXAMPLE 2 *Express* $a\sqrt[3]{8a^3b}+c\sqrt[3]{-c^3b}-\sqrt[3]{a^6b}$ *in simplest form.*

The expression $=a\cdot2a\sqrt[3]{b}+c(-c)\sqrt[3]{b}-a^2\sqrt[3]{b}$
$$=2a^2\sqrt[3]{b}-c^2\sqrt[3]{b}-a^2\sqrt[3]{b}=(a^2-c^2)\sqrt[3]{b}$$

363 Unlike surds cannot be collected.

Thus the sum of $2\sqrt{3}$ and $5\sqrt{2}$ is $2\sqrt{3}+5\sqrt{2}$, and can only be further simplified by substituting the numerical values of $\sqrt{3}$ and $\sqrt{2}$.

EXAMPLES XXXI b

1 Read off the following surds in their simplest form

$\sqrt{8}$, $\sqrt{18}$, $\sqrt{50}$, $\sqrt{27}$, $\sqrt{ab^2}$, $\sqrt{16a}$, $\sqrt{8x^4y}$,
$\sqrt[3]{16}$, $\sqrt[3]{27x}$, $\sqrt[3]{54a^3}$, $\sqrt[3]{27a^4}$, $\sqrt[4]{32}$, $\sqrt[4]{48}$, $\sqrt[5]{64a^6}$

2 Read off as entire surds

$2\sqrt{2}$, $3\sqrt{3}$, $10\sqrt{5}$, $2\sqrt[3]{2}$, $5\sqrt[3]{5}$, $a\sqrt{b}$, $2a\sqrt{3b}$

Express in the simplest form

3 $\sqrt{98}$ 4 $\sqrt{125}$ 5 $\sqrt{384}$ 6 $\sqrt{720}$

7 $\sqrt[3]{432}$ 8 $\sqrt[3]{375}$ 9 $\sqrt[4]{567}$ 10 $\sqrt[5]{160}$

11 $2\sqrt{175}$. 12 $5\sqrt{726}$ 13 $\sqrt{72a^2}$ 14 $3\sqrt{8a^3}$

15 $\sqrt[3]{54a^4b}$ 16 $2x\sqrt[3]{-16y^6}$ 17 $\sqrt{27m^2n^3}$ 18 $2n\sqrt{50n^5p^2}$

19 $\sqrt{2x^2-4xy+2y^2}$ 20 $\sqrt{(a^2-b^2)(a+b)}$ 21 $\sqrt{3ax^2+18ax+27a}$

Express as entire surds

22 $11\sqrt{5}$ 23 $4\sqrt[3]{4}$ 24 $-6\sqrt[3]{2}$ 25 $3a^2x\sqrt{2x^3}$

26. $7\sqrt{\frac{3}{7}}$. 27. $\frac{2}{3}\sqrt{27}$ 28 $\frac{3}{5}\sqrt{75}$ 29. $\frac{11}{18}\sqrt{\frac{54}{55}}$.

30 $\dfrac{m}{n^2}\sqrt{\dfrac{3n^3}{m}}$ 31 $\dfrac{4a}{5x^3}\sqrt{3ax^5}$ 32 $\dfrac{a}{3x}\sqrt[3]{\dfrac{9x^4}{a^2}}$ 33. $\dfrac{2a}{b}\sqrt[4]{\dfrac{b^4}{8a^3}}$

Find the value of

34 $\sqrt{50}+\sqrt{32}-\sqrt{18}$ 35 $\sqrt{108}-\sqrt{48}+\sqrt{75}$

36 $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$ 37 $\sqrt[3]{54}+\sqrt[3]{128}-\sqrt[3]{432}$

38 $\sqrt[3]{81}+\sqrt[3]{-375}-\sqrt[3]{-192}$ 39 $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}$

40 $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$ 41 $3\sqrt{147}-7\sqrt{\frac{1}{27}}-\frac{11}{3}\sqrt{\frac{1}{3}}$

42 $2a\sqrt{a^2x}+3a^2\sqrt{4x}-a\sqrt{9a^2x}$ 43 $\sqrt{18p^3q^3}-p\sqrt{8pq^3}-q\sqrt{50p^3q}$

Simplify the following surds, and find their numerical values, correct to two places of decimals, given $\sqrt{2}=1·4142$, $\sqrt{3}=1·7321$, $\sqrt{5}=2·2361$, $\sqrt{7}=2·6458$

44 $\sqrt{98}$ 45. $\sqrt{125}$ 46 $\sqrt{147}$ 47 $\sqrt{512}$ 48 $\sqrt{175}$

49 $\sqrt{128}-\sqrt{50}$ 50 $\sqrt{252}+\sqrt{63}$ 51 $3\sqrt{500}-2\sqrt{243}$

52 $5\sqrt{243}-2\sqrt{363}+\sqrt{192}$ 53 $4\sqrt{128}-6\sqrt{108}+5\sqrt{75}$

Multiplication and Division of Simple Surds.

364 *To find the product of two surds of the same order: multiply separately the rational factors and the irrational factors*

For $$a\sqrt[n]{x}\times b\sqrt[n]{y}=ax^{\frac{1}{n}}\times by^{\frac{1}{n}}=abx^{\frac{1}{n}}y^{\frac{1}{n}}$$
$$=abx^{\frac{1}{n}}y^{\frac{1}{n}}=ab\sqrt[n]{xy}$$

NOTE The product of two *like quadratic* surds is rational.

EXAMPLES (i) $3\sqrt{2}\times5\sqrt{5}=15\sqrt{10}$; (ii) $5\sqrt{a}\times6\sqrt{a}=30a$;

(iii) $\sqrt[3]{x-y}\times\sqrt[3]{x-y}=\sqrt[3]{(x+y)(x-y)}=\sqrt[3]{x^2-y^2}$;

(iv) $2\sqrt{8}\times\sqrt{125}\times\sqrt{40}=2\cdot2\sqrt{2}\times5\sqrt{5}\times2\sqrt{10}$
$=40\sqrt{2}\cdot\sqrt{5}\cdot\sqrt{10}$
$=40\times10=400$

In Example (iv) the surds are first reduced to simplest form

365 *To multiply surds not of the same order reduce them to surds of the same lowest order, and proceed as before*

EXAMPLE. *Find the product of $2\sqrt[3]{4}$ and $7\sqrt{3}$*

The product$=2\sqrt[6]{4^2}\times7\sqrt[6]{3^3}=2\times7\sqrt[6]{4^2\times3^3}=14\sqrt[6]{144}$.

366. **Rationalizing Factors.** A factor by which any irrational expression is multiplied so as to give a rational product is called a rationalizing factor

Thus $$\frac{3}{\sqrt{8}}=\frac{3}{2\sqrt{2}}=\frac{3\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{3}{4}\sqrt{2}.$$

Here the factor $\sqrt{2}$, introduced into numerator and denominator, gives a result with a *rational denominator*

EXAMPLE. *To find the numerical value of $\frac{\sqrt{5}}{\sqrt{7}}$*

The approximate value could be found by dividing 2·2361 by 2·6458, but some labour will be saved if we *rationalize the denominator.*

Thus $\frac{\sqrt{5}}{\sqrt{7}}=\frac{\sqrt{5}\times\sqrt{7}}{\sqrt{7}\times\sqrt{7}}=\frac{\sqrt{35}}{7}=\frac{5\cdot9161}{7}=0\cdot845$, to three decimal figures.

Not only is this latter method the simpler of the two, but it has the advantage of using an integral divisor instead of a non-terminating decimal. Thus the answer can be obtained correct to any required number of figures

A fraction with an irrational denominator should always be expressed as an equivalent fraction the denominator of which is rational.

Thus $$\frac{a\sqrt{b}}{\sqrt{c}}=\frac{a\sqrt{b}\times\sqrt{c}}{\sqrt{c}\times\sqrt{c}}=\frac{a\sqrt{bc}}{c}.$$

EXAMPLE *Find the value of* (i) $5\sqrt{27} \div 3\sqrt{24}$, (ii) $\sqrt[3]{a} \div \sqrt[3]{b^2}$.

(i) The quotient$=\dfrac{5\sqrt{27}}{3\sqrt{24}}=\dfrac{5\times3\sqrt{3}}{3\times2\sqrt{2}\times\sqrt{3}}=\dfrac{5}{2\sqrt{2}}=\dfrac{5\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\dfrac{5\sqrt{2}}{4}$.

(ii) The quotient$=\dfrac{\sqrt[3]{a}}{\sqrt[3]{b^2}}=\dfrac{\sqrt[3]{a}\times\sqrt[3]{b}}{\sqrt[3]{b^2}\times\sqrt[3]{b}}=\dfrac{\sqrt[3]{ab}}{\sqrt[3]{b^3}}=\dfrac{\sqrt[3]{ab}}{b}$

EXAMPLES XXXI c

Find the value of

1. $2\sqrt{10}\times\sqrt{15}$
2. $\sqrt{12}\times\sqrt{75}$
3. $3\sqrt{14}\times\sqrt{35}$
4. $\sqrt{96}\times2\sqrt{12}$
5. $3\sqrt{33}\times2\sqrt{77}$
6. $x^2\sqrt{y}\times y\sqrt{x^3}$.
7. $3\sqrt{8}\div4\sqrt{12}$
8. $2\sqrt{75}\div5\sqrt{56}$
9. $2\sqrt{63}\div3\sqrt{35}$
10. $\sqrt[3]{a+b}\times\sqrt[3]{a-b}$
11. $6\sqrt{28}\div5\sqrt{288}$
12. $21\sqrt{384}\div8\sqrt{98}$
13. $5\sqrt[3]{2}\times2\sqrt{5}$
14. $\sqrt[4]{48}\times\sqrt{27}$
15. $\dfrac{2\sqrt{15}}{3\sqrt{7}}\div\dfrac{4\sqrt{18}}{9\sqrt{35}}$
16. $\dfrac{3a}{5}\sqrt{\dfrac{10x^3}{21a^2}}\times\dfrac{1}{6}\sqrt{\dfrac{7a}{3x}}$
17. $\dfrac{1}{3cx}\sqrt{\dfrac{1}{4cx^3}}\times c^2x^2\sqrt{c^7x^5}\div\dfrac{x^2}{3c}\sqrt{\dfrac{x^3}{8c^3}}$

Given $\sqrt{2}=1.4142$, $\sqrt{3}=1.7321$, $\sqrt{5}=2.2361$, $\sqrt{6}=2.4495$, $\sqrt{7}=2.6458$; find correct to three places of decimals the value of the following surds:

18. $\dfrac{9}{\sqrt{3}}$
19. $\dfrac{15}{\sqrt{18}}$
20. $\dfrac{20}{3\sqrt{5}}$
21. $\dfrac{2\sqrt{3}}{3\sqrt{2}}$
22. $\dfrac{3\sqrt{3}}{\sqrt{21}}$.
23. $\dfrac{20\sqrt{24}}{\sqrt{75}}$
24. $\dfrac{5\sqrt{12}}{7\sqrt{3}}\times\dfrac{\sqrt{14}}{6\sqrt{2}}$
25. $\dfrac{3\sqrt{48}}{5\sqrt{112}}\div\dfrac{6\sqrt{84}}{\sqrt{392}}$.

367. **Compound Surds** An expression which involves two or more simple surds is called a **compound surd**

Thus $3\sqrt{5}-\sqrt{2}$, $2\sqrt{3}-\sqrt{7}+\sqrt{2}$, $\sqrt[3]{a}-\sqrt[3]{b}$ are compound surds

368. In finding the product of compound surds we follow the same arrangement as in dealing with rational expressions

EXAMPLE 1 *Find the product of* $2\sqrt{x}-7$ *and* $5\sqrt{x}$

The product$=5\sqrt{x}(2\sqrt{x}-7)=10x-35\sqrt{x}$

EXAMPLE 2 *Multiply* $2\sqrt{3}-6\sqrt{a}$ *by* $\sqrt{3}+\sqrt{a}$.

The product$=(2\sqrt{3}-6\sqrt{a})(\sqrt{3}+\sqrt{a})$

$$=2\sqrt{3}\cdot\sqrt{3}-6\sqrt{3}\cdot\sqrt{a}+2\sqrt{3}\cdot\sqrt{a}-6\sqrt{a}\cdot\sqrt{a}$$
$$=6-4\sqrt{3a}-6a$$

[*Examples* XXXI d 1-20, *page* 339, *may conveniently be taken here.*]

369 Conjugate surds Two *binomial quadratic* surds which differ only in the sign which connects their terms are said to be conjugate. *The product of two conjugate surds is always rational*

Thus $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})=(a\sqrt{x})^2-(b\sqrt{y})^2=a^2x-b^2y$

$$(3\sqrt{7}+2\sqrt{3})(3\sqrt{7}-2\sqrt{3})=(3\sqrt{7})^2-(2\sqrt{3})^2=63-12=51$$

370 The only important case in division of compound surds is that in which the divisor is a binomial quadratic surd. If the operation of division is expressed in fractional form, the denominator can always be rationalized by multiplying numerator and denominator by the surd which is conjugate to the divisor

EXAMPLE 1 *Divide* $7-2\sqrt{6}$ *by* $5-2\sqrt{6}$, *and find the value of the quotient to three decimal figures*

$$\text{The quotient}=\frac{7-2\sqrt{6}}{5-2\sqrt{6}}=\frac{7-2\sqrt{6}}{5-2\sqrt{6}}\times\frac{5+2\sqrt{6}}{5+2\sqrt{6}}$$
$$=\frac{35-10\sqrt{6}+14\sqrt{6}-24}{25-24}=11+4\sqrt{6}$$
$$=11+4\times2\ 4495=20\ 798$$

EXAMPLE 2 *Express* $\dfrac{b^2}{\sqrt{a^2+b^2}+a}$ *with rational denominator*

$$\text{The expression}=\frac{b^2}{\sqrt{a^2+b^2}+a}\times\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}$$
$$=\frac{b^2(\sqrt{a^2+b^2}-a)}{(a^2+b^2)-a^2}=\sqrt{a^2+b^2}-a$$

EXAMPLE 3 *Divide* $\dfrac{\sqrt{5}+2}{\sqrt{3}-1}$ *by* $\dfrac{2+\sqrt{3}}{\sqrt{5}-2}$

$$\text{The quotient}=\frac{\sqrt{5}+2}{\sqrt{3}-1}\times\frac{\sqrt{5}-2}{2+\sqrt{3}}=\frac{(\sqrt{5})^2-4}{2\sqrt{3}-2+3-\sqrt{3}}=\frac{1}{\sqrt{3}+1}$$
$$=\frac{1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{\sqrt{3}-1}{2}$$

EXAMPLE 4 *Express* $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$ *with rational denominator*

Here the work of rationalization is performed in two steps

$$\frac{1}{1+\sqrt{2}-\sqrt{3}}=\frac{1}{(1+\sqrt{2})-\sqrt{3}}\times\frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})+\sqrt{3}}=\frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-3}=\frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}$$
$$=\frac{(1+\sqrt{2}+\sqrt{3})\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}+2+\sqrt{6}}{4}$$

EXAMPLES XXXI. d.

Find the value of

1. $(2\sqrt{a}-1)\times 3\sqrt{a}$ 2. $\sqrt{5x}(\sqrt{x}-\sqrt{5})$ 3. $(4+3\sqrt{p})\times 2\sqrt{p}$

4. $(3\sqrt{5}+4\sqrt{2})(\sqrt{5}+\sqrt{2})$ 5. $(2\sqrt{7}+\sqrt{5})(3\sqrt{5}-\sqrt{7})$

6. $(5+2\sqrt{3})^2$ 7. $(3\sqrt{5}-\sqrt{11})^2$ 8. $(5\sqrt{7}-7\sqrt{2})^2$

9. $(5+3\sqrt{2})(5-3\sqrt{2})$ 10. $(3\sqrt{10}+4\sqrt{7})(\sqrt{10}+\sqrt{7})$

11. $(7\sqrt{11}-23)(7\sqrt{11}+23)$ 12. $(\sqrt{a+1}-\sqrt{a})\times\sqrt{a+1}$

13. $(6\sqrt{5}+3\sqrt{7})(6\sqrt{5}-3\sqrt{7})$ 14. $(9\sqrt{2}-\sqrt{15})(3\sqrt{2}-2\sqrt{15})$

15. $(\sqrt{a+x}+\sqrt{a})(\sqrt{a+x}-\sqrt{a})$ 16. $(\sqrt{2p-3q}-2\sqrt{q})(\sqrt{2p+3q}+2\sqrt{q})$

Write down the square of

17. $\sqrt{p+q}+\sqrt{p-q}$ 18. $3\sqrt{a^2+b^2}-2\sqrt{a^2-b^2}$

19. $\sqrt{2x-1}+\sqrt{x+3}$ 20. $\sqrt{3x-5}-\sqrt{2x-1}$

Find the value of

21. $1\div(7-4\sqrt{3})$ 22. $1\div(8+3\sqrt{7})$ 23. $17\div(3\sqrt{7}-2\sqrt{3})$

24. $(5+2\sqrt{6})\div(5-2\sqrt{6})$ 25. $(4+3\sqrt{2})\div(5-3\sqrt{2})$

26. $(2\sqrt{ab}-b)\div(2a-\sqrt{ab})$. 27. $(\sqrt{5}-\sqrt{3})\div(4-\sqrt{15})$

28. $\dfrac{\sqrt{2}}{3-\sqrt{2}}\div\dfrac{2+3\sqrt{2}}{7}$ 29. $\dfrac{22}{3\sqrt{2}-\sqrt{7}}\div(\sqrt{18}+\sqrt{7})$

Express with rational denominators

30. $\dfrac{1}{2\sqrt{2}-\sqrt{3}}$ 31. $\dfrac{7\sqrt{6}+3\sqrt{5}}{4\sqrt{6}+\sqrt{5}}$ 32. $\dfrac{40}{9\sqrt{5}-5\sqrt{7}}$

33. $\dfrac{2\sqrt{a}}{5\sqrt{a}-3\sqrt{x}}$ 34. $\dfrac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}}$ 35. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

36. $\dfrac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$ 37. $\dfrac{4}{2+\sqrt{3}+\sqrt{7}}$

Express in the simplest form

38. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}-\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ 39. $\dfrac{18}{\sqrt{3}+\sqrt{2}}+\dfrac{12}{\sqrt{3}-\sqrt{2}}$ 40. $\dfrac{7+\sqrt{5}}{7-\sqrt{5}}-\dfrac{7-\sqrt{5}}{7+\sqrt{5}}$

41. $\dfrac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y}-\dfrac{\sqrt{x^2+y^2}-y}{x-\sqrt{x^2-y^2}}$ 42. $\dfrac{4(\sqrt{3}+1)}{\sqrt{3}-1}-\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$, $\sqrt{6}=2.44949$, find correct to four places of decimals the value of the following expressions.

43. $\dfrac{5\sqrt{2}-3}{5\sqrt{2}+3}$ 44. $\dfrac{\sqrt{5}-2}{9-4\sqrt{5}}$ 45. $\dfrac{5+2\sqrt{3}}{7-4\sqrt{3}}$ 46. $\dfrac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

Some Properties of Quadratic Surds.

371. Theorem I. *A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd*

If possible let $\sqrt{n}=a\pm\sqrt{m}$,

where a is rational and $\sqrt{n}$ and $\sqrt{m}$ are quadratic surds Then by squaring both sides, $n=a^2+m\pm 2a\sqrt{m}$,

$$\pm\sqrt{m}=\frac{n-a^2-m}{2a};$$

that is, a quadratic surd is equal to a rational quantity, which is impossible

372 Theorem II. *If* $x+\sqrt{y}=a+\sqrt{b}$, *where* x *and* a *are both rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are both irrational, then will* $x=a$ *and* $y=b$

For if x is not equal to a, let $x=a+m$,

then $a+m+\sqrt{y}=a+\sqrt{b}$,

that is, $\sqrt{b}=m+\sqrt{y}$,

which is impossible, by Theorem I

Hence $x=a$,

and consequently, $y=b$

If therefore $x+\sqrt{y}=a+\sqrt{b}$,

it follows that $x-\sqrt{y}=a-\sqrt{b}$

373 It appears from the preceding article that in any equation of the form

$$X\pm\sqrt{Y}=A\pm\sqrt{B}$$

we may equate the rational parts on each side and also the irrational parts, so that the above equation is really equivalent to *two independent equations* $X=A$ and $Y=B$ But it must be noticed that this principle of equating rational and irrational parts separately can only be applied when $\sqrt{Y}$ and $\sqrt{B}$ are *essentially irrational and not merely irrational in form*

Thus it would be absurd to apply the principle to the statement

$$7+\sqrt{5x-4}=2+\sqrt{5x+61}$$

This is really a *conditional equation* only true when $x=4$, and when x has this value, $\sqrt{5x-4}$ and $\sqrt{5x+61}$ are both rational

374 Theorem III *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, *then*

$$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$$

For by squaring we obtain

$$a+\sqrt{b}=x+y+2\sqrt{xy},$$

$$a=x+y, \quad \sqrt{b}=2\sqrt{xy} \qquad \text{[Art. 372]}$$

Hence $a-\sqrt{b}=x+y-2\sqrt{xy}$,

that is, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

375 Square Root of Binomial Quadratic Surds From the formula $(\sqrt{a}\pm\sqrt{b})^2=a+b\pm2\sqrt{ab}$ we can write down the square root of an expression of the form $X+2\sqrt{Y}$, if we can find two quantities, a and b, such that their sum is X and their product is Y

EXAMPLE 1 *Find the square root of* $12+2\sqrt{35}$

Writing the expression in the form $7+5+2\sqrt{7\times5}$ we see that the square root is $\sqrt{7}+\sqrt{5}$

NOTE Since every quantity has two square roots, here $-\sqrt{7}-\sqrt{5}$ is also a square root of the given expression, but in this and similar cases we confine our attention to the positive root

EXAMPLE 2 *Find the square root of* $18-8\sqrt{5}$

We must first write the expression so that the coefficient of the surd is 2

Thus $18-8\sqrt{5}=18-2\sqrt{5\times16}$, which may be written

$$18-2\sqrt{5\times2\times8}, \text{ or } 10+8-2\sqrt{10\times8}$$

Hence the square root is $\sqrt{10}-\sqrt{8}$, or $\sqrt{10}-2\sqrt{2}$

When the numbers are inconveniently large, we may make use of Theorem III

EXAMPLE 3 *Find the square root of* $41+6\sqrt{32}$

Assume $$\sqrt{41+6\sqrt{32}}=\sqrt{x}+\sqrt{y}, \qquad (1)$$

then $$\sqrt{41-6\sqrt{32}}=\sqrt{x}-\sqrt{y}$$

By multiplication, $$\sqrt{41^2-36\cdot32}=x-y, \qquad (2)$$

$$x-y=\sqrt{529}=23 \qquad (3)$$

Squaring (1), and equating the rational parts (Art 372),

$$x+y=41, \qquad (4)$$

whence, from (3) and (4), $x=32, \quad y=9$

$$\therefore \text{the square root of } 41+6\sqrt{32}=\sqrt{32}+\sqrt{9}=4\sqrt{2}+3$$

In finding $x-y$ from (2) it is sufficient to take the positive value of $\sqrt{529}$, as we assume x to be greater than y

EXAMPLE 4 *Find the square root of* $\sqrt{48}-\sqrt{45}$

Here we have the difference of two quadratic surds, and we proceed as follows

$$\sqrt{48}-\sqrt{45}=\sqrt{3}(\sqrt{16}-\sqrt{15})=\sqrt{3}(4-\sqrt{15})$$

$$=\sqrt{3}\times\frac{8-2\sqrt{15}}{2}, \text{ writing a coefficient 2 before } \sqrt{15},$$

$$=\sqrt{3}\times\frac{5+3-2\sqrt{5\times3}}{2},$$

$$\therefore \sqrt{\sqrt{48}-\sqrt{45}}=\sqrt[4]{3}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}=\sqrt[4]{3}\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)$$

EXAMPLES XXXI e.

Find the square root of the following binomial surds

1. $4+2\sqrt{3}$ 2. $5+2\sqrt{6}$ 3. $10+2\sqrt{24}$ 4. $7-4\sqrt{3}$

5. $9-2\sqrt{18}$ 6. $16+2\sqrt{55}$ 7. $17+6\sqrt{8}$ 8. $28-6\sqrt{3}$

9. $50+8\sqrt{6}$ 10. $38-12\sqrt{10}$ 11. $49-20\sqrt{6}$ 12. $\sqrt{32}-\sqrt{24}$

13. $248+32\sqrt{60}$ 14. $\sqrt{175}-\sqrt{147}$ 15. $26+2\sqrt{165}$

16. $12a+2b-4\sqrt{6ab}$ 17. $9m+8n+12\sqrt{2mn}$

18. $3x-1+2\sqrt{2x^2-3x-2}$ 19. $2m+2\sqrt{m^2-9n^2}$

Express in the simplest form

20. $\sqrt{45}+\sqrt{8}-\sqrt{80}+\sqrt{18}+\sqrt{7-\sqrt{40}}$

21. $\sqrt{19+4\sqrt{21}}+\sqrt{7}-\sqrt{12}-\sqrt{29-2\sqrt{28}}$

22. $\sqrt{27}-\sqrt{8}+\sqrt{17+12\sqrt{2}}-\sqrt{28-6\sqrt{3}}$

23. Shew that $\sqrt{a+\sqrt{b}}$ cannot be expressed in the form $\sqrt{x}+\sqrt{y}$ unless a^2-b is a perfect square

Irrational Equations.

376 An **Irrational Equation** is one which has one or more terms involving the unknown under a radical sign

377 In the following examples the *positive value* of the square root is always taken Thus a term such as $\sqrt{3x-5}$, when $x=3$, means the positive value of $\sqrt{4}$, or $+2$ Also all the terms are supposed to be *real*, so that a term of the form $\sqrt{ax+b}$ only admits of values which make $ax+b$ positive

The method of solution is illustrated in the following examples

EXAMPLE 1 *Solve the equation* $3\sqrt{x}+\sqrt{9x+13}-13=0$

Transpose so as to have a single surd term on one side,

thus $$\sqrt{9x+13}=13-3\sqrt{x}$$

Squaring both sides, $9x+13=169-78\sqrt{x}+9x$,

that is, $$78\sqrt{x}=156,$$

$$\sqrt{x}=2, \text{ and } x=4$$

[*Verification* When $x=4$, the left side$=3.2+7-13=0$]

NOTE Unless $\sqrt{x}$ and $\sqrt{9x+13}$ are both regarded as positive, the equation is *not* satisfied by $x=4$

EXAMPLE 2 *Solve* $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Squaring both sides,

$$x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$$

Transposing, and dividing by 2,

$$\sqrt{(x+7)(x+2)}=2x+2 \tag{1}$$

Squaring both sides, $x^2+9x+14=4x^2+8x+4$,

that is, $3x^2-x-10=0$,

or $(x-2)(3x+5)=0$

$$x=2, \text{ or } -\tfrac{5}{3}$$

[*Verification* When $x=2$, L.S. $=\sqrt{9}+\sqrt{4}=5$; R.S. $=\sqrt{25}=5$

When $x=-\frac{5}{3}$, L.S. $=\sqrt{7-\frac{5}{3}}+\sqrt{2-\frac{5}{3}}$

$$=\frac{4}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\frac{5}{\sqrt{3}},$$

R.S. $=\sqrt{-10+13}=\sqrt{3}$

Thus the equation is satisfied by $x=2$, but not by $x=-\frac{5}{3}$]

The latter value will be found on trial to satisfy the given equation if the sign of the second radical is changed.

Thus $\sqrt{x+7}-\sqrt{x+2}=\sqrt{6x+13}$

After squaring and reduction, this leads to

$$-\sqrt{(x+7)(x+2)}=2x+2 \tag{2}$$

By comparing the lines marked (1) and (2) it will be seen that in each case the next step gives rise to the quadratic whose roots are 2 and $-\frac{5}{3}$.

378 From the above example we see that when each side is squared, the resulting rational equation is not an *equivalent* one (Art 255), and gives rise to an *extraneous solution* which will only satisfy the original equation in a modified form

More generally in solving irrational equations we first get rid of surds by one or more preliminary steps The rational equation so obtained will contain all the roots of the original equation *if there are any*, but it may happen that one or both of the roots of the rational equation will not satisfy the original equation if we adhere to the limitations mentioned in Art 377

In solving irrational equations every root must be tested by substitution

[*Examples* XXXI f 1-25, *page* 344, *may be taken here*]

379 The following example deserves special attention

EXAMPLE *Solve* $2x^2-3\sqrt{2x^2-7x+7}=7x-3$

Transposing, $(2x^2-7x)-3\sqrt{2x^2-7x+7}=-3$

Put y for the *positive value* of the radical, so that $y=+\sqrt{2x^2-7x+7}$, then $2x^2-7x+7=y^2$, and $2x^2-7x=y^2-7$

By substitution we obtain

$$y^2-7-3y=-3, \text{ or } y^2-3y-4=0$$

The roots of this equation are $y=4$, or -1 The latter value we reject, since y is positive

Thus $\sqrt{2x^2-7x+7}=4$, or $2x^2-7x+7=16$,

whence we obtain $x=\frac{9}{2}$, or -1

These values should be checked by substitution

It will be found on trial that the values of x derived from $y=-1$ satisfy the equation obtained by changing the sign of the radical in the given equation

EXAMPLES XXXI. f.

Solve the equations

1 $\sqrt{2x+3}=3$ 2. $\sqrt{12x+1}=7$ 3. $7\sqrt{3x-5}=28$

4. $5+\sqrt[3]{x-2}=12$ 5. $6+\sqrt[4]{x-2}=9$ 6 $3\sqrt{8x}=2\sqrt{15x+6}$

7 $\sqrt{3x+10}=x$ 8 $\sqrt{x+7}+\sqrt{x}=7.$ 9 $2\sqrt{x}-1=\sqrt{4x-11}$

10 $\sqrt{4x+1}-\sqrt{x-2}=\sqrt{x+3}$ 11. $\sqrt{14+25x}=\sqrt{7+9x}+\sqrt{1+4x}$

12. $\sqrt{3x+4}+\sqrt{3x-5}=9$ 13. $2\sqrt[3]{5x-35}=5\sqrt[3]{2x-7}$

14. $\sqrt{x+5}-\sqrt{x-11}=\sqrt{x-16}$ 15 $\sqrt{2x-1}=\sqrt{8x-4}-\sqrt{x+4}$

16. $\sqrt{x+a}+\sqrt{x-a}=\sqrt{a}$ 17. $\sqrt{x+a}+\sqrt{x+b}=\sqrt{a+b}$

18 $\sqrt{x-5}+\sqrt{x+4}=\dfrac{45}{\sqrt{x+4}}$ 19 $\sqrt{x+a}+\sqrt{x+b}=\dfrac{a}{\sqrt{x+a}}$

20 $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x}+6}$. 21. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$

22. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$ 23 $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$

24. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}}=2$ 25 $x+\sqrt{a^2+x^2}=\dfrac{5a^2}{\sqrt{a^2+x^2}}$

26. $x^2+6\sqrt{x^2-2x+5}=11+2x$ 27. $4x+1-2\sqrt{x^2-6x+2}=x^2-2x$

28 $x^2+18=8x+6\sqrt{x^2-8x+9}$ 29. $3x-\sqrt{2x^2+6x+1}=1-x^2$

30. $9x-3x^2+4\sqrt{x^2-3x+5}=11$ 31. $(x+1)^2=5(\sqrt{x^2+2x+2}-1)$

32 $6-4\sqrt{(x+2)(x+3)}=x^2+5x$ 33. $30-\dfrac{16x}{5}=2x^2-3\sqrt{5x^2+8x-21}$

CHAPTER XXXII

LOGARITHMS

380 SUPPOSE x and y are two numbers connected by the relation $y=2^x$, and that values of y are obtained by ascribing different values to x. For each value so obtained we have an *algebraical name* in reference to the corresponding value of x, for it is *the x^{th} power of 2*. But with reference to any value of y we have no convenient name at present for the corresponding value of x. We can only say that x *is the index of the power to which 2 must be raised in order to give the value of y*.

For example $8=2^3$, $16=2^4$, $32=2^5$, $64=2^6$, ... and we want a name for the *indices* 3, 4, 5, 6, ... with reference to the *numbers* 8, 16, 32, 64, ... They are called **logarithms**.

Thus 3 is called the logarithm of 8 to the base 2,
5 „ „ 32 „ 2,
and so on.

381 **General Definition.** If a number N can be expressed in the form a^x, the index x is called the **logarithm** of the **number** N to the base a.

EXAMPLES. (i) Since $81=3^4$, the logarithm of 81 to base 3 is 4,
(ii) Since $10^1=10$, $10^2=100$, $10^3=1000$, ...
the natural numbers 1, 2, 3, ... are respectively the logarithms of 10, 100, 1000, ... to base 10.

382 The logarithm of N to base a is usually written $\log_a N$, so that the same relation between the number and its logarithm is expressed by the two equations

$$a^x=N, \qquad x=\log_a N.$$

Thus since $7^3=343$, $\quad 3=\log_7 343$,

and since $10^{-3}=\frac{1}{10^3}=0\cdot001$, $\quad -3=\log_{10}0\cdot001$.

Any number might be taken as base, and the logarithms of all positive numbers to any given base are known as a **System of Logarithms**. In all practical calculations the system in use is that which has 10 for its base. The advantages of this system will appear later.

Logarithms to the base 10 are known as **Common Logarithms**; this system was first introduced in 1615 by Briggs, a contemporary of Napier the inventor of logarithms.

383 Before proceeding to some general properties of logarithms we will briefly indicate how a system of logarithms to base 2 may be derived from the graph of $y=2^x$

When $\qquad x=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \infty,$

we have $\qquad y=1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \infty$

By means of fractional indices we can use intermediate approximate values

Thus $\qquad x=\frac{1}{2}$ gives $y=2^{\frac{1}{2}}=\sqrt{2}=1{\cdot}41$, very nearly,

$\qquad x=\frac{3}{2}$ „ $y=2^{\frac{3}{2}}=2\sqrt{2}=2{\cdot}82$, „

$\qquad x=\frac{5}{2}$ „ $y=2^{2\frac{1}{2}}=4\sqrt{2}=5{\cdot}65$, „

and so on

Taking 1 inch as unit for x and one-tenth of an inch as unit for y, the graph will be as in Fig 31 on the opposite page.

In this curve each abscissa is an index, and the corresponding ordinate is the value of 2 raised to that index. Or in other words, each abscissa is the logarithm to base 2 of the number denoted by the corresponding ordinate

We may now read off the logarithm to base 2 of any number between 1 and 35

Thus at P, when $y=10$, $x=3{\cdot}32$, approximately,

that is, $\qquad \log_2 10=3{\cdot}32$

Again, at Q, when $x=4{\cdot}75$, $y=26{\cdot}9$, approximately,

that is, $\qquad \log 26{\cdot}9=4{\cdot}75$

384 The figure is limited by the size of the page, and we have only considered positive values of x The pupil should draw his own diagram on a larger scale, and extend the curve to the left of OY It will then readily be seen that it extends to infinity in the negative direction, but never crosses the axis of x Hence the range of *logarithms* from $-\infty$ to $+\infty$ applies only *to positive numbers*

The following points may be noticed

(i) The logarithms of all numbers greater than 1 are positive

(ii) The logarithm of 1 is 0

(iii) The logarithm of all numbers less than 1 are negative

385 The base 2 is not an essential part of the above graphical representation Hence we infer the possibility of forming a system of logarithms for all positive numbers to any suitable base

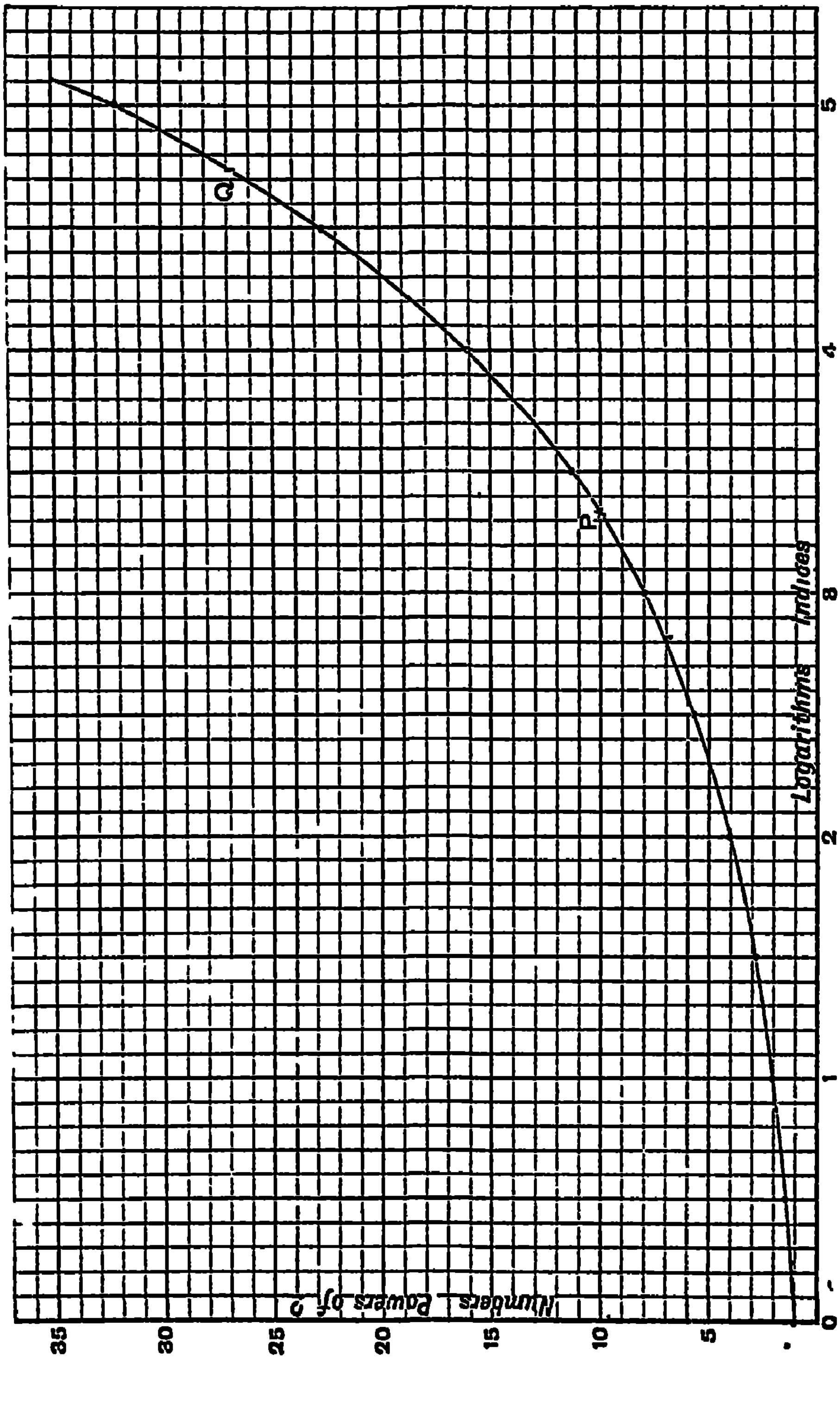

Q
P
Logarithms Indices
Numbers Powers of 2
0
1
2
3
4
5
0
5
10
15
20
25
30
35

FIG 31

386 The following general propositions are applicable to all logarithms independently of any particular base

387 *The logarithm of* 1 *is* 0, *and the logarithm of the base is* 1

For $a^0=1$ for all values of a, therefore $\log_a 1=0$, whatever the base may be Again $a^1=a$, therefore $\log_a a=1$

388 *To find the logarithm of a product*

Let M and N be two numbers such that $M=a^x$, $N=a^y$

Then $\qquad x=\log_a M, \quad y=\log_a N$

The product $\qquad MN=a^x\times a^y=a^{x+y}$,

whence, by definition, $\qquad \log_a MN=x+y$

$$=\log_a M+\log_a N$$

Similarly $\qquad \log_a MNP=\log_a M+\log_a N+\log_a P$;

and so on for any number of factors

EXAMPLE $\log_a 42=\log_a(2\times 3\times 7)=\log_a 2+\log_a 3+\log_a 7$

389 *To find the logarithm of a quotient, or a fraction*

As before suppose $\qquad M=a^x, \qquad N=a^y$,

so that $\qquad x=\log_a M, \quad y=\log_a N$

The fraction $\qquad \dfrac{M}{N}=\dfrac{a^x}{a^y}=a^{x-y}$,

whence, by definition, $\qquad \log_a \dfrac{M}{N}=x-y$

$$=\log_a M-\log_a N$$

EXAMPLE $\log_a(2\frac{1}{7})=\log_a\frac{15}{7}=\log_a 15-\log_a 7$

$$=\log_a(3\times 5)-\log_a 7=\log_a 3+\log_a 5-\log_a 7$$

390 *To find the logarithm of a number raised to any power, integral or fractional*

Suppose $M=a^x$, so that $x=\log_a M$, and suppose it is required to find the value of $\log_a(M^p)$

We have $\qquad M^p=(a^x)^p=a^{px}$,

whence, by definition, $\qquad \log_a(M^p)=px$

$$=p\log_a M$$

Similarly $\qquad \log_a(M^{\frac{1}{r}})=\dfrac{1}{r}\log_a M$

EXAMPLE. $\qquad \log_{10}\dfrac{3^5\ \sqrt[3]{2}}{\sqrt[4]{5}}=\log_{10}(3^5\ 2^{\frac{1}{3}})-\log_{10}5^{\frac{1}{4}}$

$$=\log_{10}3^5+\log_{10}2^{\frac{1}{3}}-\log_{10}5^{\frac{1}{4}}$$

$$=5\log_{10}3+\tfrac{1}{3}\log_{10}2-\tfrac{1}{4}\log_{10}5$$

391 The results we have proved may be summarised as follows

(i) the logarithm of a **product** is equal to the **sum** of the logarithms of its factors,

(ii) the logarithm of a **fraction** is equal to the logarithm of the numerator **minus** the logarithm of the denominator,

(iii) the logarithm of the p^{th} **power** of a number is equal to the logarithm of the number **multiplied by** p,

(iv) the logarithm of the r^{th} **root** of a number is equal to the logarithm of the number **divided by** r

Thus by the use of logarithms the operations of **multiplication** and **division** may be replaced by those of **addition** and **subtraction**; the operations of **involution** and **evolution** by those of **multiplication** and **division**

392 The following examples will serve to illustrate the laws of operation above established

When any particular system is in use and no ambiguity is likely to arise, the suffix denoting the base may be omitted Thus in the case of *common logarithms* we usually write $\log 2$, $\log 3$, instead of $\log_{10} 2$, $\log_{10} 3$

EXAMPLE 1 *Express* $\log \frac{324}{\sqrt[5]{64}}$ *in terms of* $\log 2$ *and* $\log 3$

$$\log \frac{324}{\sqrt[5]{64}} = \log 324 - \log 64^{\frac{1}{5}}$$

$$= \log (81 \times 4) - \tfrac{1}{5} \log 64$$

$$= \log (3^4 \times 2^2) - \tfrac{1}{5} \log 2^6$$

$$= \log 3^4 + \log 2^2 - \tfrac{1}{5} (6 \log 2)$$

$$= 4 \log 3 + 2 \log 2 - \tfrac{6}{5} \log 2$$

$$= 4 \log 3 + \tfrac{4}{5} \log 2$$

EXAMPLE 2 *Shew that* $\log \frac{26}{51} + \log \frac{119}{91} = \log 2 - \log 3$

By Art 388, $\log \frac{26}{51} + \log \frac{119}{91} = \log \left(\frac{26}{51} \times \frac{119}{91} \right)$

$$= \log \left(\frac{2 \times 13 \times 7 \times 17}{3 \times 17 \times 7 \times 13} \right)$$

$$= \log \frac{2}{3}$$

$$= \log 2 - \log 3$$

EXAMPLES XXXII. a.

Read off the following equations in the logarithmic form (e.g. $x=\log_a N$)

1. $2^8=256$ 2. $3^5=243$ 3. $6^3=216$ 4. $8^3=512$

Express the following equations in the index form (e.g. $a^x=N$)

5. $\log_3 729=6$ 6. $x=\log_a y$ 7. $\log_{10} 0{\cdot}0001=-4$ 8. $p=\log_b q$.

Express in terms of $\log a$, $\log b$, and $\log c$

9. $\log\dfrac{ab}{c}$ 10. $\log\dfrac{a^3}{b^2c}$ 11. $\log\dfrac{ac^{\frac{1}{2}}}{b^{\frac{1}{3}}}$ 12. $\log\dfrac{\sqrt[3]{a}\,\sqrt[4]{b}}{\sqrt{c^3}}$.

13. Express in terms of $\log 2$ and $\log 3$

(i) $\log 18$; (ii) $\log 48$, (iii) $\log 6^5$; (iv) $\log\sqrt[4]{192}$;

and find their numerical values, assuming

$$\log 2=0{\cdot}301,\quad \log 3=0{\cdot}477$$

14. Shew that $\log\sqrt{54}\times\sqrt[3]{243}=\frac{19}{6}\log 3+\frac{1}{2}\log 2$

15. Shew that $\log\frac{9}{16}+\log\frac{40}{81}=\log 5-\log 18$

16. Shew that $\log\left(\frac{217}{33}-\frac{31}{366}\right)=2\log 7$

17. From the graph of $y=2^x$ (Art. 383), find

(i) $\log_2 7$; (ii) $\log_2 13$, (iii) $\log_2 15$, each to the nearest tenth

Thence by Arts. 388, 389, find $\log_2 91$, $\log_2\frac{13}{7}$, $\log_2 105$

Common Logarithms.

393 Since $10^1=10$, $10^2=100$, $10^3=1000$, $10^4=10000$, ..., we see that the numbers

$$1,\quad 2,\quad 3,\quad 4,\quad \ldots$$

are the logarithms of $10,\ 100,\ 1000,\ 10000,\ \ldots$, respectively

Also since $10^{-1}=\dfrac{1}{10}=\cdot 1$, $10^{-2}=\dfrac{1}{10^2}=\cdot 01$, $10^{-3}=\dfrac{1}{10^3}=\cdot 001$, ...

we see that the logarithms of $\cdot 1,\ \cdot 01,\ \cdot 001,\ \ldots$

are respectively $-1,\ -2,\ -3,\ \ldots$

Hence the logarithms of all numbers which are *exact powers* of 10 are *integers* either positive or negative In the case of all other numbers, the logarithms will not be integral they may be wholly fractional or partly integral and partly fractional

The integral part of a logarithm is called the **characteristic**, and the fractional part when expressed as a positive decimal is called the **mantissa**.

394. *The characteristics may always be written down by inspection.*

A number which has one digit before the decimal point, such as 3·27, is greater than 10^0 and less than 10^1;

$\therefore$ its logarithm lies between 0 and 1; that is, *its logarithm is wholly fractional.*

Thus the characteristic of the logarithm of any number less than 10 is 0.

The following example should be studied very carefully.

EXAMPLE. Suppose we know that $\log 3·27 = 0·5145$; then

$$\log 32·7 = \log (3·27 \times 10) = \log 3·27 + \log 10 = 1 + ·5145,$$

$$\log 32700 = \log (3·27 \times 10^4) = \log 3·27 + \log 10^4 = 4 + ·5145,$$

$$\log ·0327 = \log (3·27 \times 10^{-2}) = \log 3·27 + \log 10^{-2} = -2 + ·5145,$$

$$\log ·000327 = \log (3·27 \times 10^{-4}) = \log (3·27 + \log 10^{-4}) = -4 + ·5145,$$

and so on.

From this example we infer that the logarithms of all numbers which have the same sequence of digits (*i.e.* differing only in the position of the decimal point) can be written so that they have the same *positive* mantissa, but that the characteristics are different, and may be positive, negative, or zero.

Also we see that by introducing a suitable power of 10 the digits of all numbers can be expressed in one standard form in which *the decimal point always stands after the first significant digit.* When a number is written in this form the characteristic is given at once by the index of the power of 10.

By examining the cases given above we may also deduce the following verbal rules for writing down the characteristics.

The characteristic of the logarithm of a number greater than unity is positive, and is less by one than the number of digits before *the decimal point.*

EXAMPLE. The characteristics of

	log 314,	log 87·263,	log 2·78,	log 3500
are	2,	1,	0,	3, respectively.

The characteristic of the logarithm of a number less than one is negative, and is numerically greater by one than the number of ciphers immediately after *the decimal point.*

EXAMPLE. The characteristics of

	log ·4,	log ·3725,	log ·000135,	log ·08
are	-1,	-1,	-4,	-2, respectively.

395 The logarithms of integers have been found and tabulated. For most practical purposes a system which gives the logarithms to four places of decimals, commonly called **Four-figure Tables**, will give results sufficiently accurate

396 Common Logarithms have two great advantages:

(i) *The characteristics can be written down by inspection.*

Hence the characteristics need not be tabulated.

(ii) *The mantissæ are the same for the logarithms of all numbers which have the same significant digits*

Hence the Tables need only contain the mantissæ of the logarithms of *integers*

In order to secure these advantages we arrange our work so as *always to keep the mantissa positive*, and the minus sign is written over a negative characteristic and not before it, so as to indicate that the characteristic alone is negative.

Thus $\bar{5}\cdot4771$ which is the logarithm of $\cdot00003$ is equivalent to $-5+\cdot4771$, and must be distinguished from $-5\cdot4771$, in which both the integer and the decimal are negative

397 In the course of work we sometimes meet with a logarithm wholly negative In such a case a rearrangement is necessary in order to write the logarithm with a positive mantissa. A result such as $-3\cdot5229$ may be transformed by subtracting 1 from the integral part and adding 1 to the decimal part

$$\text{Thus}\qquad -3\cdot5229 = -3-1+(1-\cdot5229)$$
$$= -4+\cdot4771, \text{ or } \bar{4}\cdot4771$$

In adjusting the mantissa, so as to make it positive, we have to *add* unity to the negative decimal This is most easily done by *subtracting* each digit from 9, except the last on the right, which is subtracted from 10 The negative characteristic is numerically greater by 1 than the integral part of the negative logarithm.

Thus we may write down at once

$$-4\cdot3247 = \bar{5}\cdot6753, \quad -0\cdot4239 = \bar{1}\cdot5761.$$

EXAMPLE 1. *From the sum of* $\bar{3}\cdot9605$ *and* $1\cdot2135$ *subtract*

(i) $3\cdot7234$, (ii) $\bar{4}\cdot7234$

$$\begin{array}{rr}
 & \bar{3}\cdot9605 \\
 & \underline{1\cdot2135} \\
\text{(i)} & \bar{1}\cdot1740 \\
 & \underline{3\cdot7234} \\
 & \bar{5}\cdot4506
\end{array}
\qquad
\begin{array}{rr}
\text{(ii)} & \bar{1}\cdot1740 \\
 & \underline{\bar{4}\cdot7234} \\
 & 2\cdot4506
\end{array}$$

Here after adding the decimal figures we have 1 to carry. Thus at the first stage the characteristic is the algebraic sum of 2 and -3, which is written $\bar{1}$.

In (i) when we get to the integral part we have to subtract 3+1 from -1, the result is -5, and is written $\bar{5}$

In (ii) we have to subtract $-4+1$ (or -3) from -1. The result is 2.

EXAMPLE 2. (i) *Multiply* $\bar{1}.8173$ *by* 3. (ii) *Divide* $\bar{4}.8134$ *by* 3.

(i) $\bar{1}.8173$
$\underline{\quad\quad 3}$
$\bar{1}.4519$

On multiplying 8 by 3 we have to carry 2 to the product of -1 and 3. Hence the characteristic is $-3+2$, or -1.

(ii) $\bar{4}.8134=\bar{6}+2.8134$,

$\frac{1}{3}(\bar{4}.8134)=\frac{1}{3}(\bar{6}+2.8134)$
$=\bar{2}.9378$.

Here we cannot divide $\bar{4}.8134$ as it stands by 3. By a suitable adjustment the characteristic is written so that its negative part is a multiple of the given divisor. A similar artifice is always employed in dividing a logarithm with a negative characteristic.

398. The logarithm of 5 and its powers can always be obtained from log 2.

Thus $\log 5=\log\frac{10}{2}=\log 10-\log 2=1-\log 2$.

EXAMPLE. *Given* $\log 2=0.3010$, $\log 7=0.8451$, *find the number of digits in* 875^{10}.

$$\log 875^{10}=10\log(7\times 125)=10(\log 7+3\log 5)=10(\log 7+3-3\log 2)$$
$$=10\times 2.9421=29.421;$$

hence the number of digits in 875^{10} is 30. [Art. 394.]

EXAMPLES. XXXII. b.

1. Find by inspection the characteristics of the logarithms of

213, 4271, 5, 3.680, ·275, ·0008, 25.62.

2. The mantissa of log 4562 is ·6592; write down the logarithms of 45.62, ·004562, 4562000, ·4562.

3. The logarithm of 7.561 is 0.8786; write down the logarithms of 756.1, 7.561×10^4, 7.561×10^{-4}, ·07561.

Also write down the numbers whose logarithms are

5.8786, 1.8786, $\bar{3}.8786$, $\bar{1}.8786$, 15.8786.

Find (to four decimal figures) the values of

4. $\bar{1}.3681\times 3$. 5. $\bar{2}.0068\times 7$. 6. $\bar{4}.9832\times 12$.

7. $\bar{2}.4320\div\bar{1}.3971$. 8. $\bar{3}.6583-\bar{4}.7241$. 9. $\bar{2}.4871\div 4.3970$.

10. $\bar{4}.5703\div 5$. 11. $\frac{1}{6}(\bar{3}.8123)$. 12. $\frac{3}{4}(\bar{2}.1305)$.

13. Given $\log 2=0.3010$, $\log 3=0.4771$, find the number of digits in the 20^{th} power of 72.

14. Given $\log 2=0.3010$, shew that $(\frac{1}{40})^{16}$ is a decimal beginning with 25 ciphers.

15. In Art. 329 it is shewn that the sum of

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots \quad \text{to } n \text{ terms}=2-2^{-n+1}.$$

Given $\log 2=0.3010$, shew that when $n=1000$ the sum of the series differs from 2 by a decimal beginning with 300 ciphers.

399 We shall now give an example to indicate briefly how the graph of $y=10^x$ may be used to find logarithms to base 10 The full details of the work are left as an exercise for the pupil, who should draw a larger diagram and extend it further than is here possible

Example *By repeated evolution find the values of* $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$, $10^{\frac{1}{16}}$ *Evaluate* $10^{\frac{1}{8}} \times 10^{\frac{1}{16}}=10^{\frac{3}{16}}$, $10^{\frac{1}{4}} \times 10^{\frac{1}{16}}=10^{\frac{5}{16}}$, $10^{\frac{1}{4}} \times 10^{\frac{1}{8}}=10^{\frac{6}{16}}$, *and so on* *Use the values of* $10^{\frac{1}{16}}$, $10^{\frac{2}{16}}$, $10^{\frac{3}{16}}$, $10^{\frac{9}{16}}$ *to plot a portion of the curve* $y=10^x$ *on a large scale, and thence find the values of* log 1 68, log 2 24, log 3, *correct to three places of decimals*

Also from the graph find the product of 1 68 *and* 2 24

If we take 10 inches and 1 inch as units for x and y respectively, a diagonal scale will give values of x correct to three decimal places and values of y correct to two The graph is given in Fig 32 on the opposite page

Expressing the given fractional indices in decimal form, it will be found that the corresponding values of x and y are approximately as in the annexed table

x	y
0 000	1 00
0 063	1 15
0 125	1 33
0 188	1 54
0 250	1 78
0 313	2 05
0 375	2 37
0 438	2 74
0 500	3·16
0 563	3 65

Since $x=\log y$ we have to find x when y has the values 1 68, 2 24, 3

The values required are the abscissæ of P, Q, R, viz 0 225, 0 35, 0 477

Again from the graph

$$1\ 68=10^{\cdot 225}, \quad 2\ 24=10^{\cdot 35},$$

$$\therefore \quad 1\ 68 \times 2\ 24=10^{\cdot 225} \times 10^{\cdot 35}=10^{\cdot 575},$$

and when $x=\cdot 575$, $y=3\ 76$

$$1\ 68 \times 2\ 25=3\ 76$$

The graph may be used in the following way to find the logarithms of some numbers which are not directly given within the limits of the figure

Thus suppose we want the value of log 196

When $y=1\ 96$, $x=\cdot 29$, so that $\log 1\ 96=\cdot 29$

And $196=1\ 96 \times 10^2$, $\quad \log 196=2\ 29$

400 It will be noticed that the curve bends very slowly, so that for any two points, a short distance apart, the intervening portion of the curve is almost a straight line, the *slope* of which is constant In other words, when the difference between two ordinates is small compared with either of them the difference between their abscissæ is very nearly proportional to the difference between the ordinates

Hence *when any small change is made in a number there is an approximately proportional change in the corresponding logarithm* This is known as the Principle of Proportional Differences, and will be referred to again when we come to discuss the use of Logarithmic Tables

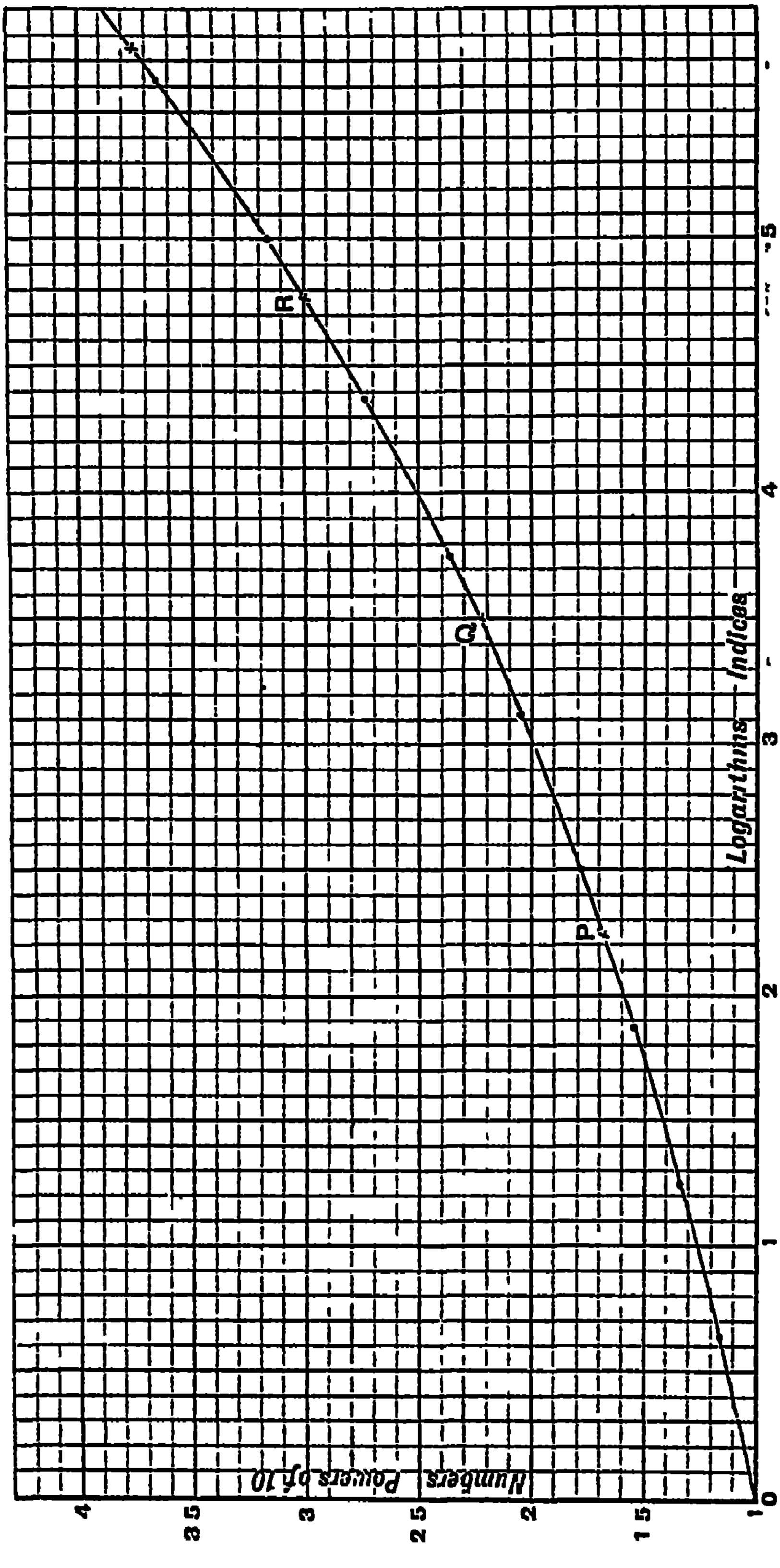


FIG 62.

Use of Four-Figure Tables.

401 *To find the logarithm of a given number from the Tables*

EXAMPLE 1 *Find* log 38, log 380, log 0038

The following is an extract from the Table on page 364

No	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10

We first find the number 38 in the left hand column Opposite to this we find the digits 5798 This, with the decimal point prefixed, is the mantissa for the logarithms of all numbers whose significant digits are 38 Hence, prefixing the characteristics, we have

$$\log 38 = 1\ 5798, \quad \log 380 = 2\ 5798, \quad \log\ 0038 = \bar{3}\ 5798$$

EXAMPLE 2 *Find* log 3 86, log 0386, log 386000

The same line as before will give the mantissa of the logarithms of all numbers which begin with 38. From this line we choose the mantissa which stands in the column headed 6 This gives 5866 as the mantissa for all numbers whose significant digits are 386 Hence, prefixing the characteristics, we have

$$\log 3\ 86 = \ 5866, \quad \log\ 0386 = \bar{2}\ 5866, \quad \log 386000 = 5\ 5866$$

402 Similarly the logarithm of any number consisting of not more than 3 significant digits can be obtained directly from the Tables When the number has 4 significant digits, use is made of the *Principle of Proportional Differences* [Art 400] The differences in the logarithms corresponding to small differences in the numbers have been calculated, and are printed ready for use in the *difference columns* at the right hand of the Tables The way in which these differences are used is shewn in the following example

EXAMPLE *Find* (i) log 3 864, (ii) log 003868

Here, as before, we can find the mantissa for the sequence of digits 386 This has to be *corrected* by the addition of the figures which stand underneath 4 and 8 respectively in the difference columns

$$\begin{array}{llrl} \text{(i)} & \log 3\ 86 & = 0\ 5866 \\ & \text{diff for} \quad 4 & 5 \\ \hline & \log 3\ 864 & = 0\ 5871 \end{array} \qquad \begin{array}{llrl} \text{(ii)} & \log\ 00386 & = \bar{3}\ 5866 \\ & \text{diff for} \quad 8 & 9 \\ \hline & \log\ 003868 & = \bar{3}\ 5875 \end{array}$$

NOTE In printing the differences non-significant ciphers are omitted Thus the differences used above are really 0005 and 0009 This accounts for the position of the digits 5 and 9 in making the necessary 'correction' With a little practice the correction from the difference columns can be performed mentally

403 The number corresponding to a given logarithm is called its **antilogarithm** Thus in the last example 3 864 and ·003868 are the numbers whose logarithms are 0 5871 and $\bar{3}$ 5875

Hence antilog 0 5871 = 3 864, antilog $\bar{3}$ 5875 = 003868

404 *To find the antilogarithm of a given logarithm*

In using the Tables of antilogarithms on pages 366, 367, it is important to remember that we are seeking *numbers* corresponding to *given logarithms* Thus in the left hand column we have the first two digits of the given *mantissæ*, with the decimal point prefixed The characteristics of the given logarithms will fix the position of the decimal point in the numbers taken from the Tables

EXAMPLE *Find the antilogarithms of* (i) 1 583, (ii) $\bar{2}$ 8249

The following is an extract from the Table on page 367

Log	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8

(i) We first find 58 in the left hand column, and pass along the horizontal line and take the number in the vertical column headed by 3 Thus 583 is the mantissa of the logarithm of a number whose significant digits are 3828 Hence antilog 1 583 = 38 28

$$\begin{aligned} &\text{(ii) antilog } \bar{2}\ 824 = 06668 \\ &\quad \text{diff for } \underline{\quad 9} \quad \underline{\quad 14} \\ &\quad \text{antilog } \bar{2}\ 8249 = 06682 \end{aligned}$$

Here corresponding to the first 3 digits of the mantissa we find the sequence of digits 6668, and the decimal point is inserted in the position corresponding to the characteristic $\bar{2}$ To the number so found we add 14 from the difference column headed 9

405 Examples illustrating the use of Logarithmic Tables

EXAMPLE 1 *Find the product of* 72 38 *and* 5689

$$\begin{aligned} &\log 72\ 3 \quad = 1\ 8591 \\ &\text{diff for} \quad 8 \quad\quad 5 \\ &\log \quad 568 = \bar{1}\ 7543 \\ &\text{diff for} \quad 9 \quad \underline{\quad 7} \\ &\log \textit{product} = 1\ 6146 \end{aligned}$$

$$\begin{aligned} &\text{antilog } 1\ 614 = 41\ 11 \\ &\text{diff for} \quad 6 \quad \underline{\quad 6} \\ &\text{antilog } 1\ 6146 = 41\ 17 \end{aligned}$$

Thus the product is 41 17

NOTE Accuracy beyond four significant figures can never be secured with four-figure logarithms Moreover we cannot always rely on the accuracy of the last figure In the present case, if the product of 72 38 and 5689 is obtained by contracted multiplication it will be found that the result to four significant figures is 41 18

EXAMPLE 2 *Find the value of* $\dfrac{15.38 \times .0137}{27.64 \times .0038}$ *to four significant digits*

By Art 389, log *fraction* = log *numerator* − log *denominator*

Numerator	Denominator
$\log 15.3 = 1.1847$	$\log 27.6 = 1.4409$
diff for $8 \quad 23$	diff for $4 \quad 6$
$\log .0137 = \bar{2}.1367$	$\log .0038 = \bar{3}.5798$
log *numerator* $= \bar{1}.3237$	log *denominator* $= \bar{1}.0213$
$\bar{1}.3237$	antilog $0.302 = 2.004$
subtract $\bar{1}.0213$	diff for $4 \quad 2$
log *fraction* $= 0.3024$	antilog $0.3024 = 2.006$

Thus $$\frac{15.38 \times .0137}{27.64 \times .0038} = 2.006$$

406 Care must be taken not to attempt a greater degree of accuracy than is obtainable from the Tables In some cases the first step of the work will be to adapt the *data* to the Tables

EXAMPLE 1 *Find as accurately as possible from four-figure Tables the product of* 3784.8 *and* 40869

Here the data must first be adapted to the Tables

Now 3784.8 = 3785 correct to four significant figures,

and 40869 = 40870 „ „

$\log 3785 = 3.5781$	Now antilog $.1896 = 1.547$
$\log 40870 = 4.6115$	the required product $= 1.547 \times 10^8$
log product $= 8.1896$	$= 154700000,$

the fourth significant digit being open to doubt, and this is the closest approximation that can be obtained by the use of four figure Tables

NOTE When very large or very small approximate results are under consideration it is best to express them in the standard form suggested in Art 394, namely, as the product of some power of 10 and a number with one integral digit

Thus if the distance of the earth from the sun is 92,000,000 miles, true to the nearest million, the approximate distance is conveniently represented by 9.2×10^7 miles

EXAMPLE 2 *Find the cube root of* .027476 *from the Tables*

$.027476 = .02748$ correct to four significant figures

Let $x = \sqrt[3]{.02748}$, or $(.02748)^{\frac{1}{3}}$,

then $$\log x = \tfrac{1}{3}\log(.02748)$$
$$= \tfrac{1}{3}(\bar{2}.4391), \text{ from the Table of Logs}$$
$$= \bar{1}.4797,$$
$$x = .3018, \text{ from the Table of Antilogs.}$$

EXAMPLE 3 *Find the value of* $\dfrac{(275 \times \frac{1}{63})^5}{\sqrt[4]{35 \times 2\ 983}}$ *to the nearest integer*

Denote the expression by x, then

$$\log x = 5(\log 275 - \log 63) - \tfrac{1}{4}(\log 35 + \log 2\ 983)$$

$$\begin{array}{rl} \log 275 = & 2\ 4393 \\ \log\ \ 63 = & \underline{1\ 7993} \\ & 0\ 6400 \\ & \underline{5} \\ & 3\ 2000 \\ \text{subtract} & \underline{0\ 5047} \end{array} \qquad\qquad \begin{array}{rl} \log 35 = & 1\ 5441 \\ \log 2\ 983 = & \underline{0\ 4746} \\ 4 & \underline{\big|\ 2\ 0187} \\ & 0\ 5047 \end{array}$$

$$\log x = 2\ 6953 = \log 495\ 8, \text{ from the Tables}$$

$$x = 496, \text{ to the nearest integer}$$

EXAMPLE 4 *Find x to two places of decimals from the equation*

$$6^{x+1}\ 3^{5x-2} = 21$$

Taking logarithms of both sides, we have

$$(x+1)\log 6 + (5x-2)\log 3 = \log 21,$$

$$x(\log 6 + 5\log 3) = \log 21 - \log 6 + 2\log 3;$$

$$x = \frac{\log 21 - \log 6 + 2\log 3}{\log 6 + 5\log 3}$$

$$= \frac{1\ 3222 - \ 7782 + \ 9542}{7782 + 2\ 3855}$$

$$= \frac{1\ 4982}{3\ 1637} = 0\ 47$$

$$\begin{array}{r} 0\ 47 \\ 3\ 16{,}4\ \big|\ 1\ 4982 \\ \underline{2326} \\ 111 \end{array}$$

EXAMPLES XXXII. c

1. Read off from the Tables the logarithms of

$2\ 3,\quad 54\ 7,\quad 6345,\quad 6\ 345 \times 10^4,\quad 032,\quad \cdot 0326 \times 10^2$

2. Read off from the Tables the antilogarithms of

$3\ 723,\quad 10\ 723,\quad \bar{4}\ 723,\quad 0\ 6451,\quad \bar{1}\ 4325,\quad 15\ 835,\quad \overline{11}\ 5623$

Find the values of the following quantities, to four significant figures

3. $24\ 32 \times 7\ 681$ 4. $396 \cdot 2 \times \ 0017$ 5. $27\ 35 \times 310\ 6$

6. $2\ 8 \times 3\ 9 \times \ 056$ 7. $56\ 2 \times 1\ 72 \times 8\ 7$ 8. $27 \times \cdot 034 \times 9\ 08$

9. $\dfrac{2\ 803}{0634}$ 10. $\dfrac{16\ 83}{24\ 76}$ 11. $\dfrac{30\ 56}{4\ 105}$ 12. $\dfrac{2\ 391}{30\ 72}$ 13. $\dfrac{487}{6398}$.

14. $\dfrac{21\ 43 \times 3\ 721}{8\ 532}$.   15. $24\ 37 \times \dfrac{3\ 94}{897}$ 16. $\dfrac{7\ 859}{46 \cdot 28}$ of $3 \cdot 01$

Evaluate the following expressions, to four significant figures

17. $(\cdot097)^{\frac{1}{5}}$. 18. $(1\cdot73)^{11}$. 19 $\sqrt{51}$ 20. $\sqrt[5]{27\cdot2}$ 21. $\sqrt[11]{1772}$.

Solve the following equations, giving the values of x correct to two decimal figures:

22. $3^{x-1}=405$ 23 $2^{2x-7}=10^{3x-5}$. 24. $2^{x-1}\cdot 3^{5x-1}=250$

25 $30^{x-1}=20^{3x-1}$. 26 $5^x\cdot 6^{2x-1}=30$ 27 $7^x\cdot 5^{x+3}=27^{x-1}$

Find as accurately as the Tables permit the value of the following expressions, giving the results in standard form [Art 406, Ex. 1, Note]

28 $11^{\frac{1}{6}}$. 29. $13^{\frac{1}{5}}$ 30. $17^3\times 29^2$ 31 $(\cdot089)^{\frac{4}{5}}$. 32. $(2\cdot301)^{\frac{1}{5}}$.

33 $703\times\sqrt{\cdot52086}$ 34. $(5\cdot398)^2\div(\cdot079)^3$. 35 $(25\cdot06)^{1\cdot23}$

36. $\dfrac{52\cdot45\times 378\cdot4\times\cdot020857}{87\cdot32\times\cdot58443}$ 37. $\dfrac{38\cdot54\times\sqrt[3]{\cdot035776}}{\sqrt{\cdot5164}\times 431\cdot04}$.

38. Find, to the nearest integer, the values of

$$\text{(i)}\ \frac{(330\times\frac{1}{47})^4}{\sqrt[3]{22\times 70}};\qquad \text{(ii)}\ \sqrt{\frac{678\times 9\cdot01}{\cdot0234}}.$$

407 It is sometimes necessary to transform logarithms from one base to another.

Suppose for example that the logarithms of all numbers to base a are known and tabulated and it is required to find the logarithms to base b

Let N be any number whose logarithm to base b is required.

Let $x=\log_b N$, so that $b^x=N$,

$$\log_a(b^x)=\log_a N,$$

that is,

$$x\log_a b=\log_a N\,;$$

$$x=\frac{1}{\log_a b}\times\log_a N,$$

or

$$\log_b N=\frac{1}{\log_a b}\times\log_a N. \qquad\qquad (1)$$

Now since N and b are given, $\log_a N$ and $\log_a b$ are known from the Tables, and thus $\log_b N$ may be found.

Hence it appears that to transform logarithms from base a to base b we have only to multiply them all by $\frac{1}{\log_a b}$; this is a constant quantity and is given by the Tables, it is known as the *modulus*.

COR. If in equation (1) we put a for N, we obtain

$$\log_b a=\frac{1}{\log_a b}\times\log_a a=\frac{1}{\log_a b};$$

$$\therefore \log_b a\times\log_a b=1.$$

Applications of Logarithms

408 If a sum of money represented by £P is put out at compound interest at r per cent, the amount at the end of n years is given by the formula

$$A = P\left(1+\frac{r}{100}\right)^n$$

If R is the *amount* of £1 in 1 year, $R = 1 + \frac{r}{100}$

Hence $$A = PR^n$$

$$\log A = \log P + n \log R$$

Thus any of the four quantities involved in the formula may be found when the other three are known The Tables should be used in all cases where r or n is required, and their use will also be found convenient in finding A or P whenever the number of years is large

EXAMPLE 1 *In how many years at compound interest will £342 amount to £1000 at 3 p c per annum?*

Let n denote the number of years, then

$$1000 = 342(1.03)^n$$

Hence $$\log 1000 = \log 342 + n \log 1.03,$$

or
$$n = \frac{\log 1000 - \log 342}{\log 1.03} = \frac{.4660}{.0128} = 36.4$$

$$\begin{array}{r} \log 1000 = 3.0000 \\ \log 342 = \underline{2.5340} \\ 0.4660 \end{array}$$

Thus the required time is about $36\frac{1}{2}$ years

EXAMPLE 2 *If a water pipe is* L *yards long,* d *inches in diameter, and one end is* H *feet higher than the other, then* $\sqrt{(3d)^5 \times H \div L}$ *gallons of water will flow through the pipe in a minute Use this formula to find how many gallons per minute will flow through a pipe a mile long,* $4\frac{1}{4}$ *inches in diameter, one end being* 38 *feet higher than the other*

Let n be the number of gallons required, then substituting $d = 4.25$, $H = 38$, $L = 1760$ in the formula, we have

$$n = \sqrt{(12.75)^5 \times 38 \div 1760}$$

$$\log n = \tfrac{1}{2}(5 \log 12.75 + \log 38 - \log 1760)$$

$$= 1.9309,$$

$$x = 85.29 \text{ from the Tables}$$

$$\begin{array}{r} 5 \log 12.75 = 5.5275 \\ \log 38 = \underline{1.5798} \\ 7.1073 \\ \log 1760 = \underline{3.2455} \\ 2\,\boxed{3.8618} \\ 1.9309 \end{array}$$

Thus the pipe supplies a little more than 85 gallons per minute

EXAMPLES XXXII d

1. Shew that the logarithm of any number to base a is found by dividing its *common logarithm* by $\log_{10}a$ Hence find $\log_2 6$, $\log_2 15$, $\log_2 20$ to two decimal figures and check the results roughly by the graph on page 347

2. Find the multiplier which will convert logarithms to base a into logarithms to base a^2 Hence find $\log_{100}27\,3$, $\log_{100}4\,068$ from the Tables

3. If a, b, c are in G P prove that $\log_a N$, $\log_b N$, $\log_c N$ are in H.P

4 Find, to the nearest pound, the amount at compound interest of £350 in 25 years at 3 % per annum

5 Find, to the nearest pound, the sum to which £1000 will amount in 40 years at 4 % compound interest, payable half-yearly

6. How many pounds must be put out at 4 % compound interest so as to amount to £1000 in 17 years?

7. In how many years will £1130 amount to £3000 at 5 % compound interest?

8. Given 1 metre $=3\,2808$ feet, find to the nearest hundredth the number of square feet in a square metre

9. One gallon of water weighs 10 lbs One litre of water weighs 1 kilogram One kilogram $=2\,2046$ lbs *nearly* Find the equivalent of 1 gallon in litres

10 Find, to the nearest integer, the 60^{th} term of a G P of which the first term is 5 and the seventh is 8

11 If the population of a city was 465,000 on Jan 1, 1905, and 527,000 on Jan 1, 1910, find what it was on Jan 1, 1908, assuming that the rate of increase is uniform from year to year

12. In a certain research the value of $\dfrac{bRt}{v-b}$ was required, find its value when $b=1\,53$, $R=2\,835$, $t=532$, $v=10\,07$

13. Knowing the number of pounds in a cubic inch of a substance, the number of kilograms in a cubic centimetre can be found by multiplying by $\dfrac{0{\cdot}4536}{(2\,54)^3}$ Express this multiplier as a decimal to 3 places If steel weighs 488 lbs per cubic foot, how many kilograms per cubic centimetre does it weigh?

14. In the formula $V=\frac{4}{3}\pi r^3$, which gives the volume of a sphere whose radius is r, find, as accurately as possible,

(i) V, in cubic centimetres, when $r=27\,3$ cm ,

(ii) r, in inches, when $V=1$ cubic foot

15. A cubical block of metal, each edge of which is 23 8 cm , is melted down into a sphere Find the diameter of the sphere as accurately as possible

16 Find the weight, to the nearest kilogram, of an iron girder which is 5·4 m long, 0·36 m wide, and 0·22 m thick, having given that a cubic centimetre of iron weighs 7·76 grams

17 Steel wire is made to bear a strain of 215,000 lbs per square inch of section Find, to the nearest cwt, the weight that can be suspended from such a steel wire of diameter 0·104 of an inch

The area of the section is $0·785 \times d^2$, where d is the diameter

18 The pressure of water on a given area at a given depth may be found from the formula

$$P = 0·4335 \times H \times A,$$

where P denotes the pressure in *pounds*, H the depth in *feet*, and A the area in *square inches*

Find, to the nearest pound, the pressure on the circular end of a cylinder, 9 inches in diameter, submerged at a depth of 10 feet, having given that the area of a circle, d inches in diameter, contains $d^2 \times 0·7854$ square inches

19 The gas service pipe to a house 75 feet from the main is $\frac{7}{8}$ in in diameter, for how many burners, each taking 5 cubic feet of gas per hour, will this serve? [The number of cubic feet per hour delivered by a pipe on that main is $1000\sqrt{\dfrac{d^5}{0·45L}}$, where d is the diameter of the pipe in inches, and L is the length of the pipe in yards]

20 A cask is 4 ft 6 in deep, its greatest diameter is 2 ft 3 in., and the diameter of each end is 2 ft Calculate the number of gallons which it will hold [The volume of a cask of depth h feet, of which the greatest and least diameters are D and d feet, is approximately $0·7854 \times h \times \left(\dfrac{D+d}{2}\right)^2$ cubic feet, and a cubic foot is 6·23 gallons]

21 The velocity of water in a rectangular mill stream whose breadth is A feet, and depth D feet, is $\dfrac{520 \times R}{0·516+\sqrt{R}}$ feet per minute, when $R = \dfrac{AD}{2D+A}$ Calculate the amount of water which would pass along such a stream when A=4 and D=3

22 A projectile, whose weight is w pounds and diameter d inches, strikes a wrought iron plate when moving at the rate of v feet per second Assuming that the penetration p (in inches) is given by the formula

$$p = \frac{v}{608·3}\sqrt{\frac{w}{d}} - 0·14d,$$

find the penetration when $d=13·5$, $w=1250$ and $v=2016$

No	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	6	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

$\log \pi = 0{\cdot}49715$ $\log \frac{\pi}{4} = \bar{1}{\cdot}8951$.

No	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

$\text{Log}\ \frac{4\pi}{3} = 0.6221$ $\text{Log}\ \frac{\pi}{6} = \bar{1}.7190$

Log	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Log	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

MISCELLANEOUS EXAMPLES VII.

Exercises for Revision

A

1 Simplify (i) $\left(\dfrac{a^{-\frac{4}{3}}b^{-\frac{1}{3}}}{a^{2}b^{\frac{1}{2}}}\right)^{\frac{3}{5}}-\left(\dfrac{a^{-\frac{2}{3}}b^{\frac{3}{4}}}{a^{\frac{1}{2}}b^{\frac{2}{3}}}\right)^{6}$, (ii) $\dfrac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}$

2 Find the 15th term of the series $-2,\ -1\frac{5}{6},\ -\frac{5}{3},\ \ldots$ Which term of the series is zero?

3. Evaluate by means of logarithms

(i) $\sqrt[5]{19\cdot63}$, (ii) $(6\cdot297)^2-(0\cdot089)^3$

4. Solve the equations

(i) $\dfrac{ac}{bx}-\dfrac{a}{2}=\dfrac{bc}{ax}+\dfrac{b}{2}$, (ii) $(2x+1)^2+4(2x+1)=5$

5 A man's income is £1500 a year On this he pays £63 10*s* as income tax, partly at the rate of 1*s* in the £, and partly at 9*d* Find the amount on which he pays at the two rates

6 Draw on the same axes the graphs of x^3 and $x+1\cdot5$ from $x=-2$ to $x=2$

Thence solve the equation $2x^3-2x-3=0$ graphically

B

7. If $\dfrac{x}{2}+y=9$, and $\dfrac{x}{6}=y-1$, find the value of $x-y$

8. Express in the simplest form

(i) $2\sqrt{63}-3\sqrt{\frac{1}{5}}-\sqrt{\frac{9}{7}}+\frac{1}{5}\sqrt{45}$,

(ii) $(2\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})(3\sqrt{3}-\sqrt{2})$

9 Solve the equations

(i) $\sqrt{x+5}+\sqrt{x-4}=\sqrt{4x+1}$, (ii) $(x^2+2x)-\dfrac{8}{x(x+2)}=7$

10. Sum to 10 terms each of the series

(i) $5+10+15+20+\ldots$, (ii) $5-10+20-40+\ldots$

11. The perimeter of a rectangle is 302 yards and its area is 5460 square yards find the diameter of the circumscribing circle

12 The difference between two numbers is 3 and the difference between their arithmetic and harmonic means is $\frac{3}{4}$ Find the numbers

C

13 Find the factors of

(i) $(a^2+1)y^2-y^4-a^2$, (ii) $12x^2+19ax-18a^2$.

14. Simplify $\dfrac{a(b+c)}{(a-b)(a-c)}+\dfrac{b(c+a)}{(b-c)(b-a)}+\dfrac{c(a+b)}{(c-a)(c-b)}$.

15. Solve the equations

$$\text{(i)}\ (x^2-3x-5)(x^2-3x+3)+7=0,$$

$$\text{(ii)}\ \sqrt{4x^2+2x+7}=12x^2+6x-119$$

16. Simplify $\dfrac{\sqrt[3]{x^4}+2x^{\frac{2}{3}}-3}{3x^{\frac{4}{3}}+2\sqrt[3]{x^2}-5}$, and express it in a form free from fractional indices

17. Find the sum of n terms of the progression

$$3+2\tfrac{1}{2}+2\tfrac{1}{12}+\ldots$$

18 Write down the sum of an infinite G P, the first term being a and the common ratio r If a and b are proper fractions, and if $a>b$, shew that if

$$(1+a+a^2+\ldots)^2=(1+b+b^2+\ldots),$$

then $$1+\frac{b}{a}+\frac{b^2}{a^2}+\ldots=-1-a-a^2-\ldots$$

and $$1+\frac{b}{2a}+\frac{b^2}{4a^2}+\frac{b^3}{8a^3}+\ldots=\frac{2}{a},$$

each series being continued to infinity

D

19 If $\dfrac{pc-a}{c}=\dfrac{a-(p+1)b+pc}{c-b}$, prove that $\dfrac{a}{c}=\dfrac{a-b}{b-c}$

20 Express $n^2(n+1)^2-6n(n+1)(2n+1)+36n(n+1)$ as a continued product, and find the simplest form of

$$6x^2+13xy+6y^2+y-x$$

when $2x+3y=1$

21. Insert 7 arithmetic means between -3 and 17

22 Enunciate the Remainder Theorem If 3 is one root of

$$x^3-49x+a=0,$$

find the other roots

23. Sum to $2n$ terms each of the series:

(i) $1-3+9-27+\ldots$; (ii) $1-3+5-7+\ldots$;

and write down the last term of each series

24. A has an old motor-car and travels 16 miles an hour, stopping 6 minutes at the end of each hour to overhaul his car. B on a new car travels continuously at 32 miles an hour, and starts 1 hour after A. Find graphically when B overtakes A.

E

25. If $xy=3(x-3)$ and $yz=3(y-3)$, prove that $xz=3(z-3)$.

26. Find, without unnecessary calculation, the coefficient of x^3 in the product of

$$5x^3+2x^2-7x-8 \text{ and } 2x^3-4x^2-10x+6$$

27. Find the H C F of x^4+x^2+1 and $x^{10}+x^5+1$.

28. Reduce to their simplest forms

$$\text{(i) } \frac{2^{n+4}-2\times 2^n}{2^{n+2}\times 4}, \quad \text{(ii) } \frac{(a+b)^{\frac{3}{2}}}{(a-b)^{\frac{1}{2}}}\times\sqrt{a^2-b^2}, \quad \text{(iii) } \frac{(a-b)^{\frac{1}{3}}\sqrt[3]{a^2+2ab+b^2}}{\sqrt[3]{a^2-b^2}\times(a+b)^{-\frac{2}{3}}}$$

29. If a, b, c, d are four consecutive terms in A P, shew that $a^2-3b^2+3c^2-d^2=0$. [See Art 320, Ex 2, Note.]

30. A tourist starts for a walk at noon and allows himself seven hours to reach a certain town. After walking two thirds of the distance at $3\frac{1}{2}$ miles an hour, he rests for an hour and a quarter, and then finds that he must walk 4 miles an hour if he is to reach the town by 7 o'clock. What was the length of the walk?

F

31. The expression $x^4+ax^3+5x^2+bx+6$ when divided by $x-2$ leaves the remainder 16, and when divided by $x+1$ leaves the remainder 10. Find the values of a and b.

32. With £x a man buys equal amounts of p per cent. stock at a, and q per cent. stock at b; what is the income so derived?

33. If the n^{th} term of a series is $3n+2$, find the sum of n terms.

34. Solve the equations

$$\text{(i) } \sqrt{\frac{x}{a}}+\sqrt{\frac{a}{x}}=\sqrt{\frac{x}{b}}+\sqrt{\frac{b}{x}}, \quad \text{(ii) } \left.\begin{array}{r} 7y^2+5xy=468 \\ 5x+4y=-3 \end{array}\right\}$$

35. Given $\log_{10}2=x$, $\log_{10}3=y$, find the logarithms of $\frac{2}{9}$, 75, 0·0015 in terms of x and y.

36. Shew by drawing graphs that the values of x and y which satisfy the equations

$$3x+y=2, \quad 7x+4y=3,$$

also satisfy the equations

$$2x+5y+3=0, \quad 5x+2y-3=0$$

G

37 If $2y-\frac{x-3}{5}=4$, and $3x+\frac{y-2}{3}=9$, find the value of $\frac{x+y}{x-y}$.

38 Reduce $\frac{7-\sqrt{7}}{(5+\sqrt{7})(\sqrt{7}-2)}$ to simplest form with rational denominator, and calculate its value given $\sqrt{7}=2.646$.

39 Solve the equations

(i) $(a^2-b^2)(1-x^2)=4abx$, (ii) $\frac{x}{x-a}-\frac{a-b}{x}=\frac{x+a+b}{x+a}$.

40 Three numbers are in H P If their sum is 37, and the sum of their reciprocals is $\frac{1}{4}$, what are they?

41 A line 12 inches long is divided into two parts so that the square on one part is equal to three times the rectangle contained by the whole line and the other part find to the nearest hundredth of an inch the lengths of the two segments

Explain the negative solution

42 Find the maximum value of $5-2x-3x^2$ when x is real

H

43 Find, from the Tables, the values of (i) $\sqrt[5]{29}$, (ii) $(987.6)^{\frac{2}{5}}$

44 If for all values of n the sum of n terms of an A P is n^2+6n, find the n^{th} term

45 What is the price of gold per ounce if a rise of $3d$ reduces by 5 the number of ounces that can be bought for £57.57?

46 Find the square root of

(i) $2a-x+2\sqrt{a^2-ax-6x^2}$, (ii) $a+b+\frac{4}{a+b}-4$

47 Solve the equations

(i) $\frac{x+a}{3x-1}+\frac{a-x}{1+3x}=\frac{3a+1}{9x^2-1}$, (ii) $2\sqrt{x}+\frac{1}{2}=\sqrt{5x+1}$

48 A hare runs ten times as fast as a tortoise which it is pursuing The hare at A is 100 yards behind the tortoise at B, when the hare is at B the tortoise has advanced to C, and when the hare is at C the tortoise has advanced to D, and so on an endless number of times Find by adding up the pieces AB, BC, CD, etc, how far the hare will have gone when it overtakes the tortoise Find the answer also by solving an equation, and shew that the two results agree

CHAPTER XXXIII

RATIO AND PROPORTION.

409 THE ratio of one quantity to another *of the same kind* is he multiple or fraction that the first is of the second *when both are xpressed in terms of the same unit*

Thus the ratio of £1 5s to £2 13s $= \frac{25}{53}$,
and the ratio of 1 ft 3 in to 1 yd. $= \frac{15}{36} = \frac{5}{12}$

Every ratio is an *abstract number*, whole or fractional, since it is the quotient of one number by another

410 The ratio of two quantities a and b is usually written $a : b$ Thus $\frac{a}{b}$ and $a : b$ have the same meaning The quantity a is called the first term or antecedent of the ratio, and b the second term or consequent.

411 The properties of ratios are the same as those of fractions Thus since $\frac{a}{b} = \frac{ma}{mb}$, a ratio remains unaltered in value when each of its terms is multiplied or divided by the same quantity

Again, to compare two ratios $a : b$ and $c : d$, we have only to express the equivalent fractions with a common denominator, and compare the numerators

412 Two or more ratios are said to be **compounded** when the antecedents are multiplied together to form a new antecedent, and the consequents to form a new consequent That is, the compounded ratio is the product of the fractions which represent the ratios

Thus the ratio compounded of $2 : 3$, $9a : 16b$, $4ab : 3c^2$

$$= \frac{2}{3} \times \frac{9a}{16b} \times \frac{4ab}{3c^2} = \frac{a^2}{2c^2}, \text{ or } a^2 : 2c^2$$

413 When a ratio is compounded with itself the result is called its **duplicate ratio.**

Thus $a^2 : b^2$ is the duplicate ratio of $a : b$

Similarly $a^3 : b^3$ is the triplicate ratio of $a : b$

414 Two quantities whose ratio cannot be expressed as the ratio of two integers are said to be *incommensurable*.

For example, since no exact numerical equivalent can be found for $\sqrt{2}$, the ratio of $\sqrt{2}$ to 1 cannot be expressed as the ratio of two integers In other words $\sqrt{2}$ cannot be expressed as an exact *multiple of unity* Hence the origin of the terms "irrational," "incommensurable" as applied to all surd quantities

EXAMPLE 1 *If* $\dfrac{2(x+y)}{y}=\dfrac{7x-8y}{x-y}$, *find the ratio of* x *to* y

We have $$2(x^2-y^2)=y(7x-8y),$$

that is, $$2x^2-7xy+6y^2=0,$$

or $$2\left(\frac{x}{y}\right)^2-7\left(\frac{x}{y}\right)+6=0; \quad \text{whence } \frac{x}{y}=\frac{3}{2}, \text{ or } 2$$

Thus the required ratio is $3 : 2$, or $2 : 1$

EXAMPLE 2 *A's age is to B's in the ratio of* $5 : 8$ *In* 9 *years' time the ratio of their ages will be* $8 : 11$. *Find their ages*

Let A's and B's ages be $5x$ years and $8x$ years respectively,

then $$\frac{5x+9}{8x+9}=\frac{8}{11}, \quad \text{whence } x=3$$

Hence the ages are 15 years and 24 years

EXAMPLES XXXIII a

1. Express the following ratios in their simplest fractional form

(i) £2 4s : £3 17s , (ii) 510 metres : 1 105 kilometres;

(iii) $46x^3y^4z^5 : 69x^2y^3z^4$, (iv) $5a^3b+10a^2b^2 : 3a^2b^2+6ab^3$.

2 Find the ratio compounded of the three ratios

$$3a^2b : 4b^3c, \quad 2c^2 : 8a^3, \quad 16b^2x : 6ac$$

3 Divide 13 tons 10 cwt into parts having the ratio of $7 : 11$.

4 Find the ratio compounded of the duplicate ratio of $3 : 7$ and the ratio of $35 : 27$

5 If $a : b=3 : 4$, find the values of

(i) $2a-b : 3a-2b$, (ii) $a^2-ab-2b^2 : a^2-4b^2$

Find the ratio of $x : y$ from the following equations

6. $4(2x-y)=3(2y+x)$ 7. $3x^2-7xy+2y^2=0$

8. $\dfrac{x-3y}{2y}=\dfrac{6x-5y}{3x}$. 9. $\dfrac{2ax+by}{2ax-by}=\dfrac{3by}{ax}$

10. What number must be added to each term of the ratio $5 : 7$ that it may become $10 : 11$?

11. If $a \;\; b=10 \;\; 3$, find the value of $2a-5b \;\; a-3b$

12. Two men's ages are in the ratio of 2 3 In 7 years' time they will be in the ratio of 3 4 Find their ages

13. Find two numbers in the ratio of 4 7, and differing by 39

14. If $m \;\; n$ is the duplicate ratio of $m+x \;\; n+x$, shew that $x^2=mn$

15 If $a-x \;\; b-x$ is the duplicate ratio of $a \;\; b$, shew that $2x$ is the harmonic mean between a and b

16. If a and b are unequal, and $ab(c^2+d^2)=b^2c^2+a^2d^2$, shew that the ratio of a to b is the duplicate ratio of c to d

415 The ratio $a \;\; b$ is said to be of *greater inequality* if $a>b$,

"	"	*equality*	if $a=b$,
"	"	*less inequality*	if $a<b$

416 *A ratio of greater inequality is diminished, and a ratio of less inequality is increased by adding the same positive quantity to both its terms*

Let $\dfrac{a}{b}$ be the ratio, and let $\dfrac{a+x}{b+x}$ be a new ratio formed by adding x to both its terms

Now
$$\frac{a}{b}-\frac{a+x}{b+x}=\frac{ax-bx}{b(b+x)}=\frac{x(a-b)}{b(b+x)},$$

and $a-b$ is positive or negative according as a is greater or less than b

Hence if $a>b$, $$\frac{a}{b}>\frac{a+x}{b+x},$$

and if $a<b$, $$\frac{a}{b}<\frac{a+x}{b+x},$$

which proves the proposition

Thus if to each term of the ratio 7 5 we add 5, the new ratio becomes 12 10, or 6 5, which is clearly less than 7 5

Again if to each term of the ratio 3 4 we add 2, the new ratio becomes 5 6 which is greater than 3 4

Both cases of the above theorem may be included in a single statement which is easily remembered

Any ratio (or fraction) is made more nearly equal to unity by adding the same positive quantity to each of its terms

417 The following theorem may be proved as in the last article

A ratio of greater inequality is increased, and a ratio of less inequality is diminished by taking the same positive quantity from both its terms.

418 *If the ratios* $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$ *are all equal, then each of these ratios is equal to* $\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}$, *for all values of* p, q, r, ... n.

Let k stand for the value of the equal ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$;

then $$a=bk, \quad c=dk, \quad e=fk, \ldots;$$

whence $$pa^n=pb^nk^n, \quad qc^n=qd^nk^n, \quad re^n=rf^nk^n, \ldots;$$

$$\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}=\frac{pb^nk^n+qd^nk^n+rf^nk^n+\ldots}{pb^n+qd^n+rf^n+\ldots}$$
$$=k^n,$$

$$\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$$

By giving different values to $p, q, r, \ldots n$ many particular cases of this general proposition may be deduced.

Thus, when $n=1$, and $p=q=r=\ldots$,

$$\text{if } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots, \text{ each of these ratios}=\frac{a+c+e+\ldots}{b+d+f+\ldots},$$

a result which may be quoted verbally as follows:

When a series of fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.

EXAMPLE 1. *If* $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, *prove that* $\sqrt{\frac{2a^4b^2+3a^2e^2-5e^4f}{2b^6+3b^2f^2-5f^5}}=\frac{ac}{bd}$.

Let $k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$; then $a=bk, c=dk, e=fk$;

$$\text{the first side}=\sqrt{\frac{2k^4b^6+3k^4b^2f^2-5k^4f^5}{2b^6+3b^2f^2-5f^5}}=\sqrt{k^4}=k^2$$
$$=\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

EXAMPLE 2. *If* $$\frac{x}{cm-bn}=\frac{y}{cl-an}=\frac{z}{bl-am},$$

shew that $$ax-by+cz=0.$$

Denote each of the given ratios by k; then

$$x=(cm-bn)k, \quad y=(cl-an)k, \quad z=(bl-am)k;$$
$$\therefore ax-by+cz=k\{a(cm-bn)-b(cl-an)+c(bl-am)\}$$
$$=k\times 0$$
$$=0.$$

419 *The fraction* $\dfrac{a+c+e+\dots}{b+d+f+\dots}$ *lies between the greatest and least of the fractions* $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \dots$ *when these fractions are unequal and all the denominators are positive*

Suppose $\dfrac{a}{b}$ is the greatest of the fractions, and let $\dfrac{a}{b}=k$

then $\dfrac{c}{d}<k, \dfrac{e}{f}<k, \dots$

$$a=bk, \quad c<dk, \quad e<fk, \quad \dots;$$

$$a+c+e+\dots<bk+dk+fk+\dots$$

$$<k(b+d+f+\dots),$$

$$\frac{a+c+e+\dots}{b+d+f+\dots}<k$$

$$<\frac{a}{b}$$

Similarly it may be shewn that $\dfrac{a+c+e+\dots}{b+d+f+\dots}$ is greater than the least of the fractions $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \dots$

As a particular case consider the two unequal fractions $\dfrac{a}{b}, \dfrac{x}{y}$

Then $\dfrac{a+x}{b+y}$ lies between $\dfrac{a}{b}$ and $\dfrac{x}{y}$

If $y=x$, $\dfrac{a+x}{b+x}$ lies between $\dfrac{a}{b}$ and 1, which is the result obtained in Art 416

[*Examples* XXXIII b 1–12, *page* 378, *may be taken here*]

420 *To find the ratios of* x y z *from the equations*

$$a_1x+b_1y+c_1z=0, \qquad (1)$$

$$a_2x+b_2y+c_2z=0 \qquad (2)$$

By writing these equations in the form

$$a_1\left(\frac{x}{z}\right)+b_1\left(\frac{y}{z}\right)+c_1=0,$$

$$a_2\left(\frac{x}{z}\right)+b_2\left(\frac{y}{z}\right)+c_2=0,$$

we can solve for $\dfrac{x}{z}, \dfrac{y}{z}$, and obtain

$$\frac{x}{z}=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \qquad \frac{y}{z}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1},$$

that is,

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1} \qquad (3)$$

Hence from two equations of the form (1) and (2) we may write down the ratios $x : y : z$ in terms of the coefficients by the following rule:

Write down the coefficients of x, y, z in order, beginning with those of y; repeat these last, as in the scheme below.

$$\begin{tikzcd} b_1 \arrow[dr] & c_1 \arrow[dr] & a_1 \arrow[dr] & b_1 \\ b_2 \arrow[ur] & c_2 \arrow[ur] & a_2 \arrow[ur] & b_2 \end{tikzcd}$$

Multiply the coefficients across in the way indicated by the arrows, remembering that any product formed by *descending* is *positive*, and any formed by *ascending* is *negative*. Then the three results

$$b_1c_2 - b_2c_1, \quad c_1a_2 - c_2a_1, \quad a_1b_2 - a_2b_1$$

are the denominators for x, y, z respectively.

This is called the Rule of Cross Multiplication.

421 If we put $z=1$, in the equations of Art. 420, we have

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0;$$

and (3) becomes $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1},$$

or $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

Hence any two linear simultaneous equations in two unknowns may be solved by the rule of cross multiplication.

Example. *Find the ratios of* $x : y : z$ *from the equations*

$$4x = 7y + 5z, \quad 2x + y = z.$$

By transposition, we have $4x - 7y - 5z = 0$,

$$2x + y - z = 0.$$

Write down the coefficients according to the rule, thus

$$\begin{array}{cccc} -7 & -5 & 4 & -7 \\ 1 & -1 & 2 & 1; \end{array}$$

whence we obtain the products

$$(-7)\times(-1) - (-5)\times 1, \quad (-5)\times 2 - (-1)\times 4, \quad 4\times 1 - (-7)\times 2,$$

or $$12, \quad -6, \quad 18;$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{18};$$

that is, $$\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}.$$

EXAMPLES XXXIII. b.

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each of these ratios is equal to

(i) $\frac{5a-7c+3e}{5b-7d+3f}$; (ii) $\sqrt{\frac{4a^2-5ace+6e^2f}{4b^2-5bde+6f^3}}$

2. If $\frac{p}{q}=\frac{r}{s}=\frac{t}{u}$, prove that

(i) $\frac{p^2-pr+t^2}{q^2-qs+u^2}=\frac{pt}{qu}$, (ii) $\frac{r^3-p^2tu}{s^3-q^2u^2}=\frac{prt}{qsu}$

3. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, prove that

(i) $x+y+z=0$, (ii) $(b+c)x+(c+a)y+(a+b)z=0$

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each ratio is equal to

(i) $\sqrt[3]{\frac{4ac^2-3ce^2+2ace}{4bd^2-3cf^2+2bdf}}$, (ii) $\sqrt[5]{\frac{6a^2c^2e-c^4ef+7ac^5}{6b^2d^2f-d^4f^2+7ad^5}}$.

5. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, prove that $\sqrt{5x^2+8y^2+7z^2}=5y$

6. The sides of a triangle are as 1 : $1\frac{1}{2}$: $1\frac{3}{4}$, and the perimeter is 221 yards : find the sides

7. If $\frac{x}{lm-n^2}=\frac{y}{mn-l^2}=\frac{z}{nl-m^2}$, shew that

$$lx+my+nz=0, \text{ and } mx+ny+lz=0$$

8. If $\frac{p}{bz-cy}=\frac{-q}{cx+az}=\frac{-r}{ay+bx}$, shew that

$$ap+bq-cr=0, \text{ and } xp-yq+zr=0$$

9. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that the square root of

$$\frac{a^6b-2c^5e+3a^4c^3e^2}{b^7-2d^5f+3b^4cd^2e^3} \text{ is equal to } \frac{ace}{bdf}$$

10. Prove that the ratio $la+mc+ne : lb+md+nf$ is equal to each of the ratios $a : b$, $c : d$, $e : f$, if these are all equal, and that it will be intermediate in value between the greatest and least of these ratios if they are not all equal

11. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, then will each of these fractions be equal to $\frac{x}{y}$, unless $b+c=0$

12. If $\frac{2x-3y}{3z+y}=\frac{z-y}{z-x}=\frac{x+3z}{2y-3x}$, prove that each of these ratios is equal to $\frac{x}{y}$; hence shew that either $x=y$, or $z=x+y$

Find the ratios of $x : y : z$ from the following equations

13. $ax+by+cz=0$, $lx+my+nz=0$ 14. $4x-2y-7z=0$, $x+y-4z=0$ 15. $3x-2y=3z$, $10y-6z=x$

Solve the following equations by cross multiplication

16 $2x+y-10=0$, $7x+8y-53=0$ 17 $5x-3y=1$, $x+2y=12$ 18 $px-qy=r$, $rx-py=q$.

[*In Examples* 19, 20, *from the first two equations obtain* $\frac{x}{}=\frac{y}{}=\frac{z}{}$, *put each ratio equal to* k, *and find* k *by substituting in the third equation*]

19 $2x+3y-7z=0$, $5x-2y-8z=0$, $3x^2-4y^2+z^2=9$

20 $3x-4y+7z=0$, $2x-y-2z=0$, $3x^3-y^3+z^3=18$

21 If $ax+cy+bz=0$, $cx+by+az=0$, $bx+ay+cz=0$,

shew that $a^3+b^3+c^3=3abc$.

422 Proportion A statement expressing the equality of two ratios is called a **proportion**, the four quantities compared are the **terms** of the proportion, the first and last terms are called the **extremes**, and second and third are called the **means**

Thus a, b, c, d are in proportion if $\frac{a}{b}=\frac{c}{d}$, or $a : b=c : d$

The proportion $a : b=c : d$ is sometimes written $a : b :: c : d$

Obviously a and b must be of one kind also c and d must be of one kind

423 If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$

Hence, when four numbers are in proportion, *the product of the extremes is equal to the product of the means* Hence when any three terms of a proportion are given, the fourth may be found

Conversely, if there are any four quantities, a, b, c, d, such that $ad=bc$, then a, b, c, d are proportionals, a and d being the extremes, b and c the means, or vice versa

424 Quantities are said to be **in continued proportion** when the first is to the second, as the second is to the third, as the third is to the fourth, and so on

Thus $a, b, c, d, \ldots$ are in continued proportion when $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\ldots$

425 If *three* quantities a, b, c, of the same kind, are in continued proportion, then $a : b=b : c$, whence $b^2=ac$ [Art 423]

In this case b is said to be a **mean proportional** between a and c; and c is said to be a **third proportional** to a and b

426 If $\dfrac{a}{x}=\dfrac{x}{y}=\dfrac{y}{b}$, the quantities x and y are said to be *two mean proportionals* between a and b

The values of x and y can be found in terms of a and b, for since we have three equal fractions, their product is equal to the cube of each.

Thus
$$\frac{a}{b}=\frac{a^3}{x^3}=\frac{y^3}{b^3}$$
$$x^3=a^2b, \quad\text{and}\quad y^3=b^2a,$$
that is,
$$x=\sqrt[3]{a^2b}, \quad\text{and}\quad y=\sqrt[3]{b^2a}$$

427 If four quantities a, b, c, d form a proportion, many other proportions may be deduced from them Some of these which are given below are frequently referred to by the annexed Latin names borrowed from geometry

I. *If* $a : b = c : d$, *then* $b : a = d : c$ [*Invertendo*]

For $\dfrac{a}{b}=\dfrac{c}{d}$, therefore $1 \div \dfrac{a}{b}=1 \div \dfrac{c}{d}$, that is, $\dfrac{b}{a}=\dfrac{d}{c}$,

hence
$$b : a = d : c$$

II *If* $a : b = c : d$, *then* $a : c = b : d$ [*Alternando*]

For $ad=bc$, therefore $\dfrac{ad}{cd}=\dfrac{bc}{cd}$, that is, $\dfrac{a}{c}=\dfrac{b}{d}$,

hence
$$a : c = b : d$$

NOTE This alternation is only admissible when all the four quantities a, b, c, d are of the same kind

III *If* $a : b = c : d$, *then* $a+b : b = c+d : d$ [*Componendo*]

For $\dfrac{a}{b}=\dfrac{c}{d}$, therefore $\dfrac{a}{b}+1=\dfrac{c}{d}+1$, that is, $\dfrac{a+b}{b}=\dfrac{c+d}{d}$,

hence
$$a+b : b = c+d : d$$

IV *If* $a : b = c : d$, *then* $a-b : b = c-d : d$ [*Dividendo*]

For $\dfrac{a}{b}=\dfrac{c}{d}$, therefore $\dfrac{a}{b}-1=\dfrac{c}{d}-1$, that is, $\dfrac{a-b}{b}=\dfrac{c-d}{d}$,

hence
$$a-b : b = c-d : d$$

V *If* $a : b = c : d$, *then* $a+b : a-b = c+d : c-d$

For by III, $\dfrac{a+b}{b}=\dfrac{c+d}{d}$, and by IV, $\dfrac{a-b}{b}=\dfrac{c-d}{d}$,

hence by division
$$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$
that is,
$$a+b : a-b = c+d : c-d$$

This result is sometimes referred to as *Componendo et Dividendo.*

428 Miscellaneous Examples on Ratio and Proportion A large number of examples are readily solved by the 'k method' of Art 418 Others depend upon suitable applications of the results proved in Art 427

EXAMPLE 1 *Find a third proportional to* $3(a+b)^2$ *and* $6(a^2-b^2)$

Let x be the required third proportional, then

$$3(a+b)^2 : 6(a^2-b^2)=6(a^2-b^2) : x$$

$$\therefore \quad 3x(a+b)^2=36(a^2-b^2)^2,$$

whence $$x=12(a-b)^2$$

EXAMPLE 2 *If* $a : b=x : y$, *shew that*

$$pa^2+qax+rx^2 : pb^2+qby+ry^2=a^2+x^2 : b^2+y^2$$

Let $k=\dfrac{a}{b}=\dfrac{x}{y}$, then $a=bk$, $x=yk$,

$$\therefore \quad \frac{pa^2+qax+rx^2}{pb^2+qby+ry^2}=\frac{pb^2k^2+qbyk^2+ry^2k^2}{pb^2+qby+ry^2}=k^2$$

Again, $$\frac{a^2+x^2}{b^2+y^2}=\frac{b^2k^2+y^2k^2}{b^2+y^2}=k^2$$

$$\therefore \quad \frac{pa^2+qax+rx^2}{pb^2+qby+ry^2}=\frac{a^2+x^2}{b^2+y^2}$$

EXAMPLE 3 *Solve the equation* $\dfrac{x^2-x+2}{x-2}=\dfrac{4x^2-5x+6}{5x-6}$.

We have $$\frac{x^2}{x-2}=\frac{4x^2}{5x-6} \quad \text{[Componendo, Art 427, III.]}$$

$$\therefore \text{ either } x=0, \text{ or } \frac{1}{x-2}=\frac{4}{5x-6}, \text{ whence } x=-2$$

Thus the roots are $0,\ -2$.

EXAMPLE 4 *If* $b^2+bx+x^2 : a^2+ay+y^2=b^2-bx+x^2 : a^2-ay+y^2$,
prove that either $x : a=b : y$,
or $x : b=y : a$

We have $$\frac{b^2+bx+x^2}{b^2-bx+x^2}=\frac{a^2+ay+y^2}{a^2-ay+y^2} \quad \text{[Alternand]}$$

Also $$\frac{2bx}{2b^2+2x^2}=\frac{2ay}{2a^2+2y^2} \quad \text{[Componendo et dividendo]}$$

$$\therefore \quad bx(a^2+y^2)=ay(b^2+x^2),$$

$$bxa^2+bxy^2-ayb^2-ayx^2=0,$$

$$by(xy-ab)-ax(xy-ab)=0,$$

$$\therefore \quad (by-ax)(xy-ab)=0$$

$\therefore$ either $by-ax=0$, whence $x : b=y : a$,

or $xy-ab=0$; whence $x : a=b : y$

EXAMPLE 5 *If* $\dfrac{2y+2z-x}{a}=\dfrac{2z+2x-y}{b}=\dfrac{2x+2y-z}{c}$,

shew that $$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}$$

Multiply the numerators and denominators of the three given ratios by -1, 2, and 2 respectively. Then, by addition of the new numerators and denominators,

$$\text{each ratio}=\frac{-(2y+2z-x)+2(2z+2x-y)+2(2x+2y-z)}{-a+2b+2c}$$
$$=\frac{9x}{2b+2c-a}.$$

Similarly each of the given ratios may be shewn equal to

$$\frac{9y}{2c+2a-b} \text{ and } \frac{9z}{2a+2b-c},$$

which gives the required result.

EXAMPLES XXXIII. c.

Find a fourth proportional to

1. a^2b, b^2c, c^2a^2. 2. $12x^3, 9ax^2, 8a^3x$. 3. 21·6 ft., 28·8 ft., 7·2 lbs.

Find a mean proportional between

4. $9p^3q, 4q^3$. 5. $27ab^2c^3, 75a^3b^2c$. 6. $2\sqrt{18}, 3\sqrt{128}$.

Find a third proportional to

7. $9y^2, 3xy$. 8. 5·6, 0·84. 9. $5\sqrt{3}, \sqrt{15}$.

Find the missing terms in each of the following proportions:

10. 44 : 15 = 9·24 tons : [ ]. 11. 945 miles : [ ] = £13 6s : £4.

Solve the equations:

12. $2x+1 : x+5 = 6x-7 : 3x+5$. 13. $x : y = 3 : 4 = x+y : 3x+1$.

If a, b, c are three proportionals, shew that

14. $a : a+b = a-b : a-c$. 15. $a^3+b^3 : b^3+c^3 = ab : c^2$.

16. $ma+nb : mb+nc = ma-nb : mb-nc$.

17. $a^2-ab+b^2 : a^2+ab+b^2 = b^2-bc+c^2 : b^2+bc+c^2$.

18. $(ab+bc)^2=(a^2+b^2)(b^2+c^2)$.

19. $(b^2+bc+c^2)(ac-bc+c^2)=b^4+ac^3+c^4$.

20. If $b+c$ is a mean proportional between $a+b$ and $c+a$, shew that
$$b+c : c+a = c-a : a-b.$$

21. If $\dfrac{x}{y}=\dfrac{a}{a+b}$, then $\dfrac{x^2-xy+y^2}{a^2+ab+b^2}=\dfrac{x^2}{a^2}$.

If $a : b = c : d$, prove that

22. $ab+cd : ab-cd = a^2+c^2 : a^2-c^2$.

23. $a : b = \sqrt{2a^2+3c^2} : \sqrt{2b^2+3d^2}$.

24. $\frac{a}{m}+\frac{b}{n} : b = \frac{c}{m}+\frac{d}{n} : d$.

25. $\frac{a}{a-b} : \frac{a+b}{b} = \frac{c}{c-d} : \frac{c+d}{d}$.

26. $\frac{(a-c)b^2}{(b-d)cd} = \frac{a^2-b^2-ab}{c^2-d^2-cd}$.

27. $\frac{ab^3-c^3d}{b^2c^2-c^3d} = \frac{b^3}{ad^2}+1$.

28. If $(a-b-3c-3d)(2a-2b-c+d)=(2a+2b-c-d)(a-b-3c+3d)$, prove that a, b, c, d are proportionals.

29. If a, b, c, d are in continued proportion, prove that

(i) $a : d = a^3+b^3+c^3 : b^3+c^3+d^3$;

(ii) $b+c$ is a mean proportional between $a+b$ and $c+d$.

30. If $12 : x = x : y = y : z = z : 18$, calculate the value of x to two places of decimals, and shew that

$$x^4+y^4+z^4=(x^2+y^2+z^2)(x^2-y^2+z^2).$$

31. If $a+x : a-x$ is the duplicate ratio of $a+b : a-b$, then

$$x-b : a-x = b(a+b) : a(a-b).$$

32. If $a : b = x-2y : y+2x$, then $x : y = a+2b : b-2a$.

33. If a and b are unequal, and $ab(c^2+d^2)=b^2c^2+a^2d^2$, prove that the ratio of $a : b$ is the duplicate ratio of $c : d$.

Solve the following equations as concisely as possible:

34. $\frac{x^2+x-2}{x^2-x+2}=\frac{x+2}{x-2}$.

35. $\frac{3x-7}{x^2-3x+7}=\frac{x+3}{x^2-x-3}$.

36. $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$.

37. $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$.

38. If $\frac{y+z-x}{a}=\frac{z+x-y}{b}=\frac{x+y-z}{c}$, prove that

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

39. If $x+y : 3a-b = y+z : 3b-c = z+x : 3c-a$, prove that

$$x+y+z : a+b+c = ax+by+cz : a^2+b^2+c^2.$$

40. If $\frac{x+y}{2a+b}=\frac{y+z}{2b+c}=\frac{z+x}{2c+a}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{(b+c)x+(c+a)y+(a+b)z}{2(ab+bc+ca)}.$$

41. If $\dfrac{x}{a(b+c-a)}=\dfrac{y}{b(c+a-b)}=\dfrac{z}{c(a+b-c)}$, prove that

$$(y+z-x)(b+c-a)=(z+x-y)(c+a-b)=(x+y-z)(a+b-c)$$

42. Divide 60 into two parts so that their product shall be to the sum of their squares as 2 to 5

43. The incomes of A and B are in the ratio of 3 to 2, and their expenditures are in the ratio of 5 to 3. Each saves £1000 a year; find their incomes.

44. Find the ratio of the value of a gold coin to a silver coin when 12 gold coins together with 10 silver coins are worth twice as much as 3 gold coins together with 65 silver coins.

45. Find four proportionals such that the sum of the extremes is 13, the sum of the means 11, and the sum of the squares of all four numbers is 170.

46. Two chests A and B were filled with coffee and chicory mixed in A in the ratio of 5 : 3, and in B in the ratio of 7 : 3. What quantity must be taken from each to form a mixture which shall contain 6 lbs. of coffee and 3 lbs. of chicory?

47. One vessel contains water and another spirits: half the water is poured into the spirits, and an equal quantity of the mixture is poured back into the water. If the first vessel then contains water and spirits in the ratio of 9 : 4, compare the quantities of water and spirits at first.

48. In a certain country the consumption of tea is three times the consumption of coffee. If a per cent. more tea and b per cent. more coffee were consumed, the aggregate amount consumed would be $5c$ per cent. more; but if b per cent. more tea and a per cent. more coffee were consumed, the aggregate amount consumed would be $3c$ per cent. more: find the ratio of $a : b$.

49. Two vats contain mixtures of spirit and water. In the first vat there are 8 parts of spirit to 3 of water; and in the second there are 5 parts of spirit to 1 of water. A 35 gallon cask is filled from these vats so as to contain a mixture of 4 parts of spirit to 1 of water. How many gallons are taken from the first vat?

50. An alloy of zinc, tin, and copper contains 90 per cent. of copper, 7 of zinc, and 3 of tin. A second alloy of copper and tin only is melted with the first, and the mixture contains 85 per cent. of copper, 5 of zinc, and 10 of tin. Find the percentages in the second alloy.

CHAPTER XXXIV

VARIATION

429 Direct Variation One quantity y is said to vary directly as another x, when the two quantities are so related that any change in the value of x produces a *proportionate* change in the value of y

For instance suppose a train travels uniformly over a certain distance in a certain time In *double* the time *double* the distance would be covered, if we *halve* the time we must *halve* the distance, and so on In fact, if we multiply the time by any number (whole or fractional), we must also multiply the distance by the same number

In this case the distance covered is said to be directly proportional to, or to vary directly as, the time

430 The symbol $\propto$ is used to denote variation, so that $y \propto x$ is read "y varies as x"

NOTE The word "directly" is often omitted in cases of direct proportion, but it will be convenient to retain it for the present

431 In the illustration of Art 429, if y represents the number of miles covered in a time represented by x hours, the ratio $\frac{y}{x}$ is the same in all cases so long as the speed of the train is the same

And, more generally, when y varies directly as x,

$$\frac{\text{any value of } y}{\text{the corresponding value of } x} \text{ is always the same},$$

that is, y and x are connected by an equation of the form

$$\frac{y}{x}=k, \quad \text{or} \quad y=kx,$$

where k is a *constant quantity*

The symbol k is called the variation constant, and its value can be found when we know one pair of corresponding values of the connected variables

EXAMPLE *If* y *varies directly as* x, *and* x$=60$ *when* $y=28$, *find the relation between* y *and* x *Also find the value of* y *when* x$=\frac{10}{21}$

Since $y \propto x$, $y=kx$, where k is constant

When $x=60$, $y=28$, $\quad 28=k\times 60$, whence $k=\frac{7}{15}$

Hence x and y are connected by the relation $y=\frac{7}{15}x$

From this equation, when $x=\frac{10}{21}$, we obtain $y=\frac{2}{9}$

432 Two quantities which are so related that when one is increased or diminished, the other is also increased or diminished, are not necessarily proportional

For instance when we increase the *side* of a square, we increase the *area*, but the side and area of a square are not proportional, for on *doubling* the side, we multiply the area by 4, and on *trebling* the side, we multiply the area by 9

Again, a body dropped from rest falls 16 feet roughly in the first second, but not 32 feet in the first 2 seconds, nor 48 feet in the first 3 seconds, for the speed is continually increasing Thus the distance traversed does not vary directly as the time

EXAMPLE *The area of a circle varies directly as the square of its radius If the area is 13·86 sq ft when the radius is 2 1 ft, find the area of a circle of radius 1 ft 9 in*

Let A be the area (in square feet) of a circle of radius r feet; then

$$A = kr^2, \text{ where } k \text{ is constant}$$

Now it is given that $A = 13\ 86$ when $r = 2\ 1$

$$13\ 86 = k \times (2\ 1)^2, \quad \text{whence} \quad k = \tfrac{1386}{441} = \tfrac{22}{7};$$

that is,

$$A = \tfrac{22}{7} r^2$$

Hence, when $r = 1\tfrac{3}{4}$, $A = \tfrac{22}{7} \times (1\tfrac{3}{4})^2$,

$$A = \tfrac{22}{7} \times \tfrac{7^2}{16} = \tfrac{77}{8} = 9\tfrac{5}{8}$$

Thus the required area $= 9\ 625$ sq ft

433 An equation of the form $y = kx$ is represented graphically by *a straight line through the origin* Hence the ordinate of every point on such a line is directly proportional to the abscissa

Conversely, if corresponding values of two variables x and y are plotted as abscissæ and ordinates, and the resulting points are found to lie on a straight line through the origin, we infer that y varies directly as x

434 **Inverse Variation** One quantity y is said to vary inversely as another x, when y varies directly as the reciprocal of x Thus if y varies inversely as x, $y = \dfrac{k}{x}$, where k is constant

For instance in travelling a certain distance, the *greater* the speed, the *less* will be the time required To *double* the speed would *halve* the time, and to *halve* the speed would *double* the time, and so on In fact, if we *multiply* the speed by any number (whole or fractional), we must *divide* the time by the same number

In this case the time is said to be inversely proportional to, or to vary inversely as, the speed.

435 When $y=\frac{k}{x}$, we have $xy=k$, hence, *when the product of two variable quantities is constant, each varies inversely as the other*

For instance in rectangles of constant area, the bases vary inversely as the corresponding heights

Again, if a fixed sum is to be spent in buying tea, the quantity bought will vary inversely as the price per pound

For suppose the fixed sum is represented by k shillings,

then $\qquad x$ lbs at y shillings per lb cost k shillings,

hence $\qquad xy=k$, where k is constant;

that is, $\qquad x$ varies inversely as y

EXAMPLE 1 *If* y *is equal to the sum of two quantities one of which varies directly as* x, *and the other inversely as* x, *and if* y=5 *when* x=1, *and* y=12½ *when* x=6, *find the relation between* x *and* y *Find the value of* y *when* x=3

Assume $y=kx+\frac{m}{x}$, where k and m are constant We have first to find the values of k and m

Since $x=1$, when $y=5$, we have $5=k+m$,

and since $x=6$, when $y=12\frac{1}{2}$, we have $12\frac{1}{2}=6k+\frac{m}{6}$.

From these equations we obtain $k=2$, $m=3$;

$\therefore$ x and y are connected by the relation $y=2x+\frac{3}{x}$

Substituting $x=3$, we get $y=7$

NOTE *Two* constants are here necessary, or we cannot assume that the rate of variation is the same in each of the two quantities whose sum is equal to y The *two* pairs of simultaneous values furnished by the question give *two* equations to find k and m

EXAMPLE 2 *If* $A \propto B$, *and* $C \propto \frac{1}{D}$, *then will* $AC \propto \frac{B}{D}$

For, by supposition, $A=mB$, $C=\frac{n}{D}$, where m and n are constants

Therefore $AC=mn\,\frac{B}{D}$; and as mn is constant, $AC \propto \frac{B}{D}$

436 An equation of the form $xy=k$ represents a rectangular hyperbola [Art 271] Hence if we plot corresponding values of two variables x and y which are *inversely proportional*, the resulting points will lie on a rectangular hyperbola But since y varies *directly* as $\frac{1}{x}$, it will sometimes be simpler to plot values of y and $\frac{1}{x}$, then if the resulting points are found to lie on a straight line through the origin, we infer that y varies inversely as x

437 Summary of foregoing results.

(i) y is *directly proportional to* x, or y *varies as* x, when the ratio y/x is constant, that is when $y=kx$

(ii) y is *inversely proportional to* x, or y *varies inversely as* x, when the ratio $y\Big/\frac{1}{x}$, or the product xy, is constant; that is when $xy=k$

In what follows we shall usually omit the word "directly" in cases of direct variation

EXAMPLES XXXIV. a.

1 Read off an equation to express each of the following statements:

(i) y varies inversely as x^2, (ii) p varies as $\frac{1}{v}$;

(iii) A varies as r^2, (iv) V is proportional to d^3,

(v) $s \propto t^2$, (vi) $t \propto \sqrt{l}$, (vii) $a^2 \propto b^3$

2 If $y=kx^2$, and $y=81$ when $x=6$, find the value of k, and the value of (i) y when $x=\frac{1}{3}$, (ii) x when $y=36$

3 If $y=\frac{m}{x}$, and $y=12$ when $x=1 8$, find m Also find the value of y when $x=3 75$

4 If x varies as the square of y, and if $x=144$ when $y=3$, find the *variation constant*, and the value of y when $x=324$

5. If $p \propto q$ and $q=6$ when $p=3\frac{1}{2}$, find the values of p corresponding to $q=9$, 24, $\frac{6}{7}$

6 The area of a circle varies as the square of its radius, if the area is $38\frac{1}{2}$ sq ft when the radius is 3 ft 6 in, find the area when the radius is 5 ft 3 in

7. Suppose a body falling from rest drops s feet in the first t seconds It has been found that s varies as t^2 If the body falls 257 6 ft in the first 4 seconds, find the equation between s and t Find to the nearest foot how far the body falls (i) in the 1st second, (ii) in the first 3 seconds

8 If S represents the breaking strain (in tons) of a steel wire, and C its circumference (in inches), it is known that S varies as C^2 If $S=4 9$ when $C=1\frac{3}{4}$, find S when C has the values $\frac{1}{2}$, $1\frac{1}{2}$, 2

9 From the following simultaneous values of x and y,

$$x=1 5,\ \ 2,\ \ 2 5,\ \ 3,\ \ 3 5,$$
$$y=\ 9,\ \ 16,\ \ 25,\ \ 36,\ \ 49,$$

shew that y varies as x^2.

10 The volume of a given quantity of gas at a constant temperature is inversely proportional to the pressure on it At a pressure of 20 lbs per square foot a certain quantity of gas occupies 4 5 cubic feet. Express in cubic feet the volume of the same quantity of gas at a pressure of 28 8 lbs per square foot

11. From the following simultaneous values of x and y,

$$x=1\ 2,\quad 1\ 5,\quad 1\ 6,\quad 1\ 8,\quad 2,\quad 2\ 4,$$

$$y=20,\quad 16,\quad 15,\quad 13,\quad 12,\quad 10,$$

find whether y varies as x, or as x^2, or inversely as x Find the variation constant

*12 The volume of a sphere varies as the cube of its radius Three metal spheres of radii 3, 4, 5 inches are melted down into a single sphere Find its radius

13 If y is equal to the sum of two quantities one of which varies directly as x, and the other inversely as x, and if $y=4$ when $x=1$, and $y=5$ when $x=2$, find the value of y when $x=4$

14 If $y=t+v$, where t varies as x and v varies as $\sqrt{x}$, find the relation between x and y, given that $x=4$ when $y=5$, and $x=9$ when $y=10$

15 The time of swing of a simple pendulum varies as the square root of the length of the pendulum If a pendulum 1 metre in length swings once in a second, find the length (in centimetres) of the pendulum which swings 75 times in 1 minute

16 A quantity x varies as the sum of two other quantities, one of which varies directly as y^2 and the other inversely as z If $x=16$ when $y=2$ and $z=1$, and if $x=5$ when $y=1$ and $z=2$, find the value of x when $y=\sqrt{3}$ and $z=4$

17. If y is equal to the sum of two quantities one of which varies directly as x and the other inversely as x, and if $y=na+b$ when $x=a$, and $y=a+nb$ when $x=b$, find two values of x which make $y=nab+1$

438 Sometimes a quantity depends on the variation of two or more other quantities which may vary independently of each other

For example in Geometry if A is the area of a triangle of height h on a base b, we know that

A varies as h, when b is constant,

and A varies as b, when h is constant

But we know that A is given by the formula $A=\frac{1}{2}hb$, and since $\frac{1}{2}$ is constant, this is the same thing as saying that A varies as the product hb when both h and b vary

This is a particular case of a general proposition which we shall now prove

439 *If* x *varies as* y *when* z *is constant, and* x *varies as* z *when* y *is constant, then will* x *vary as the product* yz *when both* y *and* z *vary*

The variation of x depends partly on that of y and partly on that of z Suppose these latter variations to take place separately, each in its turn producing its own effect on x Also let a, b, c denote certain simultaneous values of x, y, z

(i) *Let z be constant* while y changes to b, then x must undergo a *partial* change, dependent only on y, and will assume some *intermediate* value a', where

$$\frac{x}{a'}=\frac{y}{b} \qquad (1)$$

(ii) *Let y be constant*, that is, let it retain its value b, while z changes to c, then x must complete its change and pass from its intermediate value a' to its final value a, where

$$\frac{a'}{a}=\frac{z}{c} \qquad (2)$$

From (1) and (2), $\quad \frac{x}{a'}\times\frac{a'}{a}=\frac{y}{b}\times\frac{z}{c}$,

that is, $\quad \frac{x}{a}=\frac{yz}{bc}$, or $x=\frac{a}{bc}\,yz$,

or $\quad x$ varies as yz

Illustration. The amount of work done by *a given number of men* varies directly as the number of days they work, and the amount of work done *in a given time* varies directly as the number of men, therefore when the number of men and the number of days are both variable, the amount of work will vary as the product of the number of men and the number of days

440 **Joint Variation.** One quantity is said to vary jointly as a number of others when it varies directly as their product

Thus x varies jointly as y and z when $x=kyz$, where k is constant For instance, the interest on a sum of money varies jointly as the principal, the time, and the rate per cent

Again, x is said to vary directly as y and inversely as z when x varies as $y\times\frac{1}{z}$

EXAMPLE 1 *The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same The volume is 88 cubic feet when the height is 7 feet, and the radius of the base is 2 feet, what will be the height of a cylinder on a base of radius 9 feet, when the volume is 396 cubic feet?*

If the volume is 385 feet when the height is 10 feet, find the radius of the base

Let the radius of the base and the height be represented by r feet and h feet respectively Then if the volume is V cubic feet, we have

$$V = k \times r^2 h, \text{ where } k \text{ is constant}$$

Substituting the given values of V, r, and h, we have

$$88 = k \times 2^2 \times 7 ; \text{ whence } k = \tfrac{22}{7}$$

$$V = \tfrac{22}{7} r^2 h$$

Also when $V = 396$, $r = 9$,

$$396 = \tfrac{22}{7} \times 81 \times h, \text{ whence } h = 1\tfrac{5}{9}$$

Hence the required height is $1\frac{5}{9}$ feet

Again, if $V = 385$, when $h = 10$,

$$385 = \tfrac{22}{7} \times r^2 \times 10, \text{ whence } r^2 = \tfrac{49}{4},$$

or

$$r = \tfrac{7}{2}$$

Hence the required radius is $3\frac{1}{2}$ feet

EXAMPLE 2 *If* x *varies as* y *directly, and as* z *inversely, and* $x=14$ *when* $y=10$, $z=14$, *find* z *when* $x=49$, $y=45$

We have $x = k \times \frac{y}{z}$, where k is constant

Substituting the given values of x, y, and z, we have

$$14 = k \times \frac{10}{14}, \text{ whence } k = \frac{14 \times 7}{5},$$

$$x = \frac{14 \times 7}{5} \times \frac{y}{z}, \quad \text{or} \quad z = \frac{14 \times 7y}{5x}.$$

Hence when $x = 49$, $y = 45$, we have $z = \dfrac{14 \times 7 \times 45}{5 \times 49} = 18$

EXAMPLE 3 *The electrical resistance of a wire is proportional directly to its length and inversely to the square of its diameter Compare the resistance of two wires of the same material, one of which has a diameter of* 2·5 mm *and is* 6 m *long, while the other has a diameter of* 3·5 mm *and is* 9 m *long*

Let R represent the resistance of a wire l metres long and d millimetres in diameter

Then $R = \frac{kl}{d^2}$, where k is constant

$$\text{Resistance of } 1^{\text{st}} \text{ wire } (R_1) = k \times \frac{6}{(2{\cdot}5)^2} = \frac{6 \times 4}{25} k,$$

$$\text{,, } \quad 2^{\text{nd}} \text{ ,, } \quad (R_2) = k \times \frac{9}{(3{\cdot}5)^2} = \frac{9 \times 4}{49} k;$$

$$R_1 : R_2 = \frac{6}{25} : \frac{9}{49} = 98 : 75$$

NOTE It is not here necessary to find k, nor to express l and d in terms of the same unit

441. If a variable quantity y is partly constant and partly proportional to another variable x, the relation between x and y is of the form $y=ax+b$, where a and b are constants The values of a and b can be determined when two pairs of simultaneous values of x and y are known The variations of x and y can also be conveniently shewn by the linear graph of $y=ax+b$

EXAMPLE *In a certain machine a force of* P *pounds will support a load of* W *pounds, and it is known that* P *is partly constant and partly proportional to* W *If* $P=15$ *when* $W=48$, *and* $P=27$ *when* $W=96$, *draw a graph to shew the value of* P *for any load between* 40 *lbs and* 100 *lbs Find the value* (i) *of* P *when* $W=60$, (ii) *of* W *when* $P=23$

The variable part of P may be represented by aW, and the constant part by b Thus P and W satisfy the linear equation $P=aW+b$, where a and b are constants Hence the graph is a straight line.

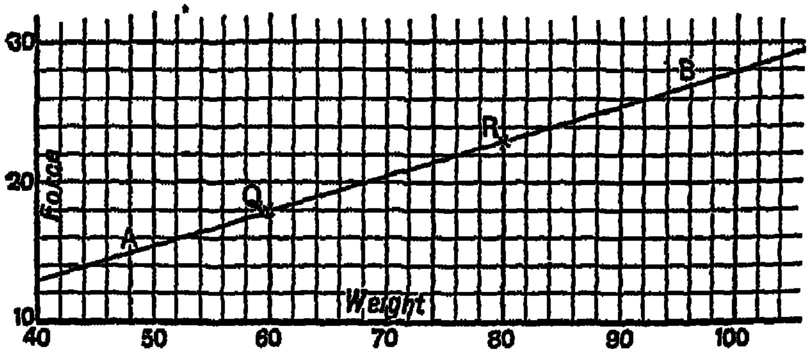


FIG 33

Plot the values of W horizontally, beginning at 40, and the values of P vertically, beginning at 10, taking 20 units to the inch in each case

When $P=15$, $W=48$, at the point A,

" $P=27$, $W=96$, " " B

Thus two points are determined, and AB is the required graph

By measurement we find that when $W=60$, $P=18$, at the point Q,

and when $P=23$, $W=80$, " " R

EXAMPLES XXXIV. b.

(*Joint Variation*)

1. Given that y varies jointly as x and z^2, and that $y=6$, when $x=9$, $z=3$, find (i) the value of y when $x=8$, $z=2$, (ii) the value of x when $y=9$, $z=\frac{1}{4}$

2. If A varies directly as B and inversely as C, and $A=\frac{1}{3}$ when $B=5$, $C=9$, find the relation between A, B, and C Hence find the value of A when $B=6$, $C=\frac{1}{5}$

3. It is known that the volume of a pyramid varies as the height when the base is the same, and as the area of the base when the height is the same When the height is 26 cm and the area of the base is 45 sq cm the volume is 390 cu cm, what is the volume of a pyramid when the height is 14 cm and the area of the base 60 sq cm?

4 The pressure of wind on a plane surface varies jointly as the area of the surface, and the square of the wind's velocity The pressure on a square foot is 1 lb when the wind's velocity is 15 miles per hour, find the velocity of the wind when the pressure on a square yard is 16 lbs

5 If A varies directly as the square root of B and inversely as the cube of C, and if $A=24$, when $B=4$, and $C=\frac{1}{2}$, find B when $A=3$ and $C=2$

Shew also that B varies jointly as C^6 and A^2

6 A boy estimates that the number of distinctions he wins in an examination varies directly as the number of subjects he takes up, and inversely as the number of competitors If he wins 3 distinctions when there are 20 competitors taking 6 subjects, how many additional subjects must he take up to win 5 distinctions when the number of competitors is 16?

7 The time of going from one place to another varies directly as the distance and inversely as the velocity Two trains describe distances which are in the ratio of 3 to 7, and the times are in the ratio of 5 to 9 Find the ratio of the velocities

8 If the cost of digging a trench is proportional to the quantity of earth taken out and the depth to which it is sunk, and if the cost of digging a trench 3 ft broad by 8 ft deep is 9*d* per yard, find to the nearest penny the cost of digging a trench 120 yds long, 5 ft broad, and 10 ft deep

9 The weight of a circular disc varies as the square of the radius when the thickness remains the same, it also varies as the thickness when the radius remains the same Two discs have their thicknesses in the ratio of 25 32, find the ratio of their radii if the weight of the first is twice that of the second

(*Miscellaneous*)

10 If y is equal to the sum of two quantities, one of which is constant and the other varies as x, and if $y=17$ when $x=\frac{1}{3}$, and $y=42$ when $x=2$, find the relation between x and y Hence find values of y corresponding to $x=\frac{1}{5}$, 3, 5

11 The expenses of a ball are partly constant (for hire of room, decorations, etc), and partly proportional to the number of guests For 80 guests the cost is £64, and for 120 guests £88 Find the cost for 200 guests Also draw a graph to shew the cost for any number of guests from 50 to 300

12. For printing a circular, a printer's estimate is 1*s* 9*d* for 50 copies or 2*s* 8*d* per 100 Presuming each of these estimates to consist of (i) a charge for setting up the type, independent of the number of copies printed, (ii) a charge for printing and paper, proportional to the number of copies printed, find what his estimate for printing 1000 copies would be

13. In a certain machine P is the force in lbs.-wt required to support a load of W lbs.-wt The following values of P and W were obtained experimentally

P=	7 1,	8 0,	9 6,	12,	13,	15 3,	17,
W=	10 5,	12,	15,	18 4,	20 3,	24 6,	27

Show by a graph that P and W are connected by a relation of the form $P=aW+b$, and find the values of a and b Find the force necessary to support a load of 1 cwt

14. If $x+y \propto x-y$, prove that $x^2+y^2 \propto xy$, and if $x \propto y$, prove that $x^2-y^2 \propto xy$.

15. If $x \propto y^2-z^2$, when $y \propto t^2$ and $z \propto t^3$, and if $x=0$ when $t=3$, and $x=8$ when $t=-1$, find x when $t=2$

16. If s varies as $t^{\frac{3}{2}}$, and if $s=31\ 6$ when $t=2\ 93$, find t when $s=8\ 4$.
[*Use logarithms*]

17 If the receipts on a railway vary as the excess of speed above 20 miles an hour, while the expenses vary as the square of that excess, find the speed at which the profits will be greatest, if at 40 miles an hour the expenses are just covered

18 If H is proportional to $D^{\frac{2}{3}}v^3$, and if $D=1810$, $v=10$ when $H=620$, find H when $D=2100$ and $v=13$ [*Use logarithms*]

19. A man spends on charitable objects an annual amount proportional to the square of his income, and spends £35 more when his income is £1200 per annum than when it is £900 per annum Find his charitable expenditure in each case

20. The tractive force of a locomotive varies as the pressure on the circular piston and the length of the stroke directly, and as the height of the driving wheel inversely Compare the tractive force exerted by two engines, in both of which the pressure of steam is 160 lbs to the square inch, but the diameters of the pistons are 18 and 20 inches, the lengths of stroke 26 and 24 inches, and the heights of the driving wheels 7 feet 6 inches and 7 feet 8 inches respectively

21. The expense of publishing a book depends partly on the cost of setting up the type, which is constant, and partly on the cost of printing, which varies as the number of copies printed, if 630 copies are sold, a loss of 10 % is incurred, and if 980 copies are sold, a gain of 12 % is made. how many copies must be sold just to pay expenses?

CHAPTER XXXV

The Theory of Quadratic Equations and Functions

442. As in Chap XXV we shall here regard the equation

$$ax^2+bx+c=0$$

as the standard form of a quadratic equation

Here a, b, c are supposed to be known quantities

In like manner ax^2+bx+c will be regarded as the standard form of a quadratic expression or function

In each case the term c, which does not involve x, is spoken of as the constant or absolute term.

443 *Every quadratic has two roots and no more*

(i) It has been shewn in Art 283 that the standard equation is satisfied by $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

Thus there are *two* roots

(ii) *To shew that there cannot be more than two roots*

If possible, let the equation $ax^2+bx+c=0$ have *three different* roots α, β, γ Then since each of these values must satisfy the equation, we have

$$a\alpha^2+b\alpha+c=0, \qquad (1)$$

$$a\beta^2+b\beta+c=0, \qquad (2)$$

$$a\gamma^2+b\gamma+c=0 \qquad (3)$$

From (1) and (2), by subtraction,

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$$

Divide out by $\alpha-\beta$ which, by hypothesis, is not zero, then

$$a(\alpha+\beta)+b=0 \qquad (4)$$

Similarly from (2) and (3), we have

$$a(\beta+\gamma)+b=0, \qquad (5)$$

by subtracting (5) from (4),

$$a(\alpha-\gamma)=0$$

But this result is impossible, since a is not zero, and α is not equal to γ

Hence there cannot be three different roots

444 The terms *unreal*, *impossible* or *imaginary*, to denote expressions which involve the square root of a negative quantity, have already been explained and illustrated in Arts 184, 280, 284, Ex. 2, 285 (ii) These articles should here be carefully revised. It is important to clearly distinguish between the terms *real* and *rational*, *imaginary* and *irrational* Thus $\sqrt{25}$ or 5, $3\frac{1}{2}$, $-\frac{5}{8}$ are rational and real, $\sqrt{7}$ is irrational but real, while $\sqrt{-7}$ is irrational and also imaginary

445 Character of the roots. The roots of the standard equation are

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

In every case the character of the roots will depend upon the value of b^2-4ac, the quantity under the radical

(i) If b^2-4ac is a *perfect square*, the roots are *rational* and *unequal*.

(ii) If b^2-4ac is *zero*, each root of the equation reduces to $-\frac{b}{2a}$ Thus the roots are *rational* and *equal*.

(iii) If b^2-4ac is *positive but not a perfect square*, the roots, though *real*, are *irrational* and *unequal*

(iv) If b^2-4ac is *negative*, the roots are *imaginary* and *unequal*

It will be convenient to refer to b^2-4ac as the **discriminant**, and to denote it shortly by the symbol Δ, as in Art 286 The pupil should be able to write down the discriminant readily for any quadratic equation

EXAMPLE 1 *Shew that the equation* $2x^2-6x+7=0$ *cannot be satisfied by any real values of* x

Here $a=2$, $b=-6$, $c=7$, so that

$$\Delta=b^2-4ac=(-6)^2-4\cdot 2\cdot 7=-20$$

Therefore the roots are imaginary

NOTE If the equation is solved graphically as in Art 285 (ii), it will be found that the graph does not cut the axis of x Thus there are no real values of x which make $2x^2-6x+7$ equal to zero

EXAMPLE 2 *For what value of* k *will the equation* $3x^2-6x+k=0$ *have equal roots?*

The roots will be equal if $\Delta=0$,

that is, if $\qquad (-6)^2-4\cdot 3\cdot k=0,$

whence $\qquad k=3$

EXAMPLE 3 *Shew that the roots of the equation*

$$x^2-2px+p^2-q^2+2qr-r^2=0$$

are rational

The roots will be rational if Δ is a perfect square

Here $$\Delta=(-2p)^2-4(p^2-q^2+2qr-r^2)$$
$$=4(q^2-2qr+r^2)=4(q-r)^2$$

Hence the roots are rational

446 Let the roots of $ax^2+bx+c=0$ be represented by α and β, so that
$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a},$$
then we have
$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$
$$=-\frac{2b}{2a}=-\frac{b}{a}.$$
$$\alpha\beta=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$
$$=\frac{(-b)^2-(b^2-4ac)}{4a^2}$$
$$=\frac{4ac}{4a^2}=\frac{c}{a}$$

These results may be quoted thus
$$\textit{sum of roots}=-\frac{\textit{coefficient of } x}{\textit{coefficient of } x^2},$$
$$\textit{product of roots}=\frac{\textit{absolute term}}{\textit{coefficient of } x^2}.$$

If we first divide by a, so that *the coefficient of* x^2 *is unity*, we have
$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$
and in this case

sum of roots = coefficient of x *with its sign changed,*

product of roots = absolute term

EXAMPLE *Find the relations between the coefficients of the equation*

$$ax^2+bx+c=0$$

in order that the roots shall be (i) *equal in magnitude and opposite in sign,* (ii) *reciprocals*

(i) The roots will be equal in magnitude and opposite in sign if their sum is zero, therefore $-\frac{b}{a}=0$, or $b=0$

(ii) The roots will be reciprocals when their product is unity, therefore

$$\frac{c}{a}=1,\quad\text{or}\quad c=a$$

447 When one root of a quadratic is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots above proved

EXAMPLE. *Solve the equation* $2m(1+x^2)-(1+m^2)(x+m)=0.$

This is a quadratic, and it is clearly satisfied by $x=m$

Also, since the equation may be written

$$2mx^2-(1+m^2)x+m(1-m^2)=0,$$

the product of the roots is $\dfrac{1-m^2}{2}$; and since one root is m, the other root is $\dfrac{1-m^2}{2m}$

448 Since $\quad -\dfrac{b}{a}=\alpha+\beta,\;$ and $\;\dfrac{c}{a}=\alpha\beta,$

the equation $\quad x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$

may be written $\quad x^2-(\alpha+\beta)x+\alpha\beta=0,$

or $\quad (x-\alpha)(x-\beta)=0 \quad\ldots\quad\ldots\quad (1)$

Hence, any quadratic may also be expressed in the form

$$x^2-(\textit{sum of roots})x+(\textit{product of roots})=0 \qquad (2)$$

449 Formation of equations with given roots. It is now easy to form an equation whose roots are known

EXAMPLE 1 *Form the equation whose roots are* 3 *and* 5

The equation is $\quad (x-3)(x-5)=0,\qquad$ [Art 448, (1)]

or $\quad x^2-8x+15=0$

EXAMPLE 2 *Form the equation whose roots are* a *and* $-\dfrac{b}{3}$

The equation is $\quad (x-a)\left(x+\dfrac{b}{3}\right)=0,$

that is, $\quad (x-a)(3x+b)=0,$

or $\quad 3x^2-3ax+bx-ab=0$

EXAMPLE 3 *Form the equation whose roots are* $3+\sqrt{5}$ *and* $3-\sqrt{5}$

When the roots are irrational it is easier to use formula (2) of Art 448.

$$\textit{Sum of roots}=6,\quad \textit{product of roots}=4,$$

the equation is $x^2-6x+4=0.$

The method of Examples 1 and 2 may be applied to equations with three or more given roots

EXAMPLE 4 *Form the equation whose roots are* $0,\ \pm 2,\ 3.$

The equation has to be satisfied by

$$x=0,\quad x=2,\quad x=-2,\quad =3;$$

therefore it is $\quad x(x-2)(x+2)(x-3)=0,$

that is, $\quad x(x^2-4)(x-3)=0,$

or $\quad x^4-3x^3-4x^2+12x=0.$

450 The *expression* $ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}$

$$=a(x-\alpha)(x-\beta),$$

where α, β are the roots of the *equation* $ax^2+bx+c=0$

Hence the *factors* of the *expression* can be written down at once when the *roots* of the *equation* are known

Hence also each of the conclusions of Art 445, with regard to the character of the *roots* of $ax^2+bx+c=0$ has a corresponding interpretation when applied to the *factors* of the *quadratic function* ax^2+bx+c

Thus the quadratic function ax^2+bx+c can be resolved into two rational factors when b^2-4ac is a perfect square [Art 445, (i)]

The function ax^2+bx+c is a perfect square with regard to x when $\alpha=\beta$, that is, when $b^2-4ac=0$ [Art 445, (ii)]

In all other cases the factors of ax^2+bx+c are irrational, being real or imaginary according as b^2-4ac is positive or negative [Art 445, (iii) and (iv)]

EXAMPLES XXXV a.

By using the Discriminant, find the nature of the roots of the following equations

1. $x^2+x-240=0$ 2 $3x^2+8=14x$ 3 $x^2-4x+1=0$

4 $4x^2-28x+49$ 5. $2x^2+7=3x$ 6 $(3x+1)^2=6x+5$

7 In each of the following quadratic functions, find whether the factors are rational or irrational, real or unreal.

(i) $x^2+5x+10$, (ii) $x^2-5x-10$,

(iii) $15x^2-11ax-14a^2$, (iv) $x^2-18cx+88c^2$

8. For what values of m will the following equations have equal roots?

(i) $4x^2+2x+m=0$, (ii) $m^2x^2+2(m+1)x+4=0$

9 Shew that the equation $3mx^2-(2m+3n)x+2n=0$ has rational roots

10. If the equation $6x^2+kx+\frac{2}{3}=0$ has equal roots, find k

11. Find the value of a which makes $9x^2-ax-x+1$ a perfect square

12 For what values of m can $x^2+(m+1)x+16$ be expressed as the product of two rational factors?

Form the equations whose roots are

13 4, -5 14 -7, -13 15 $-6a$, $7a$

16 $c+d$, $c-d$. 17 $\frac{2}{3}$, $\frac{5}{6}$. 18 3, -5, 2.

19. $\frac{a}{3b}$, $-\frac{3b}{a}$ 20 $4+\sqrt{3}$, $4-\sqrt{3}$ 21. $\frac{3\pm\sqrt{7}}{2}$.

22 0, ±5, 3 23 $-3+\sqrt{2}$, $-3-\sqrt{2}$ 24 $m+\sqrt{n}$, $m-\sqrt{n}$.

25 With as little work as possible, find the roots of the following equations

(i) $(p-q)x^2+(q-r)x+(r-p)=0$, (ii) $(a+b)x^2+cx=a+b+c$;

(iii) $ax^2-bx=c(ac-b)$, (iv) $5x^2+189x=194$

26. Write down the Discriminant of each of the equations

(i) $a(x^2-1)=(b-c)x$; (ii) $(x-a)(x-b)=c^2$

Hence shew that in each case the roots are real if a, b, c are any real quantities

27. Prove that if the roots of $ax^2+2bx+c=0$ are imaginary the roots of

$$ax^2+2(a+b)x+a+2b+c=0$$

are also imaginary

451 In examples dealing with the roots of quadratics *the roots should not be considered singly* It will usually be found far simpler to work from the sum and product of the roots, using the results of Art 446

EXAMPLE 1 *If α and β are the roots of* $x^2-px+q=0$, *find the value of* (i) $\alpha^2+\beta^2$, (ii) $\alpha^3+\beta^3$

We have $\alpha+\beta=p, \quad \alpha\beta=q$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=p^2-2q$$

Again, $$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$$
$$=p\{(\alpha+\beta)^2-3\alpha\beta\}$$
$$=p(p^2-3q)$$

EXAMPLE 2 *If α and β are the roots of the equation* $lx^2+mx+n=0$, *find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$*

For the new equation we have

$$\text{sum of roots}=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta},$$

$$\text{product of roots}=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$$

by Art 448 the required equation is

$$x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0,$$

or $$\alpha\beta x^2-(\alpha^2+\beta^2)x+\alpha\beta=0$$

Now, as in Ex 1, $\alpha^2+\beta^2=\frac{m^2-2nl}{l^2}$, also $\alpha\beta=\frac{n}{l}$

. the new equation is $$\frac{n}{l}x^2-\frac{m^2-2nl}{l^2}x+\frac{n}{l}=0$$

or $$nlx^2-(m^2-2nl)x+nl=0$$

EXAMPLE 3 *Find the relation connecting the coefficients of the equation* $px^2+qx+r=0$, *when one root is three times the other*

Let α, 3α represent the roots,

$$\text{then sum of roots} = 4\alpha = -\frac{q}{p}, \quad \text{product of roots} = 3\alpha^2 = \frac{r}{p}$$

From the first result $\alpha^2 = \frac{q^2}{16p^2}$, from the second $\alpha^2 = \frac{r}{3p}$

$$\frac{q^2}{16p^2} = \frac{r}{3p}, \quad \text{or} \quad 3q^2 = 16pr,$$

which is the required condition

452 The following example gives a result of great importance.

EXAMPLE *To find the condition that the quadratics*

$$ax^2+bx+c=0, \quad lx^2+mx+n=0$$

may have one root in common

Suppose α is a value of x which satisfies both equations, then

$$a\alpha^2+b\alpha+c=0,$$
$$l\alpha^2+m\alpha+n=0,$$

by cross multiplication (Art 420),

$$\frac{\alpha^2}{bn-cm} = \frac{\alpha}{cl-an} = \frac{1}{am-bl}$$

Since these three ratios are equal, the square of the middle one is equal to the product of the other two,

that is
$$\frac{\alpha^2}{(cl-an)^2} = \frac{\alpha^2}{bn-cm} \cdot \frac{1}{am-bl}$$

On dividing by α^2, we have

$$(cl-an)^2 = (bn-cm)(am-bl),$$

which is the required condition

NOTE This is also the condition that the two *quadratic functions* ax^2+bx+c, lx^2+mx+n may have a common *linear factor*

EXAMPLES XXXV. b

1 Without actual solution find the sum of the squares and the sum of the cubes of the roots of the following equations

(i) $x^2+7x+8=0$, (ii) $2x^2-3x+1=0$, (iii) $5x^2+x+10=0$

2 If α, β are the roots of $px^2+qx+r=0$, find the values of

(i) $\alpha^2+\beta^2$, (ii) $\alpha^3+\beta^3$, (iii) $(\alpha-\beta)^2$, (iv) $\alpha^2\beta+\alpha\beta^2$

3 If α, β are the roots of $ax^2-bx+c=0$, find the values of

(i) $\alpha^4+\beta^4$, (ii) $\alpha^3\beta+\alpha\beta^3$, (iii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$, (iv) $\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)$

4 If α, β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\alpha-\beta$, $\beta-\alpha$

5. Find the condition that one root of the equation $ax^2-bx-c=0$ may be double the other

6 If α, β are the roots of $lx^2+mx+n=0$, form the equation whose roots are $\alpha+2\beta$, $2\alpha+\beta$

7. If x_1, x_2 denote the roots of $ax^2+bx+c=0$, find the values of the following expressions in terms of a, b, c

(i) $(ax_1+b)(ax_2+b)$, (ii) $(bx_1+c)(bx_2+c)$,

(iii) $(ax_1+b)^{-2}+(ax_2+b)^{-2}$, (iv) $(ax_1+b)^{-3}+(ax_2+b)^{-3}$

[*Note that from the sum of the roots we have* $ax_1+b=-ax_2$]

8. If u, v are the roots of $x^2+x+1=0$, form the equation whose roots are $mu+nv$, $mv+nu$

9. If one root of $x^2+(3k+2)x+k^2-2k-5=0$ is three times the other, find k

10. If the equations $x^2+px+q=0$, $x^2+mx+k=0$ have a common root, prove that $(q-k)^2=(m-p)(pk-mq)$

11. If α, β are the roots of $x^2-ax+b=0$ and α^3, β^3 are the roots of $x^2-\mathrm{A}x+\mathrm{B}=0$, shew that $\mathrm{A}=a(a^2-3b)$, $\mathrm{B}=b^3$

12 If the roots of the equation $ax^2+bx+c=0$ are in the ratio of m n, prove that $(m^2+n^2)ac=mn(b^2-2ac)$

13 Find the equation whose roots are the cubes of the roots of the equation $5x^2-7x+3=0$

14 Prove that the roots of $ax^2+bx+c=0$ will both be negative if a, b, c all have the same sign, and that the roots will both be positive if a and c have like signs opposite to that of b

453 Variations in Sign and Value of Quadratic Functions. As different values are ascribed to x, the resulting values of the function ax^2+bx+c will not necessarily always have the same sign It will be found that the variations in sign depend upon the nature of the roots of the equation $ax^2+bx+c=0$

Graphical illustrations of this principle have already been given For example, in Art 285 (i) we have the graph of $y=4x^2-10x+5$ It is there shewn that the *equation* $4x^2-10x+5=0$ has real roots, approximately equal to 0 69 and 1 81, and that the *function* (represented by the ordinate y) is positive for all real values of x except such as lie between those roots The value of the function is zero when $x=0$ 69 or 1 81, and changes its sign as x passes through these values

Again, in Art 285 (ii) the equation $x^2-3x+3=0$ has no real roots, and here it is seen that the function x^2-3x+3 never becomes zero, and never changes its sign

These are particular cases of the general proposition given in the next article.

454 *To discuss the changes in sign of the quadratic function*

$$ax^2+bx+c,$$

for real values of x

Let α, β be the roots of the equation $ax^2+bx+c=0$

Then $$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

(i) *Suppose the roots are real and different* Also let α represent the greater root

Then if x is greater than α, the factors $x-\alpha$, $x-\beta$ are both positive If x is less than β, both the above factors are negative In each case the product $(x-\alpha)(x-\beta)$ is positive, and the function ax^2+bx+c has the same sign as a

But if x lies between α and β, the product $(x-\alpha)(x-\beta)$ is negative, and the sign of ax^2+bx+c is opposite to that of a

(ii) *Suppose the roots are equal* Then since $\beta=\alpha$,

$$ax^2+bx+c=a(x-\alpha)^2,$$

and $(x-\alpha)^2$ is positive for real values of x, hence ax^2+bx+c has the same sign as a

(iii) *Suppose the roots are imaginary*

Then $$ax^2+bx+c\equiv a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}\equiv a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}$$

$$\equiv a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right\}, \text{ where } \Delta \text{ is the discriminant}$$

But since Δ is negative, $-\frac{\Delta}{4a^2}$ is positive, also $\left(x+\frac{b}{2a}\right)^2$ is positive for real values of x Hence ax^2+bx+c has the same sign as a

The three cases may be included in one statement

For real values of x *the function* ax^2+bx+c *always has the same sign as* a *except when the roots of the equation* $ax^2+bx+c=0$ *are real and unequal, and* x *lies between them.*

455 Although the conclusions of the preceding article may be applied to any quadratic function, it is more instructive to deal with each special case separately, without quoting the general proposition.

EXAMPLE 1 *Find the sign of* $2x^2+5x+4$ *for real values of* x.

$$2x^2+5x+4=2\left(x^2+\tfrac{5}{2}x+2\right)$$

$$=2\left[x^2+\tfrac{5}{2}x+\left(\tfrac{5}{4}\right)^2+2-\tfrac{25}{16}\right]$$

$$=2\left[\left(x+\tfrac{5}{4}\right)^2+\tfrac{7}{16}\right],$$

which is always positive when x has any real value

EXAMPLE 2 *If* x *is real, between what values of* x *will the function* $2x^2-11x+14$ *be positive?*

We have $2x^2-11x+14=(2x-7)(x-2)=2\left(x-\frac{7}{2}\right)(x-2)$

If $x>3\frac{1}{2}$, the factors $x-\frac{7}{2}$, $x-2$ are both positive If $x<2$, both the above factors are negative In each case the product is positive and the given function is positive But if x lies between 2 and $3\frac{1}{2}$, the factors $x-\frac{7}{2}$, $x-2$ have opposite signs, and the function is negative

Hence the function is positive for all values of x except such as lie between 2 and $3\frac{1}{2}$

EXAMPLE 3 *When is the function* $10-x-3x^2$ *positive and when negative?*

$$10-x-3x^2=-(3x^2+x-10)=-(3x-5)(x+2)$$
$$=-3\left(x-\frac{5}{3}\right)(x+2)$$

Hence the function will be positive or negative according as the factors $x-\frac{5}{3}$, $x+2$ have opposite signs or like signs

If $x>\frac{5}{3}$, the factors are both positive,

if $x<-2$, „ „ negative

But if x lies between -2 and $\frac{5}{3}$, the factors have opposite signs

Hence the function is positive when x lies between -2 and $\frac{5}{3}$, and is negative for values of x outside these limits

NOTE If any difficulty is found in considering cases like $x<-2$, it may be convenient here to refer to Art 50 It is there explained that $-a$ is *less than* $-b$ when $-a-(-b)$, or $-a+b$ is negative, that is, when the absolute value of a is *greater than* that of b

Thus $-3<-2$, $-4<-3$, and so on

456 For different values of x the quadratic function ax^2+bx+c will of course vary *in value* as well as *in sign* We shall give no formal discussion of such variations, for it is better to examine the possible values of any given function independently of general results

EXAMPLE *If* x *is real, find whether* $3+x-2x^2$ *is capable of all values*

Put $3+x-2x^2=k$, then $2x^2-x+(k-3)=0$

If x is to be real, $(-1)^2-4\cdot 2(k-3)$ must be positive or zero, that is, $25-8k$ must not be negative, and this condition is satisfied by all values of k from $-\infty$ to $\frac{25}{8}$.

Thus the function is capable of all values between these limits, and its maximum value is $3\frac{1}{8}$

The maximum value may also be found as explained on p 245, or graphically as on p 244

457 It is not necessary to give graphical illustrations in detail of the foregoing examples. It will be sufficient to direct the reader's attention to Arts. 270 and 285, which furnish all that is necessary in the way of explanation.

For instance, taking Art 455, Ex 2, it will be found that the graph of $y=2x^2-11x+14$ is similar to that on Art 285. The graph cuts the axis of x where $x=2$ and $x=3\frac{1}{2}$. When x lies between these values the ordinate y is negative, and for all other values of x the ordinate is positive.

[*Examples* XXXV c. 1–11 *may be taken here*.]

458 The following examples illustrate useful applications of the properties of the Discriminant.

EXAMPLE 1. *If* x *is real, prove that the expression* $\dfrac{x^2+2x+7}{2x+3}$ *can have all numerical values except such as lie between* -3 *and* 2.

Let $$\frac{x^2+2x+7}{2x+3}=l,$$

then $$x^2+2x+7=l(2x+3)$$

that is, $$x^2+2x(1-l)+(7-3k)=0$$

This quadratic in x will have real roots if

$$4(1-l)^2-4(7-3l) \text{ is positive or zero},$$

that is, $$l^2+l-6 \quad \text{is positive or zero},$$

or $$(k+3)(l-2) \quad \text{is positive or zero}$$

Hence the two factors must not have opposite signs.

Now if $l>2$, both factors are positive,

and if $l<-3$, „ „ negative.

But if $l>-3$, and <2, the factors have opposite signs.

Hence l must not lie between -3 and 2, but may have any other value.

EXAMPLE 2. *Shew that* $\dfrac{x^2+x-1}{x^2+3x+2}$ *is capable of assuming all real values, if* x *is real*.

Let $$\frac{x^2+x-1}{x^2-3x+2}=l;$$

then $$x^2+x-1=l(x^2+3x+2);$$

that is, $$x^2(l-1)+x(3l-1)+(2l+1)=0$$

This quadratic in x will have real roots if

$$(3l-1)^2-4(l-1)(2l+1) \text{ is positive or zero},$$

that is, if $$l^2-2l+5 \text{ is positive or zero}.$$

Now $k^2-2l+5=(k-1)^2+4$, which is always positive for any real value of l.

EXAMPLES XXXV. c.

For what real values of x are the following functions negative?

1. x^2+x-12 2. $2x^2-13x+20$ 3. $3x^2+26x+16$

For what real values of x are the following functions positive?

4. $6+x-x^2$ 5. $3+11x-4x^2$ 6. $72-7x-2x^2$

7. When x is real, find the signs of

$$\text{(i) } 2x^2-6x+11, \quad \text{(ii) } 12x-3x^2-15$$

8. If x is real, shew that $7+10x-5x^2$ can be made to assume all values between $-\infty$ and 12

9. Apply Art 454 to determine the signs of

$$\text{(i) } 2x^2-x-15, \quad \text{(ii) } ab+(a-b)x-x^2, \quad \text{(iii) } 4x^2-3x+1.$$

10. Find the maximum value of $4+3x-x^2$, and the minimum value of $4x^2-4x+15$ Verify the values graphically From the graphs find when these functions are positive and when negative

11. Shew that a quadratic function will always have the same sign if its Discriminant is negative or zero

12. Shew that if x is real, $\dfrac{x^2-24}{2x-11}$ cannot lie between 3 and 8.

13. If x is real, prove that $\dfrac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6

14. Determine the limits of value between which the following functions must lie for real values of x

$$\text{(i) } \frac{x^2+10x+65}{2x+4}, \quad \text{(ii) } \frac{x^2+x+1}{x^2-x+1}, \quad \text{(iii) } \frac{x^2-3x+1}{2x^2-3x+2}$$

15. Determine the signs of the following functions for real values of x.

$$\text{(i) } \frac{x^2-6x+11}{2x^2+4x+3}, \quad \text{(ii) } \frac{2x^2+3x+3}{x^2-2x+5}, \quad \text{(iii) } \frac{6x-14-x^2}{x^2-10x+30}$$

16. Shew that if x is real, $(x^2+ab)(2x-a+b)^{-1}$ cannot lie between $-b$ and a

17. Shew that $\dfrac{(x-1)(x-3)}{(x-2)(x-4)}$ can be made to assume any real value by giving a suitable real value to x.

18. Find the maximum and minimum values of the function $\dfrac{5x^2-x+5}{x^2+x+1}$, when x is real.

19. Shew that $-\dfrac{b^2-4ac}{4a}$ is a minimum value of the function ax^2+bx+c when a is positive, and a maximum when a is negative

CHAPTER XXXVI

A CHAPTER FOR REVISION

MISCELLANEOUS THEOREMS AND EXAMPLES

459 THE present chapter will be found useful for revision purposes It contains harder applications of certain principles and processes which have hitherto only been treated in an elementary way It also includes some Miscellaneous Theorems and Examples of special importance at this stage

460 In Arts 90–92 it has been shewn that elementary processes can often be much shortened and simplified by an intelligent use of *compound* terms and coefficients We here give some further examples

EXAMPLE 1 *Divide* $a^3+b^3+c^3-3abc$ *by* $a+b+c$

$$
\begin{array}{l|l|l}
a+(b+c) & a^3 \qquad\qquad -3abc+(b^3+c^3) & a^2-a(b+c)+(b^2-bc+c^2) \\
 & \underline{a^3+a^2(b+c)} & \\
 & \quad -a^2(b+c)-3abc & \\
 & \quad \underline{-a^2(b+c)-a(b^2+2bc+c^2)} & \\
 & \qquad\qquad a(b^2-bc+c^2)+(b^3+c^3) & \\
 & \qquad\qquad \underline{a(b^2-bc+c^2)+(b^3+c^3)} &
\end{array}
$$

NOTE The work has been arranged according to descending powers of a, the divisor being considered as an expression of two terms, one simple and one compound The above compact arrangement should be compared with that in Art 175

EXAMPLE 2 *Find the* H C F *of* $2x^3-(a+6c)x^2+3(ac+b)x-9bc$ *and* $3x^3+(a-9c)x^2-(3ac+2b)x+6bc$

$$
\begin{array}{l|r|l}
2x^3-(a+6c)x^2+3(ac+b)x-9bc & 3x^3+\;(a-\;9c)x^2-\;(3ac+\;2b)x+\;6bc & \\
 & 2 \qquad\qquad\qquad\qquad\qquad\qquad & \\
\hline
 & 6x^3+(2a-18c)x^2-\;(6ac+\;4b)x+12bc & 3 \\
 & \underline{6x^3-(3a+18c)x^2+\;(9ac+\;9b)x-27bc} & \\
 & 5ax^2-(15ac+13b)x+39bc &
\end{array}
$$

Now the remainder $=5ax^2-15acx-13bx+39bc$

$$=5ax(x-3c)-13b(x-3c)$$
$$=(5ax-13b)(x-3c)$$

Also this expression contains the H.C F of the given expressions [Art 212] By the Remainder Theorem $x-3c$ is a factor of each, while the factor $5ax-13b$ clearly is not a factor of either of them.

Hence the H.C F is $x-3c$.

461 Harder Factors and Identities.

EXAMPLE 1. *Find the* H C F. *of the expressions*

$$a(a-1)x^2+(2a^2-1)x+a(a+1), \qquad (E_1)$$
$$(a^2-3a+2)x^2+(2a^2-4a+1)x+a(a-1), \qquad (E_2)$$

Here $\quad E_1=[(a-1)x+a][ax+(a+1)]$

and $\quad E_2=(a-1)(a-2)x^2+(2a^2-4a+1)x+a(a-1)$

$\quad =[(a-1)x+a][(a-2)x+(a-1)]$

Hence the H C F $=(a-1)x+a$

NOTE Here in each case it is easy to detect the requisite coefficients of x, and also the constant terms in the factors It only remains to arrange them so as to give the coefficient of the middle term correctly

[*Examples* XXXVI a 1–14, *page* 410, *may be taken here*]

EXAMPLE 2 *By means of factors find the quotient when*

$$5x(x-11)(x^2-x-156)$$

is divided by

$$x^3+x^2-132x$$

$$\text{The quotient}=\frac{5x(x-11)(x^2-x-156)}{x^3+x^2-132x}=\frac{5x(x-11)(x+12)(x-13)}{x(x+12)(x-11)}$$
$$=5(x-13)$$

EXAMPLE 3 *Find the product of*

$$(3+x-2x^2)^2-(3-x+2x^2)^2, \qquad (E_1)$$
$$(3+x+2x^2)^2-(3-x-2x^2)^2 \qquad (E_2)$$

Here $\quad E_1=(3+x-2x^2+3-x+2x^2)(3+x-2x^2-3+x-2x^2)$

$\quad =6(2x-4x^2)=12x(1-2x)$

And $\quad E_2=(3+x+2x^2+3-x-2x^2)(3+x+2x^2-3+x+2x^2)$

$\quad =6(2x+4x^2)=12x(1+2x)$

the product $=12x(1-2x)\times 12x(1+2x)=144x^2(1-4x^2)$

[*Examples* XXXVI a 15–33, *page* 411, *may be taken here*]

462 The following examples shew the advantage of a suitable arrangement of terms in factorizing a certain class of expressions. We shall use the symbol of *identical equality* [Art 100]

EXAMPLE 1 *Find the factors of*

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \qquad (E)$$

Arrange the expression according to powers of a, thus

$$E\equiv a^2(b+c)+a(b^2+c^2+2bc)+bc(b+c)$$
$$\equiv a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$\equiv (b+c)[a^2+a(b+c)+bc]$$
$$\equiv (b+c)(a+b)(a+c)$$
$$\equiv (b+c)(c+a)(a+b),$$

arranging the letters so as to preserve cyclic order [Art 243]

EXAMPLE 2 *Find the factors of*

$$a^3(b-c)+b^3(c-a)+c^3(a-b) \qquad \text{(E)}$$

Arrange the terms according to powers of a, thus

$$\begin{aligned} E &\equiv a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2) \\ &\equiv (b-c)[a^3-a(b^2+bc+c^2)+bc(b+c)] \end{aligned}$$

Now arrange the expression in square brackets according to powers of b, then

$$\begin{aligned} E &\equiv (b-c)[b^2(c-a)+bc(c-a)-a(c^2-a^2)] \\ &\equiv (b-c)(c-a)[b^2+bc-a(c+a)] \\ &\equiv (b-c)(c-a)[c(b-a)+(b^2-a^2)] \\ &\equiv (b-c)(c-a)(b-a)(c+b+a) \\ &\equiv -(b-c)(c-a)(a-b)(a+b+c), \text{ when written cyclically} \end{aligned}$$

NOTE By first arranging in powers of a, we detect the factor $b-c$, which does not contain a The next step reveals the factor $c-a$, which does not contain b, and so on

463 From Example 1, Art 460, we infer that

$$a^3+b^3+c^3-3abc\equiv(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

This important identity enables us to factorize any expression which consists of *the sum of the cubes of three quantities diminished by three times the continued product of the quantities*

Since $a^2+b^2+c^2-bc-ca-ab\equiv\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2+\frac{1}{2}(a-b)^2$,
we have $a^3+b^3+c^3-3abc\equiv\frac{1}{2}(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2]$,
a form which will often be found useful

If $a+b+c=0$, then $a^3+b^3+c^3-3abc=0$,
that is, $a^3+b^3+c^3=3abc$

EXAMPLE 1 *Resolve into factors*

(i) $a^3+b^3-c^3+3abc$, (ii) $8x^3-1-y^3-6xy$

$$\text{(i) } a^3+b^3-c^3+3abc\equiv a^3+b^3+(-c)^3-3ab(-c) \\ \equiv(a+b-c)(a^2+b^2+c^2+bc+ca-ab)$$

$$\text{(ii) } 8x^3-1-y^3-6xy\equiv(2x)^3+(-1)^3+(-y)^3-3(2x)(-1)(-y) \\ \equiv(2x-1-y)(4x^2+1+y^2+2x+2xy-y)$$

EXAMPLE 2 *If* $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, *prove that*

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc)$$

We have $x+y+z=a+b+c$
Also $y-z=2(c-b)$, so that $(y-z)^2=4(c-b)^2$
Similarly $(z-x)^2=4(a-c)^2$, and $(x-y)^2=4(b-a)^2$
Hence

$$\begin{aligned} x^3+y^3+z^3-3xyz &\equiv\tfrac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\} \\ &=4\left[\tfrac{1}{2}(a+b+c)\{(c-b)^2+(a-c)^2+(b-a)^2\}\right] \\ &\equiv4(a^3+b^3+c^3-3abc) \end{aligned}$$

EXAMPLE 3 *If* $(x+a)^2+(y+b)^2=4(ax+by)$, *and* x, y, z *are real quantities, prove that* $x=a$ *and* $y=b$

By transposition, we have $(x-a)^2+(y-b)^2=0$, and since the square of a real quantity cannot be negative, this condition can only be satisfied if $x-a$ and $y-b$ are both zero Hence $x=a$ and $y=b$

NOTE It is important to notice the difference between the conclusions to be drawn from the two statements

$$(x-a)^2+(y-b)^2=0, \tag{1}$$

and

$$(x-a)(y-b)=0 \tag{2}$$

From (1) we infer that *both* $x-a=0$ *and* $y-b=0$ *simultaneously*, while from (2) we infer that *either* $x-a=0$ *or* $y-b=0$

EXAMPLE 4 *If* $2s=a+b+c$, *prove that*

$$s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)-(s-a)(s-b)(s-c)=abc$$

$$\begin{aligned}
\text{The first side} &= s(s-c)[s-b+s-a]+(s-a)(s-b)[s-(s-c)] \\
&= s(s-c)(2s-a-b)+c(s-a)(s-b) \\
&= c[s^2-cs+s^2-(a+b)s+ab] \\
&= c[2s^2-s(a+b+c)+ab] \\
&= abc, \text{ for } s(a+b+c)=s\cdot 2s=2s^2
\end{aligned}$$

NOTE Here $2s$ is a convenient abbreviation of $a+b+c$, and the reduction is much simplified by working in terms of s instead of substituting its value at once

Factors and Identities will be further illustrated, in connection with the Remainder Theorem, in Arts 469–472

EXAMPLES XXXVI. a.

Divide

1. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ by $x^2+(a+b)x+ab$
2. $x^3-(p-q)x^2-(pq+2q^2)x+2pq^2$ by $x^2-(p-2q)x-2pq$.
3. $x^4+(a-3)x^3+(b-3a+2)x^2-(3b-2a)x+2b$ by x^2+ax+b
4. $a^3-b^3-c^3-3abc$ by $a-(b+c)$
5. $a^2x^4+(2ac-b^2)x^2+c^2$ by ax^2+bx+c
6. $x^3-(3mn+n^2)xy^2+m(m^2-n^2)y^3$ by $x+(m+n)y$

Write down the product of

7. $(a-2)x+(a+1)$ and $ax-(a-1)$
8. $(m+3)x+(m-1)y$ and $(m-3)x+(m+1)y$
9. Multiply $x^2-x(a+b)+ab$ by $x-(a-b)$

Find the H C F of

10. $x^3-3ax^2+(2a^2+b^2)x-2ab^2$ and $2x^3-3ax^2-(2a^2+b^2)x+2ab^2$.
11. $(m^2-m-6)x^2-4(2m-1)x-(m^2-m-2)$
 and $(m^2-3m)x^2-(5m-3)x-(m^2-1)$
12. $2x^3-(4a-3)x^2+6(b-a)x+9b$ and $2x^3+(2a+3)x^2+(3a-4b)x-6b$.

Find the H C F and L C M of

13 $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$

14 $p(p+1)x^2+x-p(p-1)$ and $p(p+2)x^2+2x-p^2+1$

By the use of factors, find the product of

15 $5x^2+5xy-9y^2$ and $5x^2-5xy-9y^2$

16 $x^3+2x^2y+2xy^2+y^3$ and $x^3-2x^2y+2xy^2-y^3$

17 x^3-4x^2+8x-8 and x^3+4x^2+8x+8

18 $(1+x+2x^2)^2-(1-x-2x^2)^2$ and $(1+x-2x^2)^2-(1-x+2x^2)^2$

19 $(m^2+6m-2)^2-(m^2-6m+2)^2$ and $(2m^2+3m+1)^2-(2m^2-3m-1)^2$

20 Divide $(4x+3y-2z)^2-(3x-2y+3z)^2$ by $x+5y-5z$

21 Divide the product of $x^2+7x+10$ and $x+3$ by x^2+5x+6

22 Shew that $(3a^2-7a+2)^3-(a^2-8a+8)^3$ is divisible by $2a-3$ and by $a+2$

23 Shew that the sum of the cubes of $2m^2-5m-9$ and m^2+6m-5 is divisible by the product of $3m+7$ and $m-2$

24 Find the continued product of

$$x^2+2x+2,\quad x^2-2x+2,\quad x^2+2,\quad x^2-2,$$

and express x^4+4y^4 as the product of two quadratic factors

Resolve into two or more factors

25 $ab(x^2+1)-x(a^2+b^2)$

26 $3x^2-2ab-x(b-6a)$

27 $m^3-n^3-(x^2-mn)(m-n)$

28 $a(b^2+c^2-a^2)+b(a^2+c^2-b^2)$

29 $y^2z^2(x^4-1)+x^2(y^4-z^4)$

30 $(2a^2+3y^2)x+(2x^2+3a^2)y$

31. $a(a-3)x-(a+6)x-a(a+2)$

32 $a(a+1)x^2+(a+b)x-b(b-1)$

33. Resolve into four factors

(i) $(a^4-2a^2b^2-b^4)^2-4a^4b^4$, (ii) $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$

By a suitable arrangement of terms, as in Art 462, find the factors of

34 $bc(b-c)+ca(c-a)+ab(a-b)$

35 $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

36 $bc(b+c)+ca(c+a)+ab(a+b)+2abc$

37. $a^2(b-c)+b^2(a-c)+c^2(a+b)-2abc$

38 $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$

39 Prove that

$$a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)\equiv(a-b)(b-c)(c-a)(a+b+c)$$

Write down the factors of

40 $a^3+b^3+8c^3-6abc$.

41 $a^3-27b^3+c^3+9abc$

42 $1+27x^3-8y^3+18xy$

43 $x^3-8y^3-27-18xy$

44 Prove that $(b-c)^3+(c-a)^3+(a-b)^3\equiv3(b-c)(c-a)(a-b)$

45 If $a+b+c=0$, shew that

$$(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a).$$

46 When $a=0.3$, $b=0.09$, $c=0.39$, find the value of

$$a(a^2+bc)+b(b^2+ac)-c(c^2-ab)$$

Prove the following identities

47. $(ad+bc)^2+(ac-bd)^2 \equiv (a^2+b^2)(c^2+d^2)$

48. $(a+b)(a^2-ab+b^2)-(a+c)(a^2-ac+c^2) \equiv (b-c)(b^2+bc+c^2)$

49. $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2 \equiv (x^2+y^2)(a^2+b^2+c^2)$

50. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c) \equiv 2(a^2+b^2+c^2)$.

51. $a^2(b-c)+b^2(c-a)+c^2(a-b)+(b-c)(c-a)(a-b) \equiv 0$

52 $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$

$$\equiv (a-b)(b-c)(c-a)(ab+bc+ca)$$
$$\equiv a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$$
$$\equiv -[a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)]$$

If $a+b+c=0$, prove that

53. $a^2+b^2+c^2=2(a^2+ab+b^2)=2(b^2+bc+c^2)=2(a^2+ac+c^2)$

54. $a(a+b)(a+c)=b(b+a)(b+c)=c(c+a)(c+b)$

55. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=0$

56. If $a+b+c=s$, prove that

$$(s-3a)^3+(s-3b)^3+(s-3c)^3=3(s-3a)(s-3b)(s-3c)$$

57. If $A=b+c-2a$, $B=c+a-2b$, $C=a+b-2c$, find the value of

$$A^3+B^3+C^3-3ABC$$

If $2s=a+b+c$, shew that

58. $s(s-a)+s(s-c)+(s-b)(s-c)+(s-a)(s-b)=b(a+c)$

59. $s^2+s(s-a)+s(s-b)+s(s-c)=2s^2$

60 $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$

61. $16s(s-a)(s-b)(s-c)=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

62 If $a^2+d^2=2(ab+bc+cd-b^2-c^2)$, and all the letters denote real quantities, prove that $a=b=c=d$

63. Shew that the equation $(a^2+b^2+c^2)(x^2+y^2+1)=(ax+by+c)^2$ is equivalent to $(bx-ay)^2+(cx-a)^2+(cy-b)^2=0$ Hence shew that $x=a/c$, $y=b/c$ are the only possible real solutions

464 We collect here for reference a list of useful identities, most of which have been embodied in the foregoing examples

(i) $bc(b-c)+ca(c-a)+ab(a-b) \equiv -(b-c)(c-a)(a-b)$.

(ii) $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv -(b-c)(c-a)(a-b)$

(iii) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) \equiv (b-c)(c-a)(a-b)$

(iv) $a^3(b-c)+b^3(c-a)+c^3(a-b) \equiv -(b-c)(c-a)(a-b)(a+b+c)$.

(v) $a^3+b^3+c^3-3abc \equiv (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$$\equiv \tfrac{1}{2}(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2].$$

(vi) $(b-c)^3+(c-a)^3+(a-b)^3 \equiv 3(b-c)(c-a)(a-b)$.

(vii) $bc(b+c)+ca(c+a)+ab(a+b)+2abc \equiv (b+c)(c+a)(a+b)$.

(viii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \equiv (b+c)(c+a)(a+b)$

465. The following example illustrates the advantage of arranging expressions with regard to cyclic order [Art 243]

EXAMPLE *Find the value of*

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

Writing the factors cyclically, we have

$$\text{the L C D} =(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)$$

The numerator $= -a(b-c)(x-b)(x-c)- \quad -$

$$= -a(b-c)\{x^2-(b+c)x+bc\}- \quad -$$

The coefficient of $x^2 = -a(b-c)-b(c-a)-c(a-b)=0$;

the coefficient of $x \quad =a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$

$=(b-c)(c-a)(a-b)$, [Art 464 (iii)]

the other terms $= -abc(b-c)-abc(c-a)-abc(a-b)=0$

$$\text{Hence the expression}=\frac{(b-c)(c-a)(a-b)x}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}$$

$$=\frac{x}{(x-a)(x-b)(x-c)}$$

EXAMPLES XXXVI. b.

1. If $\dfrac{1}{(a-b)(a-c)}=A$, $\dfrac{1}{(b-c)(b-a)}=B$, $\dfrac{1}{(c-a)(c-b)}=C$,

find the value of

(i) $aA+bB+cC$, (ii) $a^2A+b^2B+c^2C$; (iii) $bcA+caB+abC$.

Find the value of

2. $\dfrac{a(b+c)}{(a-b)(c-a)}+\dfrac{b(c+a)}{(b-c)(a-b)}+\dfrac{c(a+b)}{(c-a)(b-c)}$

3. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$

4. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$

5. $\dfrac{bc(a+d)}{(a-b)(a-c)}+\dfrac{ca(b+d)}{(b-c)(b-a)}+\dfrac{ab(c+d)}{(c-a)(c-b)}$

6. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}.$

7. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$

8. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}.$ 9. $\dfrac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{(a-b)(b-c)(c-a)}.$

466 Applications of the Remainder Theorem. So far this theorem has only been proved in a particular case [Art 179] We shall now give a general proof

A *rational integral function of x* has been defined in Art 177, it is always assumed that such functions are arranged in descending powers of x

467 *If any rational integral function* f(x) *is divided by* x − a *until the remainder does not contain* x, *the remainder is* f(a)

On division by $x-a$, let Q be the quotient and R the remainder, then

$$f(x)=Q(x-a)+R$$

Since R does not contain x it will remain the same whatever value we give to x, put $x=a$, then the term $Q(x-a)$ becomes zero, and we have

$$f(a)=R$$

COR. The remainder is zero when the given function is exactly divisible by $x-a$ In this case $R=f(a)=0$, hence

If a rational integral function of x *becomes equal to* 0 *when* a *is written for* x, *it contains* x − a *as a factor*

468 We can now give general proofs of the statements made in Art 176 We suppose n to be positive and integral

I When x^n-y^n is divided by $x-y$, the remainder is y^n-y^n

This is always zero, hence x^n-y^n is always divisible by $x-y$

II When x^n+y^n is divided by $x+y$, the remainder is $(-y)^n+y^n$.

(i) If n is *odd*, the remainder $=-y^n+y^n=0$

(ii) If n is *even*, the remainder $=y^n+y^n=2y^n$

Hence x^n+y^n is divisible by $x+y$ *only when n is odd*

In the same way it may be proved that x^n-y^n is divisible by $x+y$ *when n is even*, and that x^n+y^n is *never* divisible by $x-y$

469 Symmetrical and Alternating Functions. A function is said to be **symmetrical** with respect to any set of letters it contains if its value remains unaltered when any two of these letters are interchanged

Thus $x+y+z$, $bc+ca+ab$, $2(x^3+y^3+z^3)+3xyz$ are symmetrical functions of the first, second and third degrees respectively Again, $a(x^2+y^2)+bxy$ is symmetrical with respect to x and y, but not with respect to a and b

It is obvious that the sum, difference, and product of any two symmetrical functions are also symmetrical functions.

470 A function is said to be **alternating** with respect to any set of letters it contains, if its sign but not its value is altered when any two of those letters are interchanged Thus

$$x-y, \quad a^2(b-c)+b^2(c-a)+c^2(a-b), \quad (b-c)(c-a)(a-b),$$

are alternating functions

It is evident that the product of a symmetrical function and an alternating function must be an alternating function Hence if one alternating function is divided by another, the quotient must be symmetrical For example,

$$(a^3-b^3)\div(a-b)=a^2+ab+b^2$$

471 Symmetrical and alternating functions involving the *sum* of a number of quantities may be concisely denoted by writing down one of the terms and prefixing the symbol Σ, thus Σa stands for the sum of all the terms of which a is the type, Σab stands for the sum of all the terms of which ab is the type, and so on For instance, if the function contains three letters a, b, c,

$$\Sigma a \equiv a+b+c, \quad \Sigma ab \equiv ab+bc+ca,$$

$$\Sigma a^2(b-c) \equiv a^2(b-c)+b^2(c-a)+c^2(a-b)$$

472 Symmetrical and alternating functions involving the *product* of a number of quantities may be concisely denoted by writing down one of the *factors* and prefixing the symbol Π Thus $\Pi(b-c)$ stands for the product of all the factors of which $b-c$ is the type

Thus $\Sigma(b-c)^3 \equiv 3\Pi(b-c)$ is a short way of writing

$$(b-c)^3+(c-a)^3+(a-b)^3 \equiv 3(b-c)(c-a)(a-b)$$

The symbols Σ and Π are the Greek letters "Sigma" and "Pi," corresponding to the English S and P

EXAMPLE 1 *Find the factors of* $a^3(b-c)+b^3(c-a)+c^3(a-b)$ (E)

On trial E vanishes when $b=c$, therefore $b-c$ is a factor Similarly $c-a$, $a-b$ may be shewn to be factors

Thus E is divisible by $(a-b)(b-c)(c-a)$ This is an alternating function of three dimensions, while E is an alternating function of four dimensions Hence the remaining factor is symmetrical and of the first degree, and must therefore be of the form $M(a+b+c)$, where M is some numerical quantity

Hence $a^3(b-c)+b^3(c-a)+c^3(a-b) \equiv M(b-c)(c-a)(a-b)(a+b+c)$

Since M is independent of a, b, c, its value can be found by giving particular values to a, b, c. Let $a=1$, $b=2$, $c=0$, then

$$1\times2+8\times(-1)+0=M\times2\times(-1)\times(-1)\times3, \text{ whence } M=-1$$

$$a^3(b-c)+b^3(c-a)+c^3(a-b) \equiv -(b-c)(c-a)(a-b)(a+b+c)$$

NOTE Care must be taken not to select values which reduce each side to zero

EXAMPLE 2 *Prove the identity*

$$a^4(b-c)+b^4(c-a)+c^4(a-b)\equiv-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$$

Denote the expression on the left by E, then as before we find that E is divisible by $(b-c)(c-a)(a-b)$ This product is alternating and of the third degree, while E is alternating and of the fifth degree Hence the remaining factor must be symmetrical and of the second degree Now the only complete symmetrical function of the second degree in a, b, c is of the form

$$A(a^2+b^2+c^2)+B(bc+ca+ab),$$

where A and B are independent of a, b, c

Hence $a^4(b-c)+b^4(c-a)+c^4(a-b)$

$$\equiv(b-c)(c-a)(a-b)\{A(a^2+b^2+c^2)+B(bc+ca+ab)\}$$

To find A and B we shall require two equations

Putting $a=1$, $b=-1$, $c=0$, we obtain

$$1\times(-1)+1\times(-1)+0=(-1)\times(-1)\times2\{2A-B\}, \text{ or } 2A-B=-1$$

Again, putting $a=1$, $b=2$, $c=0$, we obtain

$$1\times2+16\times(-1)+0=2\times(-1)\times(-1)\{5A+2B\}, \text{ or } 5A+2B=-7.$$

From these equations we find $A=B=-1$

Hence $a^4(b-c)+b^4(c-a)+c^4(a-b)$

$$\equiv-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$$

EXAMPLES XXXVI. c

1 If x^3-4x^2-ax+3 is divisible by $x-3$, find the value of a

2. If $x-2$ is a factor of $2x^3+cx(x-1)-2$, find c

3 Find values of a and b for which x^3+3x^2+ax+b is exactly divisible by $x+2$ and $x+4$.

4. If $x-3$ is a common factor of

$$x^2-(2a+1)x+2b \quad\text{and}\quad x^2-(b+2)x+5a,$$

find the values of a and b

5. Shew that $a-b$, $b-c$, $c-a$ are factors of

(i) $a^2c-a^2b+ab^2-b^2c+bc^2-ac^2$, (ii) $\Sigma a(b^3-c^3)$

6. Write down the following expressions in full, and find their factors

(i) $\Sigma a^2(b-c)$, (ii) $\Sigma a^3(b-c)$, (iii) $\Sigma a^2(b-c)^4$

Prove the following identities

7. $x^2(y^3-z^3)+y^2(z^3-x^3)+z^2(x^3-y^3)\equiv(x-y)(y-z)(z-x)(yz+zx+xy)$

8. $(a+b)^5-a^5-b^5\equiv5ab(a+b)(a^2+ab+b^2)$

9. $\Sigma bc(b+c)+2abc\equiv\Pi(b-c)$ 10. $\Sigma a^2(b+c)+3abc\equiv(\Sigma a)(\Sigma bc)$

11. $(x+y+z)^5-x^5-y^5-z^5\equiv5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy)$.

Prove that

12 $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4 \equiv 12abc(a+b+c)$

13. $(bc+ca+ab)^3-b^3c^3-c^3a^3-a^3b^3 \equiv 3abc(b+c)(c+a)(a+b)$

14 Find the value of

(i) $\dfrac{\Sigma a^3(b-c)}{\Sigma(b-c)^3}$, (ii) $\dfrac{\Sigma a^2(b-c)^3}{\Pi(b-c)}$, (iii) $\dfrac{\Sigma(y-z)^5}{\Sigma(y-z)^3}$ [*Three letters, x, y, z, being involved*]

Undetermined Coefficients.

473 *If a rational integral function of* n *dimensions in* x *vanishes for more than* n *different values of* x, *the coefficient of each power of* x *is zero*

Suppose that $f(x)=Ax^n+Bx^{n-1}+Cx^{n-2}+ \quad +K$ is a function which vanishes when x is equal to each of the n unequal values a_1, a_2, a_3, a_n Then by the Remainder Theorem $x-a_1$, $x-a_2$, $x-a_n$ must each be a factor of $f(x)$ Now these factors are all different, and the highest power of x in their product is x^n

$$f(x)=A(x-a_1)(x-a_2)(x-a_3) \quad (x-a_n)$$

Suppose c is another value of x which makes $f(x)$ vanish, then since $f(c)=0$, we have

$$A(c-a_1)(c-a_2)(c-a_3) \quad (c-a_n)=0,$$

$$A=0, \text{ since none of the other factors is zero}$$

Hence $f(x)$ reduces to $Bx^{n-1}+Cx^{n-2}+ \quad +K$, which, by hypothesis, vanishes for more than $n-1$ values of x, and therefore $B=0$ Similarly each of the coefficients may be shewn to be zero

474 *If two rational integral functions of* n *dimensions in* x *are equal for more than* n *values of* x, *they are equal for every value of* x.

If $Ax^n+Bx^{n-1}+Cx^{n-2}+ \quad +K=ax^n+bx^{n-1}+cx^{n-2}+ \quad +l$

for more than n values of x, then

$$(A-a)x^n+(B-b)x^{n-1}+(C-c)x^{n-2}+ \quad +(K-l)$$

vanishes for more than n values of x, and therefore by the preceding article,

$$A-a=0, \quad B-b=0, \quad C-c=0, \quad ,$$

that is, $\quad A=a, \quad B=b, \quad C=c, \quad K=l$

Thus the functions are *identical*, and therefore equal for every value of x Hence the following conclusion

If two rational integral functions of x *are identically equal, we may equate the coefficients of like powers of* x

This is known as the **Principle of Undetermined Coefficients.**

475. Examples in the Use of Undetermined Coefficients.

EXAMPLE 1. *Find values of* a, b, *and* c *such that* $x^2-6x-15$ *may be equal to*

$$a(x-1)(x+1)+bx(x+1)+c(x-3)(x+2)$$

for all values of x.

Let $\quad x^2-6x-15 \equiv a(x-1)(x+1)+bx(x+1)+c(x-3)(x+2)$; .....(1)

that is, $\quad x^2-6x-15 \equiv a(x^2-1)+b(x^2+x)+c(x^2-x-6)$

$$\equiv(a+b+c)x^2+x(b-c)-(a+6c).$$

Equating coefficients of like powers, we have

$$a+b+c=1, \quad (b-c)=-6, \quad a+6c=15;$$

whence $\qquad a=3, \quad b=-4, \quad c=2$

Since (1) is true for all values of x, we may also find a, b, c by giving different numerical values to x in this identity. This method is shewn in the next example

EXAMPLE 2 *If* $\dfrac{4x^2-5x-1}{(x-2)(x-1)^2} \equiv \dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, *find the values of* A, B, C

Clearing of fractions, we have

$$4x^2-5x-1 \equiv A(x-1)^2+B(x-2)(x-1)+C(x-2).$$

A, B, C may now be found by equating coefficients; but the following method is simpler

Put $x=1$, then $4-5-1=(-1)C$, or $C=2$.

Put $x=2$, then $16-10-1=A$, or $A=5$

To find B, equate the coefficients of x^2, then $4=A+B$.

Hence $\qquad 4=5+B$, or $B=-1$

Thus $\qquad \dfrac{4x^2-5x-1}{(x-2)(x-1)^2} \equiv \dfrac{5}{x-2}-\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$

EXAMPLE 3 *Express* $6x^2-11xy+3y^2+19x-11y+10$ *as the product of two factors of the first degree*

Since $6x^2-11xy+3y^2=(3x-y)(2x-3y)$, we may assume

$$6x^2-11xy+3y^2+19x-11y+10 \equiv (3x-y+a)(2x-3y+b)$$

The terms of two dimensions are the same on each side, hence writing down only the linear and constant terms, we have

$$19x-11y+10 \equiv a(2x-3y)+b(3x-y)+ab$$
$$\equiv(2a+3b)x-(3a+b)y+ab$$

Hence $\qquad 2a+3b=19, \quad 3a+b=11, \quad ab=10$

Unless these three equations are satisfied by the same values of a and b there are no factors Here the first two equations give $a=2$, $b=5$, and these values satisfy the third

Thus the required factors are $3x-y+2$ and $2x-3y+5$

[*Examples* XXXVI d. 1-14, *page* 420, *may be taken here*]

476 Involution and Evolution by Undetermined Coefficients.

EXAMPLE 1 *To find the expanded form of* $(x+y+z)^3$.

The expression must be a homogeneous symmetrical expression of three dimensions Hence we may assume

$$(x+y+z)^3 \equiv x^3+y^3+z^3+A(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+Bxyz,$$

where A and B are independent of x, y, z

Put $z=0$, then A = coefficient of x^2y in $(x+y)^3$, that is, $A=3$

Put $x=y=z=1$, then $27=3+(3\times 6)+B$, whence $B=6$

Thus $(x+y+z)^3 \equiv x^3+y^3+z^3+3(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+6xyz$

EXAMPLE 2 *If* $27x^6+108x^5+90x^4-80x^3-60x^2+48x-8$ *is a perfect cube, find its cube root*

The cube root must be an expression of the second degree, and its first and last terms are clearly $3x^2$ and -2 Hence we may assume

$$(3x^2+ax-2)^3 \equiv 27x^6+108x^5+90x^4-80x^3-60x^2+48x-8 \qquad (1)$$

To find a it will be sufficient to expand the expression on the left as far as the term containing x^5

Now $\{3x^2+(ax-2)\}^3=(3x^2)^3+3(3x^2)^2(ax-2)+$
$\qquad =27x^6+27ax^5+$ terms in $x^4, x^3,$

Hence, equating coefficients of x^5 in (1), $27a=108$, or $a=4$

Thus the cube root is $3x^2+4x-2$

To make the solution complete, $3x^2+4x-2$ should be cubed (using Detached Coefficients) and the result compared with the given expression

EXAMPLE 3 *If* $x^4+px^3+qx^2+rx+s$ *is a perfect square for all values of* x, *prove that* $r^2=p^2s$, *and* $\left(q-\frac{p^2}{4}\right)^2=4s$

The square root must clearly be of the form x^2+Ax+B

Assume $x^4+px^3+qx^2+rx+s \equiv (x^2+Ax+B)^2$,

then, on expanding the expression on the right,

$$x^4+px^3+qx^2+rx+s \equiv x^4+2Ax^3+x^2(A^2+2B)+2ABx+B^2.$$

By equating the coefficients of like powers of x, we have

(i) $2A=p$, (ii) $A^2+2B=q$, (iii) $2AB=r$, (iv) $B^2=s$

The necessary relations between $p, q, r,$ and s will be obtained by eliminating A and B from these equations

From (i), (iii), and (iv), $\qquad \frac{r^2}{p^2}=B^2=s$,

that is, $\qquad r^2=p^2s$

Again, from (i), (ii), and (iv), $q-\frac{p^2}{4}=2B=2\sqrt{s}$:

that is, $\qquad \left(q-\frac{p^2}{4}\right)^2=4s$

477 The following examples deserve special attention

EXAMPLE 1. *If α, β, γ are the roots of the equation* $x^3+px^2+qx+r=0$,

$$\alpha+\beta+\gamma=-p, \quad \alpha\beta+\beta\gamma+\gamma\alpha=q, \quad \alpha\beta\gamma=-r$$

We have $$(x-\alpha)(x-\beta)(x-\gamma)\equiv x^3+px^2+qx+r \qquad (1)$$

But the product of the factors on the left is

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma;$$

by equating the coefficients of like powers in (1), we obtain the required result

NOTE It easily follows that if the equation is given in the form $Ax^3+Bx^2+Cx+D=0$, then

$$\alpha+\beta+\gamma=-\frac{B}{A}, \quad \alpha\beta+\beta\gamma+\gamma\alpha=\frac{C}{A}, \quad \alpha\beta\gamma=-\frac{D}{A}$$

EXAMPLE 2 *Find the sum of the cubes of the roots of* $x^3+qx+r=0$

Let α, β, γ be the roots, then, since the coefficient of x^2 is zero,

$$\alpha+\beta+\gamma=0; \quad \text{also} \quad \alpha\beta\gamma=-r$$

But $$\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma, \quad \text{when} \quad \alpha+\beta+\gamma=0 \qquad \text{[Art 463]}$$

Hence $$\alpha^3+\beta^3+\gamma^3=-3r$$

EXAMPLES XXXVI. d.

1. If $2(x^2+3x)\equiv A(x^2+1)+Bx(x-1)+C$, find A, B, and C

2. Express $4x^2+x-1$ in the form $A+B(x+1)+Cx(x+1)$

3. If $3x^2+5x+7\equiv l(x+1)(x-2)+m(x+1)+n$, for all values of x, find l, m, and n

4. Find values of A, B, and C so that
$$A(n-1)^2+B(n-1)(n+1)+C(n+1)^2$$
may be equal to $4n^2$, for all values of n

Find the values of A, B, C which make the following statements identically true

5. $\dfrac{x+10}{(x+2)(x-2)}\equiv\dfrac{A}{x-2}+\dfrac{B}{x+2}$

6. $\dfrac{2x-1}{(x+2)(x+1)}\equiv\dfrac{A}{x+2}+\dfrac{B}{x+1}$

7. $\dfrac{4x-13}{2x^2+x-6}\equiv\dfrac{A}{x+2}+\dfrac{B}{2x-3}$

8. $\dfrac{41x-40}{(2x+1)(x-5)^2}\equiv\dfrac{A}{2x+1}+\dfrac{Bx+C}{(x-5)^2}$

9. $\dfrac{5x^2+9x-32}{(x-1)(x+2)(x-3)}\equiv\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{x-3}$

10. $\dfrac{x^2+5x+14}{(x-2)(x^2-x+1)}\equiv\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+x+1}$

11. Express $3x^3-x+2$ as an integral function of $x+1$.

Find the linear factors of the following expressions

12 $x^2+2xy-8y^2+4x-2y+3$ 13 $2x^2-9xy-5y^2+22x+20$

14. For what value of k can $5x^2+13xy-6y^2-7x+13y+k$ be resolved into two linear factors?

15 Find the square roots of the following expressions

(i) $4+12x-7x^2-12x^3+34x^4-24x^5+9x^6$,

(ii) $16x^3-16x^6+8x^5-36x^4-4x^3+21x^2-10x+25$

16 Find the cube root of

$$27x^6-27x^5+171x^4-109x^3+342x^2-108x+216$$

17 For what value of A will

$$25x^4-30ax^3+49a^2x^2-24a^3x+A$$

be a perfect square?

18 Find l and m so that $9x^4-6x^3+13x^2+lx+m$ may be a perfect square

19 If x^3+px^2+qx+r is divisible by x^2+ax+b, prove that

$$q-b=a(p-a) \quad \text{and} \quad r=b(p-a)$$

20 If $4x^4+12x^3y+Px^2y^2+6xy^3+y^4$ is a perfect square, find P

21 If $x^4-ax^3+bx^2-cx+1$ is a perfect square for all values of x prove that $a=c$ and $b=\frac{a^2}{4}+2$

22 If α, β, γ are the roots of the equation $px^3+qx^2+rx+s=0$, express (i) $\alpha^2+\beta^2+\gamma^2$, (ii) $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma$, in terms of p, q, r, and s

(*Miscellaneous*)

23 Prove the identities

(i) $2(bc+ca+ab)^3-a^3(b+c)^3-b^3(c+a)^3-c^3(a+b)^3\equiv 2abc(a+b+c)$,

(ii) $(a^2+b^2+c^2)^3+2(bc+ca+ab)^3-3(a^2+b^2+c^2)(bc+ca+ab)^2$
$$\equiv(a^3+b^3+c^3-3abc)^2$$

24. If p, q, r are positive quantities, and the equations

$$x^4+px^3+qx^2+rx+1=0, \quad x^4+rx^3+qx^2+px+1=0,$$

have a common root, prove that $p+r=q+2$

25 If a, b, c are the roots of $x^3+px^2+qx+r=0$, and s denotes their sum, prove that

$$(s-a)(s-b)(s-c)=r-pq$$

26 Express $(a-d)^2(b-c)+(b-d)^2(c-a)+(c-d)^2(a-b)$ as a continued product

27 Shew that ax^3+bx^2+cx+d is a perfect cube if $b^3=27a^2d$, $c^3=27ad^2$

28. If x^3+px+r and $3x^2+p$ have a common factor, then $\frac{p^3}{27}+\frac{r^2}{4}=0$.

CHAPTER XXXVII

The Progressions and some Allied Series

478 We shall now discuss certain series which are closely allied to arithmetic and geometric progressions though not falling exactly under the rules of either. A few easy cases of such series have already been given in Chapter XXIX The formulæ on pages 304, 313, 315 should here be carefully revised

479 If we multiply together corresponding terms of an arithmetic and geometric series, such as

$$\begin{array}{l} a+(a+d)+(a+2d)+ \quad +\{a+(n-1)d\}, \\ 1+ \quad r \quad + \quad r^2 \quad + \quad + \quad r^{n-1}, \end{array}$$

we obtain a new series

$$a+(a+d)r+(a+2d)r^2+ \ \ . \ +\{a+(n-1)d\}r^{n-1},$$

which is known as an **Arithmetico-Geometric Series** A series of this kind may be summed by the same device as that used in finding the sum of a G P

480 *To find the sum of* n *terms of the series*

$$a+(a+d)r+(a+2d)r^2+ \qquad +(a+\overline{n-1}d)r^{n-1}.$$

Let $S=a+(a+d)r+(a+2d)r^2+ \quad +(a+\overline{n-1}d)r^{n-1}$;

$rS= \qquad ar+ (a+d)r^2+ \quad +(a+\overline{n-2}d)r^{n-1}+(a+\overline{n-1}d)r^n.$

By subtraction, we have

$$(1-r)S=a+(dr+dr^2+ \qquad +dr^{n-1})-(a+\overline{n-1}d)r^n$$

$$=a+\frac{dr(1-r^{n-1})}{1-r}-(a+\overline{n-1}d)r^n,$$

$$S=\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+\overline{n-1}d)r^n}{1-r}.$$

Cor Write S in the form

$$\frac{a}{1-r}+\frac{dr}{(1-r)^2}-\frac{dr^n}{(1-r)^2}-\frac{(a+\overline{n-1}d)r^n}{1-r},$$

then if $r<1$, we can make r^n as small as we please by making n sufficiently great If the term $(n-1)dr^n$ can also be made so small that it may be neglected, then in that case we obtain $\frac{a}{1-r}+\frac{dr}{(1-r)^2}$ as the sum to infinity

481 The formula for S in the last article should not be quoted Each example should be dealt with independently

EXAMPLE 1 *Sum the series* $1+2x+3x^2+4x^3+\dots$ *to* n *terms*

Let $\quad S=1+2x+3x^2+\dots+nx^{n-1},$

then $\quad xS=\quad x+2x^2+\dots-(n-1)x^{n-1}+nx^n$

By subtraction, $(1-x)S=(1+x+x^2+\dots+x^{n-1})-nx^n$

$$=\frac{1-x^n}{1-x}-nx^n;$$

$$S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$$

EXAMPLE 2 *If* $x<1$, *sum the series* $1-3x+5x^2-7x^3+\dots$ *to infinity*

Let $\quad S=1-3x+5x^2-7x^3+\dots$ to infinity,

then $\quad -xS=\quad -x+3x^2-5x^3+\dots$ „

By subtraction, $(1+x)S=1-2x+2x^2-2x^3-\dots$ „

$$=1-2x(1-x+x^2-x^3+\dots)$$

$$=1-\frac{2x}{1+x}=\frac{1-x}{1+x},$$

$$\therefore S=\frac{1-x}{(1+x)^2}$$

EXAMPLE 3 *Sum the following series to infinity*

$$1+\frac{5}{3}+\frac{12}{3^2}+\frac{22}{3^3}+\frac{35}{3^4}+\dots$$

Here the coefficients 1, 5, 12, 22, 35, ... are not in A P, but their differences 4, 7, 10, 13, ... form an A P, hence the series may be summed by a double application of the foregoing method

Let $\quad S=1+\frac{5}{3}+\frac{12}{3^2}+\frac{22}{3^3}+\frac{35}{3^4}+\dots$

then $\quad \frac{1}{3}S=\quad \frac{1}{3}+\frac{5}{3^2}+\frac{12}{3^3}+\frac{22}{3^4}+\dots$

By subtraction, $\quad \frac{2}{3}S=1+\frac{4}{3}+\frac{7}{3^2}+\frac{10}{3^3}+\frac{13}{3^4}+\dots$

Multiply again by $\frac{1}{3}$, then

$$\frac{2}{9}S=\quad \frac{1}{3}+\frac{4}{3^2}+\frac{7}{3^3}+\frac{10}{3^4}+\dots$$

By subtraction, $\left(\frac{2}{3}-\frac{2}{9}\right)S=1+\frac{3}{3}+\frac{3}{3^2}+\frac{3}{3^3}+\frac{3}{3^4}+\dots$

$$=1+\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\right).$$

That is, $\quad \frac{4}{9}S=1+\frac{1}{1-\frac{1}{3}}=\frac{5}{2};$

$$S=\frac{45}{8}=5\tfrac{5}{8}$$

482 In Art 319 we found a formula for the sum of the first n natural numbers If we denote this sum by Σn, we have

$$\Sigma n = 1+2+3+4+ \quad +n = \frac{n(n+1)}{2}$$

In the same way Σn^2 and Σn^3 may be used to denote the sum of the squares and the sum of the cubes of the first n natural numbers

Thus $$\Sigma n^2 = 1^2+2^2+3^2+4^2+ \quad +n^2,$$

$$\Sigma n^3 = 1^3+2^3+3^3+4^3+ \quad +n^3$$

483 *To find the sum of* $1^2+2^2+3^2+4^2+ \quad +n^2$

We have identically

$$n^3-(n-1)^3 = 3n^2-3n+1,$$

and by writing $n-1$ in the place of n,

$$(n-1)^3-(n-2)^3 = 3(n-1)^2-3(n-1)+1,$$

similarly $$(n-2)^3-(n-3)^3 = 3(n-2)^2-3(n-2)+1;$$

$$3^3-2^3 = 3 \cdot 3^2 - 3 \cdot 3+1,$$

$$2^3-1^3 = 3 \cdot 2^2 - 3 \cdot 2+1,$$

$$1^3-0^3 = 3 \cdot 1^2 - 3 \cdot 1+1$$

In adding these results we note that all the terms on the left disappear except n^3, on the right we consider the sums of the three columns separately

Hence $$n^3 = 3(1^2+2^2+3^2+ \quad +n^2) - 3(1+2+3+ \quad +n)+n$$

$$= 3\Sigma n^2 - \frac{3n(n+1)}{2} + n$$

$$3\Sigma n^2 = n^3 - n + \frac{3n(n+1)}{2} = n(n+1)\left(n-1+\frac{3}{2}\right)$$

$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE *Sum the series* $1 \cdot 4+2 \cdot 7+3 \cdot 10+$ *to* n *terms*

Here the n^{th} term of 1, 2, 3, is n, the n^{th} term of 4, 7, 10 . is $3n+1$, hence the n^{th} term of the given series $=n(3n+1)=3n^2+n$

By writing $n=1, 2, 3,$ in succession, we obtain

$$3(1^2+2^2+3^2+ \quad +n^2)+(1+2+3+ \quad +n)$$

$\therefore$ the required sum $=3\Sigma n^2+\Sigma n$

$$= \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2} = n(n+1)^2$$

The above summation may be shortly outlined as follows

n^{th} term $=3n^2+n$, sum to n terms $=3\Sigma n^2+\Sigma n$.

484 *To find the sum of* $1^3+2^3+3^3+4^3+\ \dots\ +n^3$

We have $\quad n^4-(n-1)^4=4n^3-6n^2+4n-1\,,$

$$(n-1)^4-(n-2)^4=4(n-1)^3-6(n-1)^2+4(n-1)-1\,,$$
$$(n-2)^4-(n-3)^4=4(n-2)^3-6(n-2)^2+4(n-2)-1\,;$$
$$\vdots$$
$$3^4-2^4=4\cdot 3^3-6\cdot 3^2+4\cdot 3-1\,,$$
$$2^4-1^4=4\cdot 2^3-6\cdot 2^2+4\cdot 2-1\,,$$
$$1^4-0^4=4\cdot 1^3-6\cdot 1^2+4\cdot 1-1$$

Hence, by addition,

$$n^4=4\Sigma n^3-6\Sigma n^2+4\Sigma n-n\,,$$
$$\begin{aligned}4\Sigma n^3&=n^4+n+n(n+1)(2n+1)-2n(n+1)\\&=n(n+1)(n^2-n+1+2n+1-2)\\&=n(n+1)(n^2+n)\,,\end{aligned}$$
$$\Sigma n^3=\frac{n^2(n+1)^2}{4}=\left\{\frac{n(n+1)}{2}\right\}^2$$

Thus *the sum of the cubes of the first* n *natural numbers is equal to the square of the sum of these numbers*

EXAMPLE *Sum to* n *terms the series whose* n[th] *term is* $n^2(2n-3)$

$$\text{The } n^{\text{th}} \text{ term}=2n^3-3n^2$$
$$\begin{aligned}\text{the required sum}&=2\Sigma n^3-3\Sigma n^2\\&=\frac{n^2(n+1)^2}{2}-\frac{n(n+1)(2n+1)}{2}\\&=\frac{n(n+1)(n^2-n-1)}{2}\end{aligned}$$

EXAMPLES XXXVII a

Find the n^{th} term and the sum to n terms of the following series. Check each result by putting $n=2$

1. $1+3x+5x^2+7x^3+\dots$
2. $1+2\cdot 3+3\cdot 3^2+4\cdot 3^3+\dots$
3. $1+4x+7x^2+10x^3+\dots$
4. $1+3\cdot 2+5\cdot 2^2+7\cdot 2^3+\dots$

Sum the following series to infinity

5. $1+2x+3x^2+\dots\quad (x<1)$
6. $1+5a+9a^2+13a^3+\dots\quad (a<1)$
7. $1-\frac{2}{5}+\frac{3}{5^2}-\frac{4}{5^3}+\dots$
8. $\frac{1}{3}-\frac{2}{3^2}+\frac{3}{3^3}-\frac{4}{3^4}+\dots$
9. $6r+11r^2+16r^3+\dots\quad (r<1)$
10. $1^2+2^2x+3^2x^2+\dots\quad (x<1)$

11. Shew that $1+\frac{2}{5}+\frac{3}{5^2}+\ \dots\ +\frac{n}{5^{n-1}}=\frac{5^{n+1}-4n-5}{16\cdot 5^{n-1}}$

Write down the n^{th} term, and thence find the sum to n terms of the following series

12. $1^2+3^2+5^2+7^2+\dots$

13. $1\cdot 2+2\cdot 3+3\cdot 4+\dots$

14. $1\cdot 3+2\cdot 5+3\cdot 7+\dots$

15. $1\cdot 4+4\cdot 7+7\cdot 10+\dots$

16. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\dots$

17. $1\cdot 4\cdot 7+2\cdot 5\cdot 8+3\cdot 6\cdot 9+\dots$

Sum to n terms the series whose n^{th} terms are

18. $n(n+3)$

19. $6n^2+2n$

20. $6n^2-2^n$.

21. $4n^3-3n^2$

22. n^3-3^n

23. $2n(n+1)(2n+1)$

Further Exercises on the Progressions.

485 The progressions furnish an endless variety of problems All of these can be shewn to depend ultimately on the fundamental properties of the three progressions as given in Chap XXIX A shorter and neater solution, however, can often be obtained by using some of the properties enumerated in the following articles

486 Three quantities a, b, c are in arithmetic, geometric, or harmonic progression according as

$$\text{(i)}\ \frac{a-b}{b-c}=\frac{a}{a}, \qquad \text{(ii)}\ \frac{a-b}{b-c}=\frac{a}{b}, \qquad \text{(iii)}\ \frac{a-b}{b-c}=\frac{a}{c}.$$

For (i) readily gives $b-a=c-b$; $\quad a, b, c$ are in A P

(ii) „ „ $b^2=ac$, $\quad a, b, c$ „ G P.

From (iii), $\quad \dfrac{a-b}{a}=\dfrac{b-c}{c}$, or $\dfrac{a-b}{ab}=\dfrac{b-c}{bc}$,

that is, $\quad \dfrac{1}{b}-\dfrac{1}{a}=\dfrac{1}{c}-\dfrac{1}{b}$, $\quad a, b, c$ are in H P

487 The following points should also be noticed·

(i) If a series of terms in A.P are all increased or all decreased by the same quantity, the resulting terms form another A P with the same common difference as before

(ii) If a series of terms in A P are all multiplied or all divided by the same quantity the resulting terms form another A.P, but with a new common difference

These two results follow at once from the definition of arithmetic progression

(iii) If a series of quantities a, b, c, d, ... are in G P they are also in continued proportion

For $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\dots=\frac{1}{r},$$

where r is the common ratio of the G P.

488 If A G, H are the arithmetic, geometric, and harmonic means between a and b, we have

$$A=\frac{a+b}{2},\quad G=\sqrt{ab},$$

$$A-G=\frac{a+b-2\sqrt{ab}}{2}=\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2,$$

which is positive if a and b are positive Hence *the arithmetic mean of any two positive quantities is greater than their geometric mean*

Again, since $AH=G^2$ (Art 327), and $A>G$, it follows that $H<G$; hence A, G, H are in descending order of magnitude

EXAMPLE 1 *If* a, b, c, d, e, f *are in* G P, *prove that*

$$\left(\frac{b-d}{c-e}\right)^3=\frac{a-c}{d-f}$$

We have
$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}$$
$$=\frac{a-c}{b-d}=\frac{b-d}{c-e}=\frac{c-e}{d-f},$$
$$\left(\frac{b-d}{c-e}\right)^3=\frac{a-c}{b-d}\cdot\frac{b-d}{c-e}\cdot\frac{c-e}{d-f}=\frac{a-c}{d-f}$$

EXAMPLE 2. *If* p, q, r *are in* G P, *then* q − p, 2q, q − r *are in* H P.

We have
$$\frac{(q-p)-2q}{2q-(q-r)}=\frac{-(p+q)}{q+r}$$

But since $\frac{p}{q}=\frac{q}{r}$, each ratio $=\frac{p+q}{q+r}=\frac{p-q}{q-r}$,

$$\frac{(q-p)-2q}{2q-(q-r)}=\frac{q-p}{q-r},$$

that is, $q-p$, $2q$, $q-r$ are in H P [Art. 486 (iii)]

EXAMPLE 3 *If* $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ *are in* H P, *prove that* a, b, c *are also in* H P

We have
$$\frac{b+c}{a},\quad \frac{c+a}{b},\quad \frac{a+b}{c}\ \text{in A P},$$
$$1+\frac{b+c}{a},\quad 1+\frac{c+a}{b},\quad 1+\frac{a+b}{c}\ \text{are in A P},$$
$$\frac{a+b+c}{a},\quad \frac{b+c+a}{b},\quad \frac{c+a+b}{c}\ \text{are in A P},$$
$$\frac{1}{a},\quad \frac{1}{b},\quad \frac{1}{c}\ \text{are in A.P.};$$

that is, a, b, c are in H P

EXAMPLES XXXVII. b.

(Miscellaneous Examples on the Progressions.)

Sum the following series:

1. $\dfrac{3+2\sqrt{2}}{3-2\sqrt{2}},\ 1,\ \dfrac{3-2\sqrt{2}}{3+2\sqrt{2}},\ \ldots$ to infinity.

2. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1},\ 2,\ \dfrac{\sqrt{3}+1}{\sqrt{3}-1},\ 2(\sqrt{3}+1),\ \ldots$ to 7 terms.

3. If p times the p^{th} term of an A. P. is equal to q times the q^{th} term, prove that the $(p+q)^{\text{th}}$ term must be zero.

4. If x, y, z are in G. P., prove that $x^2y^2z^2(x^{-3}+y^{-3}+z^{-3})=x^3+y^3+z^3$.

5. Prove that the ratio of the sum of x arithmetic means to the sum of y arithmetic means between any two numbers is $x:y$.

6. In an A. P. shew that the *sum* of any two terms equidistant from the beginning and end is constant, and that in a G. P. the *product* of two such terms is constant.

7. If p is the product of n terms in G. P. of which a is the first and l the last term, prove that $p=(al)^{\frac{n}{2}}$.

8. If a, b, c are in A. P., p, q, r in H. P., and ap, bq, cr in G. P., then
$$\frac{p}{r}+\frac{r}{p}=\frac{a}{c}+\frac{c}{a}.$$

9. The difference between two numbers is 6 and the sum of the five arithmetic means between them is 20; what are the numbers?

10. Shew that $\dfrac{2}{3}+\dfrac{3}{9}+\dfrac{4}{27}+\dfrac{5}{81}+\ldots$ to n terms $=\dfrac{5}{4}-\dfrac{1}{4}\cdot\dfrac{2n+5}{3^n}$.

11. If a^2, b^2, c^2 are in A. P., prove that $b+c,\ c+a,\ a+b$ are in H. P.

12. Prove that

(i) $a^2(b+c),\ b^2(c+a),\ c^2(a+b)$ are in A. P., if a, b, c are in A. P.;

(ii) $a(b+c),\ b(c+a),\ c(a+b)$ are in A. P., if a, b, c are in H. P.

13. Sum to n terms

(i) $x+y,\ \ x^2+xy+y^2,\ \ x^3+x^2y+xy^2+y^3,\ \ldots\ldots$;

(ii) $1+11+111+1111+\ldots\ldots$

14. A man has charge of 23 machines, each of which when started works automatically, and produces 6.5 yds. of material per hour. The first machine starts at 9 a.m., and the others at intervals of 5 minutes. Find, to the nearest yard, the length produced by 1 p.m.

15. A man pays the principal of a debt by annual instalments: £120 the first year, then each year 10% more than the year before. How much will he pay in 10 years? [*Use logarithms.*]

16. If P, Q, R are the p^{th}, q^{th}, and r^{th} terms of an A. P., then
$$p(Q-R)+q(R-P)+r(P-Q)=0.$$

17. If a, l are the first and last terms of two series, each of n terms, one in A.P. and the other in G.P., and if p, P are two corresponding terms in the two series, prove that $P^{l-a}=a^{l-p}\, l^{p-a}$.

18. If $s_1, s_2, s_3, \ldots s_{2n}$ are the sums respectively of n terms of $2n$ arithmetic progressions which have the same first term and common differences $d, 2d, 3d, \ldots 2nd$, shew that

$$(s_2+s_4+s_6+\ldots+s_{2n})-(s_1+s_3+s_5+\ldots+s_{2n-1})=\tfrac{1}{2}n^2(n-1)d.$$

19. The arithmetic mean between two numbers is 27, and their harmonic mean is 12; find the geometric mean.

20. Two men, A and B, 165 miles distant from each other, set out to meet each other; A travels one mile the first day, two the second, three the third, and so on; B travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they meet?

Give a meaning for each answer to this question.

21. A number of persons were engaged to do a piece of work which would have occupied them 24 hours if they had all begun at the same time; but instead of doing so they began at equal intervals, and then continued to work till the whole was finished, the payment being proportional to the work done by each. If the first comer received eleven times as much as the last, find the time occupied.

22. Sum the following series, each to n terms:

(i) $1.2.4+2.3.5+3.4.6+\ldots$;

(ii) $1+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2}+\frac{1}{2^2}\right)+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right)+\ldots$

23. If a, b, c are in H.P., so also are $\dfrac{a}{b-c-a}$, $\dfrac{b}{c+a-b}$, $\dfrac{c}{a-b-c}$.

24. By using the result of Art. 488, prove that

(i) $(ab+cd)(ac+bd)>4abcd$, (ii) $(b+c)(c+a)(a+b)>8abc$,

where a, b, c, d are any unequal positive quantities.

25. The sum of n terms of an A.P. is $n(b^2+x^2)-n(n-3)bx$; find the n^{th} term, and determine the series.

26. If $\qquad S_1=a+ar+ar^2+\ldots+ar^{n-1}$,

and $\qquad S_2=a^2+a^2r^2+a^2r^4+\ldots+a^2r^{2n-2}$,

shew that $\qquad (r+1)S_2-(r-1)S_1^2=2aS_1$.

27. Find the $(p+q)^{\text{th}}$ term of the H.P. whose p^{th} and q^{th} terms are P and Q.

28. If there are n quantities in G.P. whose common ratio is r, and S_n = the sum of the first n terms, prove that the sum of their products taken two at a time is $\dfrac{r}{r+1}S_n\, S_{n-1}$.

[*Note that* $(a+b+c+\ldots)^2-(a^2+b^2+c^2+\ldots)=2(ab+ac+bc+\ldots)$.]

CHAPTER XXXVIII

HARDER GRAPHS

489 THE principal graphs discussed in previous chapters may be summed up as follows.

(i) **Straight Line** Equation of the general form $y=ax+b$, and particular form $y=ax$, when the line passes through the origin [See Chap XI, and in particular Art 134]

(ii) **Circle** Equation of the form $x^2+y^2=a^2$, where a is the radius, and the origin is at the centre of the circle. [Art. 273]

(iii) **Parabola** Equation of the general form $y=ax^2+bx+c$; and particular form $y=ax^2$, when the vertex is at the origin [See Chap XXIV, and in particular Arts 265, 266]

(iv) **Rectangular Hyperbola** Equation of the form $xy=c$. [Arts 271, 272]

We shall now discuss some miscellaneous graphs which do not fall directly under any of the foregoing heads

EXAMPLE 1 *Draw the graph of* $y=x^3$ *Hence find the real roots of the equations* (i) $x^3-2.5x-3=0$, (ii) $x^3-3x+2=0$

From the form of the equation it is evident that for any point (x, y) on the curve there is a corresponding point $(-x, -y)$ which satisfies the equation Thus the following tabulated values may be used, choosing the unit for x five times as great as that for y

x	0	± 0.5	± 1	± 1.5	± 2	± 2.5	± 3
y	0	± 0.125	± 1	± 3.375	± 8	± 15.625	± 27

The graph is shewn in Fig 34 on the opposite page

To solve the given equations, we have now to find the points of intersection of

(1) $y=x^3$, $y=2.5x+3$, (2) $y=x^3$, $y=3x-2$

The line $y=2.5x+3$ joins the points $(0, 3)$ and $(2, 8)$, and meets $y=x^3$ only at the point P whose abscissa is 2 Thus 2 is the only real root of equation (i)

Again $y=3x-2$ joins the points $(1, 1)$ and $(0, -2)$; it *touches* $y=x^3$ at Q where $x=1$, and cuts it at R where $x=-2$ Corresponding to the former point equation (ii) has *two equal roots* Thus the roots are 1, 1, -2

It is evident that any equation of the form $x^3=px+q$ may be solved graphically in the same way. [Compare Art 289]

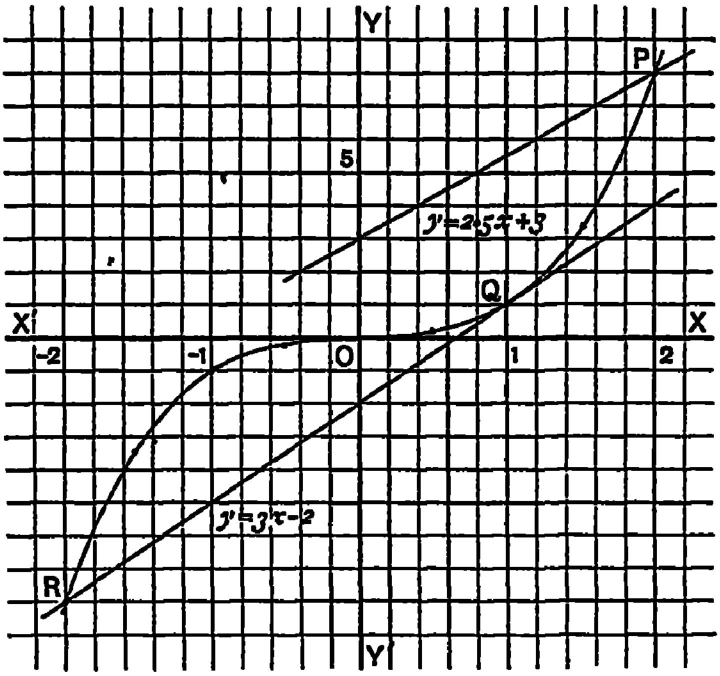


FIG 94

490 It may be noticed that the graph of $y=x^3$ touches the x-axis at O, crosses the axis at this point, and has symmetry in opposite quadrants Similar remarks apply to the graph of any equation of the form $y=ax^3$ The curve will lie in the first and third quadrants when a is positive, and in the second and fourth when a is negative

491 In the simpler cases of graphs sufficient accuracy can usually be obtained by plotting a few points, and there is little difficulty in selecting points with suitable coordinates But in other cases, and especially when the graph has infinite branches, more care is needed The graph discussed in full detail on pages 246, 247 is a case in point A revision of these pages is here recommended

The most important things to observe are (1) the values for which the function $f(x)$ becomes zero or infinite, and (2) the values which the function assumes for zero and infinite values of x In other words, we determine the *general character* of the curve in the neighbourhood of the origin, the axes, and infinity Greater accuracy of detail can then be secured by plotting points at discretion The selection of such points will usually be suggested by the earlier stages of our work

The existence of symmetry about either of the axes should also be noted When an equation contains no *odd* powers of x the graph is symmetrical with regard to the axis of y Similarly the absence of odd powers of y indicates symmetry about the axis of x.

EXAMPLE. *Draw the graph of* $y=\dfrac{2x+7}{x-4}$

We have $y=\dfrac{2x+7}{x-4}=\dfrac{2+\frac{7}{x}}{1-\frac{4}{x}}$, the latter form being convenient for infinite values of x

(i) When $y=0$, $x=-\frac{7}{2}$,
,, $y=\infty$, $x=4$,

the curve cuts the axis of x at a distance -3.5 from the origin, and meets the line $x=4$ at an infinite distance

If x is positive and very little greater than 4, y is very great and positive If x is positive and very little less than 4, y is very great and negative Thus the infinite points on the graph near to the line $x=4$ have positive ordinates to the right, and negative ordinates to the left of this line

(ii) When $x=0$, $y=-1.75$,
,, $x=\infty$, $y=2$;

the curve cuts the axis of y at a distance -1.75 from the origin, and meets the line $y=2$ at an infinite distance

By taking positive values of y very little greater and very little less than 2, it appears that the curve lies above the line $y=2$ when $x=+\infty$, and below this line when $x=-\infty$

The general character of the curve is now determined the lines PO′P′ ($x=4$) and QO′Q′ ($y=2$) are asymptotes, the two branches of the curve lie in the compartments PO′Q, P′O′Q′, and the lower branch cuts the axes at distances -3.5 and -1.75 from the origin

To examine the lower branch in detail values of x may be selected between $-\infty$ and -3.5 and between -3.5 and 4

x	$-\infty$		-16	-8	6	-3.5	-1	0	2	3		4
y	2		1.25	75	5	0	-1	-1.75	-5.5	-13		$-\infty$

The upper branch may now be dealt with in the same way, selecting values of x between 4 and ∞

x	5	5.5	6	8	10	14		∞
y	17	12	9.5	5.75	4.5	3.5		2

The graph will be found to be as represented in Fig 35 It is a rectangular hyperbola with the lines PO′P′ ($x=4$) and QO′Q′ ($y=2$) as asymptotes

In the next article the same result will be obtained in a different way.

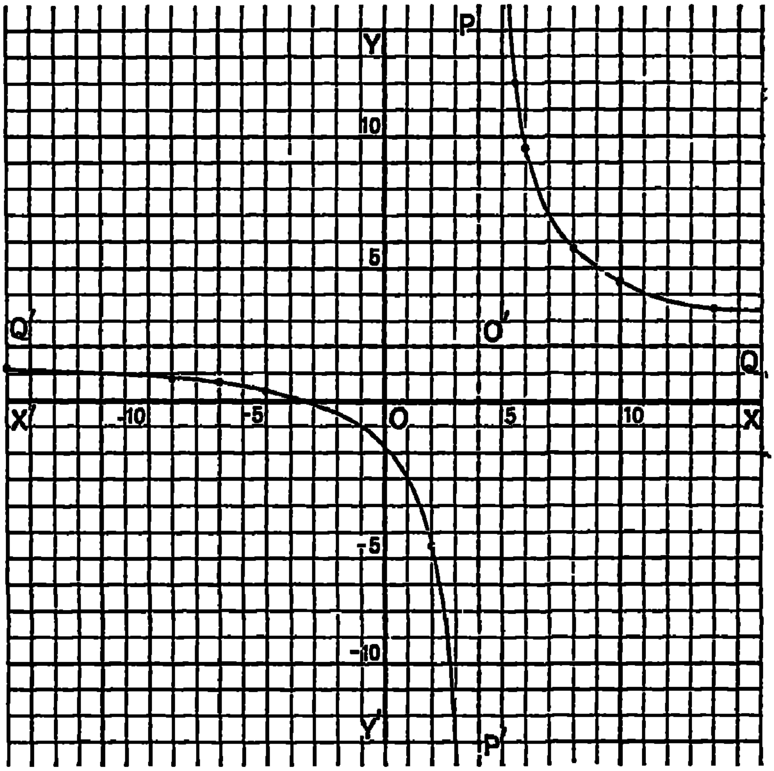


FIG 35

492 The last graph has been discussed very fully in order to emphasize some important points, but in practice it will not usually be necessary to give so much explanatory detail Moreover, there are certain devices by which the work of plotting a graph may often be shortened.

Thus in the last example, we have

$$y=\frac{2x+7}{x-4}=2+\frac{15}{x-4},$$

$$y-2=\frac{15}{x-4}, \quad \text{or} \quad (x-4)(y-2)=15$$

If we now put X for $x-4$, and Y for $y=2$, the equation becomes $XY=15$

By Art 272 this is the equation of a rectangular hyperbola which has $X=0$, $Y=0$ for its asymptotes

Hence the graph of the equation $(x-4)(y-2)=15$ is a rectangular hyperbola which has $x-4=0$ and $y-2=0$ for its asymptotes These are the lines PO'P', QO'Q' in Fig 35, and the curve might have been drawn by taking these as a new pair of axes with origin O', and plotting the graph of $XY=15$ It will be a useful exercise for the pupil to draw the graph in this way, and to compare the new diagram with Fig 35

493 EXAMPLE 1 *Draw the graph of* $y=\frac{x^2-8x+4}{x-8}$

Since $y=x+\frac{4}{x-8}$, we see that when $x=\infty$, $y=x$; thus at infinity the curve approximates to the line $y=x$, which is an asymptote This is shewn by the dotted line PO′P′ Again, when $x=8$, $y=\infty$, thus $x=8$ is an asymptote, shewn by the dotted line QO′Q′

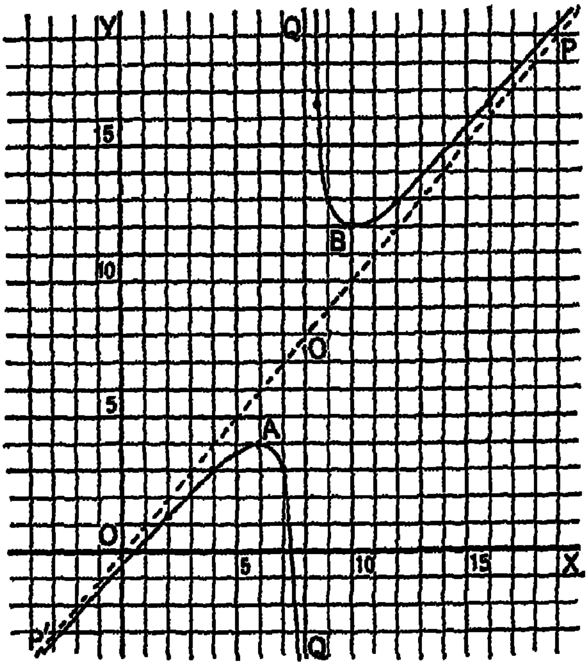


FIG 30

The position of the curve with regard to the asymptotes may be decided as in the example in Art 491, or we may proceed as in Art 458 to discover the greatest and least values of y for which x is real, and thus locate the *turning points* of the curve From the given equation we have

$$x^2-x(y+8)+8y+4=0,$$

if x is real, $(y+8)^2-4(8y+4)$ must be positive or zero;

that is, $(y-12)(y-4)$ must be positive or zero

Hence y cannot lie between 4 and 12

When $y=4$, $x=6$, and when $y=12$, $x=10$ Thus we have the turning points A and B, and the two branches of the curve lie entirely in the compartments P′O′Q′ and PO′Q

The graph may now be plotted from the following values of x and y

x	2	4	6	7	8	8 5	9	10	12	16	18
y	1 3	3	4	3	∞	16 5	13	12	13	16 5	18 4

EXAMPLE 2 *Find graphically the roots of the equation*

$$x^3-4x^2-5x+14=0$$

to three significant figures

The solution may be effected in three ways

(i) By drawing the graph of $y=x^3-4x^2-5x+14$, and noting the intercepts on the x axis,

(ii) by drawing the graphs of $y=x^3$ and $y=4x^2+5x-14$ on the same axes, and finding the abscissæ of the common points,

(iii) by drawing the graphs of $y=x^3-4x^2$ and $y=5x-14$, and finding the abscissæ of the common points This is the method we shall here adopt

We notice that $y=x^3-4x^2$ passes through the origin, and cuts the axis of x again at the point (4, 0) Other values of x and y are given below

x	-2	$-1\ 5$	-1	$-0\ 5$	0	1	2	3	4	5
y	-24	$-12\ 375$	-5	$-1\ 125$	0	-3	-8	-9	0	25

Let 1 inch represent 2 units on the x axis and 20 units on the y axis, then the graph is represented in Fig 37

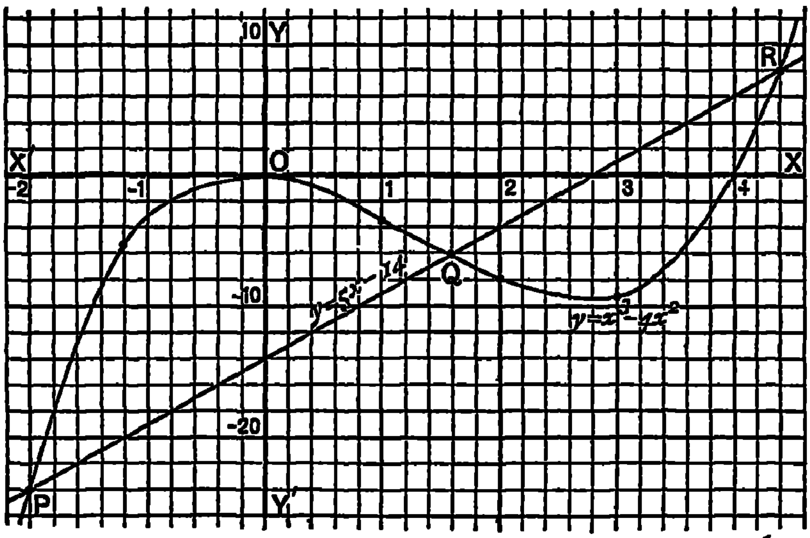


FIG 37

The line $y=5x-14$ may now be drawn by joining the points (2 8, 0) and (0, -14)

At the points of intersection, viz P, Q, R, we find the values of x are -2, 1 59, and 4 41 respectively These are the required roots

NOTE In examples of this kind it is important to choose the x unit sufficiently large In the present case, a figure drawn on twice the above scale will give the roots with great accuracy

494 Slope or gradient of a curve. We know that the lines $y=ax$, $y=ax+b$ are *parallel* Now the former of these passes through the origin, and its *direction* is clearly fixed by the value of the constant a For this reason a has been called the slope or gradient of the lines $y=ax$, $y=ax+b$ [Art 134]

If (x_1y_1), (x_2y_2) are *any* two points on the line, then $y_1=ax_1+b$, $y_2=ax_2+b$, and by subtraction $y_1-y_2=a(x_1-x_2)$, or $a=\dfrac{y_1-y_2}{x_1-x_2}$

Thus in the case of a straight line the slope or gradient is constant and is measured by the fraction

$$\frac{\text{difference of any two ordinates}}{\text{difference of the corresponding abscissæ}}$$

In the case of a curve, the direction is constantly changing, hence the gradient at any point P is defined as the gradient of the tangent to the curve at P, and it may be thus determined take a point Q on the curve near to P and find the gradient of the secant PQR When Q moves up to P along the curve, and ultimately coincides with it, the line PR becomes the tangent at P Hence the gradient of the curve at P is the limiting value of the gradient already found for the secant PQR

EXAMPLE *Find the gradient of the curve* $y=x^3$ *at the point* P(2, 8)

Let $2+h$ be the abscissa of a point Q, near to P, on the curve,

then the gradient of the secant $PQ=\dfrac{\text{diff of ordinates}}{\text{diff of abscissæ}}=\dfrac{(2+h)^3-2^3}{(2+h)-2}$

$$=\frac{12h+6h^2+h^3}{h}=12+6h+h^2.$$

Now h is very small, and ultimately vanishes when Q coincides with P, hence the gradient of the tangent at P is 12

EXAMPLES XXXVIII a.

1 Plot the graph of $y=x-x^3$ Verify it from the graphs of $y=x$, and $y=x^3$

2 Shew that the graph of $y=\dfrac{1}{x^2}$ consists of two branches lying entirely in the first and second quadrants Examine the nature and position of the graph as it approaches the axes

Draw the graphs of the following functions

3. $1+\dfrac{1}{x}$ 4. $2+\dfrac{10}{x^2}$ 5. $\dfrac{1+x}{1-x}$ 6. $\dfrac{x}{2-x}$.

7. Plot the graph of $y=x^3-3x$ Examine the character of the curve at the points $(1, -2)$, $(-1, 2)$, and shew graphically that the roots of the equation $x^3-3x=0$ are approximately $-1\,732$, 0, and $1\,732$

Draw the graphs of

8 $y=\dfrac{1+x^2}{1-x}$ 9 $y=\dfrac{x^2-15}{x-4}$ 10 $y=\dfrac{(x-1)(x-2)}{x-3}$.

11. $y=\dfrac{(x-2)(x-3)}{x-5}$ 12 $y=\dfrac{x^2+x+1}{x^2-x+1}$ 13 $y=\dfrac{x^2+5x+6}{x^2+1}$

14 $y=\dfrac{20}{x^2+2}$ 15 $y=\dfrac{40x}{x^2+10}$ 16 $y=\dfrac{x(8-x)}{x+5}$

17 As in Art 492, draw the graphs of

(i) $(x+4)(y-3)=4$,   (ii) $y-5=\dfrac{24}{x+5}$,   (iii) $(6-y)=\dfrac{10}{x-2}$

18 Plot the graphs of $y=\dfrac{15-x^2}{x}$ and $x=\dfrac{10-y^2}{y}$, and thus verify the solution of the equations $x^2+xy=15$, $y^2+xy=10$

19 Plot the graphs of

(i) $y=x^3-6x^2+11x-6$,   (ii) $10y=x^3-5x^2+x-5$

20 Draw the graphs of $y=x^3$ and $y=2x^2+x-2$ on the same axes Hence find the roots of the equation $x^3-2x^2-x+2=0$

21 Solve the equation $x^3=3x^2+6x-8$ graphically, and shew that the function x^3-3x^2-6x+8 is positive for all values of x between -2 and 1, and negative for all values of x between 1 and 4

22 Shew graphically that the equation $x^3+px+q=0$ has only one real root when p is positive

23 Find graphically the real roots of the equations

(i) $x^3+x-2=0$,   (ii) $x^3-7x+6=0$

24. As in Art 493, find the general form of the graphs of the following equations

(i) $y=\dfrac{(x-2)(x-3)}{(x-1)(x-4)}$,   (ii) $y=\dfrac{(x-1)(x-4)}{(x-3)(x-5)}$,   (iii) $y=\dfrac{2(x-1)^2}{(x-2)(x-4)}$.

25. Use the Tables to draw the graph of $y=10\log_{10}x$ between the values $x=0{\cdot}2$ and $x=10$, taking the units of x and y to be 0 5"

Find an approximate solution, other than $x=10$, of the equation $x=10\log_{10}x$

26 Find the gradient of the following curves at the points specified

(i) $y=2x^2-x+1$, when $x=4$,   (ii) $y=x^3-2x$, when $x=2$;

(iii) $y=ax^2+b$, when $x=1$,   (iv) $y=x^2-6x+7$, when $x=3$

27 Shew that the tangent to the curve $y=5+4x-2x^2$ at the point (1, 7) is parallel to the axis of x

495 In Art 156 we have shewn how in certain cases a series of plotted points may be used to determine a *linear equation* between two variables whose values have been found experimentally If the graph is not linear, it may be difficult to find its equation except by some indirect method, but there is one case of frequent occurrence in which the difficulty may be obviated by the use of logarithms

Thus suppose x and y satisfy an equation of the form $x^n y = c$, where n and c are constants By taking logarithms we have

$$n \log x + \log y = \log c$$

The form of the equation shews that $\log x$ and $\log y$ satisfy the equation to a straight line If, therefore, the values of $\log x$ and $\log y$ are plotted, a linear graph can be drawn, and the constants n and c can be found

EXAMPLE *The weight*, y *grams, necessary to produce a given deflection in the middle of a beam supported at two points*, x *centimetres apart, is determined experimentally for a number of values of* x *with results given in the following table*

x	50	60	70	80	90	100
y	270	150	100	60	47	32

Assuming that x *and* y *are connected by the equation* $x^n y = c$, *find* n *and* c

From the Tables we obtain the annexed values of $\log x$ and $\log y$ corresponding to the observed values of x and y By plotting these we obtain the graph given in Fig 38, and its equation is of the form

$$n \log x + \log y = \log c$$

$\log x$	$\log y$
1 699	2 431
1 778	2 176
1 845	2 000
1 903	1 778
1 954	1 672
2 000	1 519

To obtain n and c, choose *two extreme points through which the line passes.* It will be found that when

$$\log x = 1\ 642, \quad \log y = 2\ 6, \quad \text{at the point P},$$

and when $$\log x = 2\ 1, \quad \log y = 1\ 21, \quad \text{at the point Q}$$

Substituting these values, we have

$$2\ 6 + n \times 1\ 642 = \log c, \qquad \text{(i)}$$

$$1\ 21 + n \times 2\ 1 \quad = \log c, \qquad \text{(ii)}$$

$$1\ 39 - 0\ 458n = 0,$$

whence $$n = 3\ 04$$

from (ii) $$\log c = 6\ 38 + 1\ 21$$

$$= 7\ 59,$$

$$\therefore\ c = 39 \times 10^6, \text{ from the Tables.}$$

Thus the required equation is $x^3 y = 39 \times 10^6$

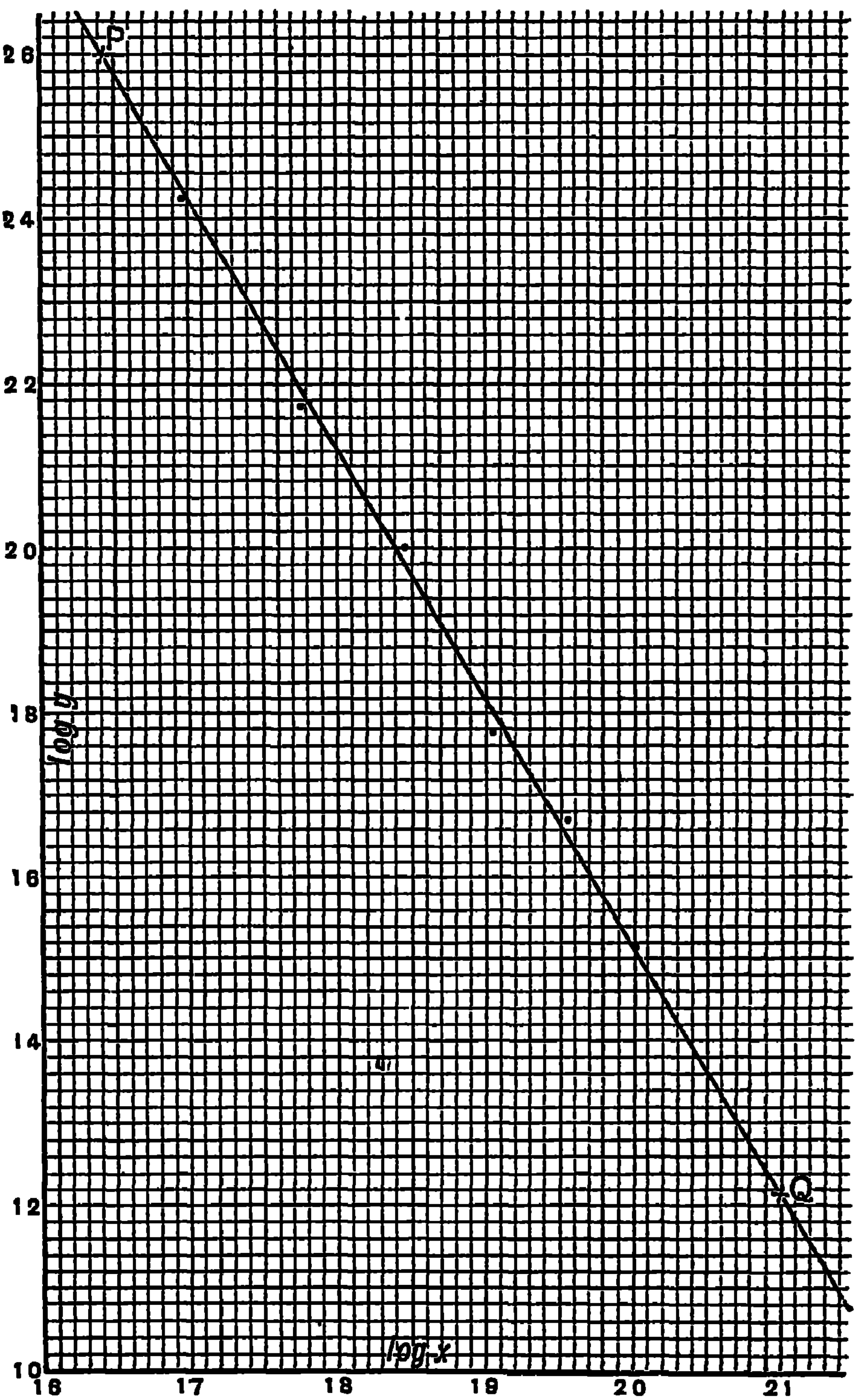


FIG 88

EXAMPLES XXXVIII. b.

1. Observed values of x and y are given as follows

$$x=100,\quad 90,\quad 70,\quad 60,\quad 50,\quad 40$$
$$y=30,\quad 31.08,\quad 33.5,\quad 35.56,\quad 37.8,\quad 40.7$$

Assuming that x and y are connected by an equation of the form $xy^n=c$, find n and c

2. The following values of x and y involve errors of observation

$$x=66.83,\quad 63.10,\quad 58.88,\quad 51.52,\quad 48.53,\quad 44.16,\quad 40.36$$
$$y=144.5,\quad 158.5,\quad 177.8,\quad 208.9,\quad 236.0,\quad 264.9,\quad 309.0$$

If x and y satisfy an equation of the form $x^ny=c$, find n and c

3. It is known that the relation of pressure to volume in saturated steam under certain conditions is of the form $pv^n=\text{constant}$ Find the value of the index n from the following data

$$p=10.2,\quad 14.7,\quad 20.8,\quad 24.5,\quad 33.7,\quad 39.2,\quad 45.5,$$
$$v=37.5,\quad 26.6,\quad 19.2,\quad 16.4,\quad 12.2,\quad 10.6,\quad 9.2,$$

where p is measured in lbs per sq in, and v is the volume of 1 lb of steam in cub ft

4. The following quantities are thought to follow a law of the form $pv^n=c$

$$v=1,\quad 2,\quad 3,\quad 4,\quad 5,$$
$$p=205,\quad 114,\quad 80,\quad 63,\quad 52$$

Ascertain if this is the case, and find the most probable values of n and c

5. In some experiments in towing a canal boat the following observations were made, P being the pull in pounds and v the speed of the boat in miles per hour

$$P=76,\quad 160,\quad 240,\quad 320,\quad 370,$$
$$v=1.68,\quad 2.43,\quad 3.18,\quad 3.60,\quad 4.03$$

By plotting $\log P$ and $\log v$, shew that P and v approximately satisfy an equation of the form $P=bv^a$, and find the best values for a and b

6. At the following draughts in sea water a particular vessel has the following displacements

Draught h feet $=$ 15, 12, 9, 6.3,
Displacement of T tons $=$ 2098, 1512, 1018, 586

By plotting $\log T$ and $\log h$ on squared paper, obtain a simple relation between T and h If one ton of sea water measures 35 cubic feet find the relation between V and h, if V is the displacement in cubic feet.

MISCELLANEOUS EXAMPLES VIII.

[*The following Examples are arranged in three sets I may be taken after* Chap XXXII, II *after* Chap XXXV, III *after* Chap XXXVIII]

I (*After* Chap XXXII)

1 Shew that $x-1$ is a common factor of x^3-3x+2 and x^3+3x^2-4 Find another common factor

2 Prove that $a^2x^2+b^2y^2+c^2z^2-a^2b^2c^2-2xyz$ is a square if $z^2=a^2b^2$

3 By the use of Detached Coefficients find the H C F of

$$2x^4+3x^3+9x^2+7x+15 \quad \text{and} \quad 4x^4+7x^3+13x^2+3x+9$$

4. Prove that $\dfrac{a}{a-b}+\dfrac{b}{b-c}+\dfrac{c}{c-a}=3-\left(\dfrac{a}{a-c}+\dfrac{b}{b-a}+\dfrac{c}{c-b}\right)$

5. Solve the equations

$$\text{(i)}\ \frac{1-2x}{1+2x}+\frac{6x}{1-x}=\frac{1}{1-2x}, \qquad \text{(ii)}\ \frac{ax+by}{2(a+b)}=c=\frac{ab(x-y)}{b^2-a^2}$$

From (i) deduce the solution of $\dfrac{x-2}{x+2}+\dfrac{6}{x-1}=\dfrac{x}{x-2}$

6 In an A P if the 8th term is twice the 13th, shew that the 2nd is twice the 10th

7 By the aid of logarithms find the value of

$$\text{(i)}\ (84\ 41)^{\frac{2}{3}}, \qquad \text{(ii)}\ \left(\frac{57\ 2\times 0\ 034}{0\ 0078}\right)^{\frac{2}{5}}$$

8 A cyclist who travels 12 miles an hour starts from A to go to B, and 3 miles beyond halfway meets another cyclist who left B one hour later, and who travels 15 miles an hour Find the distance from A to B

---

9 If $x=\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, find the value of $8x-x^2$

10 If $\dfrac{2}{x}=\dfrac{1}{a}+\dfrac{1}{b}$, find the value of $\dfrac{1}{x-a}+\dfrac{1}{x-b}$

11 Sum the following series, each to n terms

(i) $54+36+18+\ldots$, (ii) $54+36+24+\ldots$

12 If $\left(a+\dfrac{1}{a}\right)^2=3$, prove that $a^3+\dfrac{1}{a^3}=0$

13 Solve the equation $\frac{3}{7}(x-28)-\frac{7}{5}(x-3)+\frac{1}{14}(x-14)+34=0$

Deduce the solution of $\frac{3}{7}(y-27)-\frac{7}{5}(y-2)+\frac{1}{14}(y-13)+34=0$

14. Express $x^2y^3+x^2-y^3-1$ as the product of four factors

15. A man has £200 invested, partly at 4% and partly at $3\frac{1}{2}$%, and his total annual income from these investments is £7 12s How much is invested at 4%, and how much at $3\frac{1}{2}$%?

16 In a certain examination the maximum is 800 and candidates' gross marks are reduced by one quarter of the difference between their total and 800 Plot a graph to show the reduced marks corresponding to a range of gross marks from 200 to 700, and determine roughly from your diagram the gross marks of candidates whose reduced marks are given as 124 and 576

---

17. Find the factors of

$$\text{(i) } a^3-b^3-(x^2-ab)(a-b), \qquad \text{(ii) } 24(x^2y^2-1)+14xy$$

18. If $2x^3-ax^2-ax+2$ is divisible by $x+2$, find the value of a

19 If p ounces of salt are placed in a vessel along with q ounces of water so as to form brine, and r ounces of the brine are further diluted with s ounces of water, how much salt is contained in one ounce of the diluted brine?

20 Simplify

$$\text{(i) } 7\sqrt[3]{54}+3\sqrt[3]{16}-7\sqrt[3]{2}-5\sqrt[3]{128}, \qquad \text{(ii) } \sqrt{7+3\sqrt{5}}-(\sqrt{5}+1)^2$$

21 Show that the sum of $2p+1$ consecutive integers of which the smallest is p^2+1 is $p^3+(p+1)^3$

22. Find the geometric mean between $\sqrt[3]{347.3}$ and $\sqrt[5]{256.4}$, correct to three significant figures

23 Solve the equations

$$\text{(i) } \frac{3x}{2}-\frac{5}{7}=21x-\frac{1}{3}\left(2x+10\frac{3}{14}\right), \qquad \text{(ii) } \frac{2x}{x-1}+\frac{3x-1}{x+2}=\frac{5x-11}{x-2}$$

24 In a certain experiment the following observations of volume (v) and pressure (p) of a gas were made

$v=$ 3, 3.4, 4, 5.2, 6, 7.3, 8.5, 10,
$p=$ 107.3, 89.8, 71.5, 49.5, 45, 30.8, 24.9, 19.8

Draw up a table giving the values of the two variables $x=v-3$, $y=\frac{p}{20}$, and plot a graph of the results

It is thought that one observation was inaccurate, which do you regard as doubtful?

II (*After* Chap XXXV)

25 Multiply $3\sqrt{xy^3}-xy+2\sqrt{\frac{y^5}{x}}$ by $\sqrt{x^3y}-2\sqrt{xy^3}$ and express the result in a form free from radical signs

26 Solve the equations

$$\text{(i)}\ \frac{3-\sqrt{6x-x^2}}{3+\sqrt{6x-x^2}}=\frac{x}{3-x},\qquad \text{(ii)}\ \begin{aligned}x^3+y^3&=280,\\ x^2y+xy^2&=240\end{aligned}$$

27 Shew that $2x^2-5x+3$ is always positive except between the values $x=1$ and $x=\frac{3}{2}$

28 If t varies as $pv^{\frac{3}{2}}$, and $t=224$ when $p=28$ and $v=16$, what is the value of t when $p=32$ and $v=25$?

29 If the harmonic mean between two numbers is 24, and their geometric mean is 48, what is their arithmetic mean?

30 A, B, C, D are four stations on a railway, the distances AB, BC, CD being 10 miles, 10 miles, and 8 miles respectively The following is an extract from a time table

First Train	*Second Train*
A, dep, 10 57 a m	D, dep, 11 9 a m
B, dep, 11 18 a m	C, —
C, dep, 11 40 a m	B, —
D, arr, 11 55 a m	A, arr, 12 0 noon

Draw graphs to shew the positions of the trains at any intermediate time, assuming that each runs at a uniform speed between the stations, and that the first train stops 4 minutes at each of the stations B, C When and where do the trains pass each other?

[*Take one inch to represent ten minutes of time, and five miles of distance*]

31 Find the condition that the roots of the equation

$$a(1-x^2)+2bx+c(1+x^2)=0$$

may be equal

32 A man pays income tax at 1*s* in the £ on unearned income and at 9*d* in the £ on earned income, his earned income exceeds his unearned by £200, and his total income tax is £29 7*s* 6*d*, find his total income

---

33 Solve (i) $(x-y)(x-2y)=2,\quad x+2y=5,$

(ii) $\sqrt{4-x}+\sqrt{1+x}=\sqrt{11+6x}$

34. If b is a mean proportional between a and c, shew that

$$a-b\ \ b=a-c\ \ b+c$$

35 Find the factors of $(a+b)(a-b)-4c(a-c)$; and find the value of p when $x+3$ is a factor of $x^2+px-27$

36. The expenses of a boarding-house are partly constant, and partly vary with the number of boarders, each boarder paying £65 a year, the annual profits are £9 a head when there are 50 boarders, £10 13*s* 4*d* when there are 60 What is the profit per boarder when there are 80?

37 In an arithmetic progression the sum of 5 terms and the sum of 15 terms are each equal to 75, find the 10th term

38 A man buys 99 oranges at a certain price they would have cost him 1*s* less if he had obtained for each shilling four more oranges than he actually did receive What price did he pay?

39. One root of the equation $2x^2-10\,283x+12\,566=0$ is 2, find the other root

40. Trace the value of the function $x+\frac{1}{x}$ as x changes from -3 to $+3$, and represent it graphically Find its least numerical value

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41. Simplify the expressions

$$\text{(i)}\ \left(\frac{x}{1+\frac{x}{y}}+\frac{y}{1+\frac{y}{x}}\right)-\left(\frac{y}{1-\frac{y}{x}}-\frac{x}{1-\frac{x}{y}}\right),$$

$$\text{(ii)}\ \frac{n}{x-n}+\frac{n+1}{x+n+1}-\frac{2nx}{x(x+1)-n(n+1)}$$

42. Solve the following pairs of equations

$$\text{(i)}\ \frac{9}{x}+\frac{8}{y}=43, \qquad \text{(ii)}\ \frac{x}{a}+\frac{y}{b}=\frac{a+b}{2},$$

$$\frac{8}{x}+\frac{9}{y}=42, \qquad \frac{x}{b}-\frac{y}{a}=\frac{a-b}{2}$$

43 If a, b, c, d are positive quantities in continued proportion, prove that

$$\text{(i)}\ a+d>b+c, \qquad \text{(ii)}\ (a+b)(b+d)^2=(c+d)(a+c)^2$$

44 A man can make a articles of one kind in a week, of another kind he can make b articles in a week How many articles can he make in a week if the total week's work turns out the same number of each kind?

45 Find the square root of

$$(3c^2+13cd-10d^2)(2c^2+7cd-15d^2)(6c^2-13cd+6d^2)$$

46. If α, β are the roots of $mx^2+nx+l=0$, find the equation whose roots are $m\alpha+n$, $m\beta+n$

47. Using Detached Coefficients, find the first four terms of

$$(2+6x-4x^2+3x^3+x^4)(1-3x+2x^3-x^4+2x^5)$$

48. The sum of five numbers in arithmetic progression is 10, and the sum of their squares is 60, find the numbers

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49 Shew that

$$\frac{1}{2}\left(\frac{a^2-4b^2}{a^2+b^2}+\frac{a^2+4b^2}{a^2-b^2}\right)-\frac{(a+b)^2+b^2}{a^2+b^2}\equiv\frac{(a-b)^2+b^2}{a^2-b^2}$$

50 What value of a will make $4x^4-(a-1)x^3+ax^2-6x+1$ exactly divisible by $2x-1$?

51 Divide $1+4ab^{\frac{4}{3}}$ by $1+2a^{\frac{1}{4}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}b^{\frac{2}{3}}$, and verify the result when $a=16$, $b=8$.

52 Find the value of x from the equation $2^{3x}\cdot5^{2x-1}=4^{5x}\cdot3^{x+1}$.

53 Sum the following series:

(i) $7+6.5+6+\ldots$ to 14 terms; (ii) $\dfrac{1}{1.05}+\dfrac{1}{(1.05)^2}+\dfrac{1}{(1.05)^3}+\ldots$ to inf.

54. If α, β are the roots of $x^2+px+q=0$, express the function

$$\frac{(p+\alpha)(p+\beta)}{(q-\alpha^2)(q-\beta^2)}$$

in terms of p and q.

55 If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, shew that $\dfrac{x}{a}=\dfrac{y}{b}$.

56 A body is projected with a given velocity at a given angle to the horizon, and the height in feet reached after t seconds is given by the equation $h=64t-16t^2$. Find the values of h at intervals of $\frac{1}{4}$th of a second and draw the path described by the body. Find the maximum value of h, and the time after projection before the body reaches the ground.

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57 Prove that $(a+b+c)^3-a^3-b^3-c^3\equiv3(a+b)(b+c)(c+a)$.

58 Solve the equations:

(i) $\dfrac{x+a-2b}{a+b}+\dfrac{x+a-b}{a+2b}=\dfrac{2x}{3b}$; (ii) $\sqrt{x-3}+\sqrt{2x+1}=2\sqrt{x}$.

59 If $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$, prove that

$$\frac{1}{abc}-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2$$

60. Two bicyclists ride at the rate of 15 miles per hour. The circumference of the driving wheel of one bicycle is $5\frac{1}{2}$ inches less than that of the other, and it requires to be turned round once more in 5 seconds; find the circumference of the driving wheel of each bicycle.

61 If a, b, c are three proportionals prove that

(i) $a(a+b):b(b-a)=b(b+c):c(c-b)$;

(ii) $(a+b+c)(b^2-bc+c^2)=c(a^2+b^2+c^2)$.

62. Express $(27)^{\frac{2}{3}}+(16)^{\frac{3}{4}}-\dfrac{2}{(8)^{-\frac{2}{3}}}+\dfrac{\sqrt[3]{2}}{(4)^{-\frac{1}{3}}}$ as a whole number

63 If the roots of $x^2(b^2+d^2)+2x(ab+cd)+a^2+c^2=0$ are equal, shew that they will each be equal to $-\dfrac{a}{b}$

64. Taking 1 inch as the unit, draw the graph of $y=x+\dfrac{1}{x}$ Hence solve the equation $x+\dfrac{1}{x}=-3$ Verify your result by an accurate algebraical solution of this equation

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III (*After* Chap XXXVIII)

65. Find the square root of $3x+1+2\sqrt{2x^2-x-6}$

66. Find the real roots of the simultaneous equations

$$x-y=2,$$
$$x^5-y^5=2882$$

[*Put* $x=v+1$, $y=v-1$, *substitute in the second equation, and find* v.]

67. Simplify $(\sqrt[3]{54}+\sqrt[3]{250}+\sqrt[3]{128})(\sqrt[3]{54}+\sqrt[3]{250}-\sqrt[3]{128})$

68 Given that the area of a circle varies as the square of its radius, find the radius of a circle which is equal to the sum of the areas of two circles whose radii are 5 and 12 inches

69 If the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ are equal, and a, b, c are all positive, prove that a, b, c are in A.P

70 If $x+y=a$, and $x-y=b$, shew that

$$x^4-18x^2y^2+y^4=3a^2b^2-(a^4+b^4)$$

71 A square board is covered as far as possible with rows of halfpence If each side of the square had been one inch longer the weight of the halfpence would have been 1 lb 1 oz greater Find the number of halfpence in each row, given that 5 halfpence weigh 1 oz, and the diameter of a halfpenny is 1 inch

72. Prove that the function $\dfrac{(1+2x)(2+x)}{x}$ can never lie between $+1$ and $+9$ for real values of x Trace the graph from $x=-2$ to $x=1$.

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73. Simplify $\dfrac{x+4x^{\frac{1}{2}}}{x-x^{\frac{1}{2}}-20}-\left(1-\dfrac{5}{\sqrt{x}}\right)^{-1}$

74. Find 4 numbers in A.P. such that the sum of the squares of the extreme numbers is 125, and the sum of the squares of the means 89

75. Solve the equations:

(i) $x^2 - x - \dfrac{72}{x(1-x)} = 18$; (ii) $x(x-3)^2 + 2x = 6$

76. If $a+b+c=0$, prove that

$$a^2+b^2+c^2=2(a^2-bc)=2(b^2-ca)=2(c^2-ab)$$

77. Find the sum of $3a+a\sqrt{3}+a+\dfrac{a}{\sqrt{3}}+$ to 6 terms If the sum of this series to infinity is equal to 27, find the value of a

78. By the Remainder Theorem shew that x^n-y^n is divisible by $x+y$ when n is even

79 Draw the graph of $y=\frac{1}{2}(1-x)(2x+7)$ for values of x between -4 and 3 Find from the graph, or otherwise, the greatest value of y

80 A waterman rows a given distance a down-stream and back again in h hours, and finds that he can row b miles with the stream in the same time as c miles against it Prove that the stream flows at the rate of $\dfrac{a(b^2-c^2)}{2hbc}$ miles an hour

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81 Simplify $\dfrac{(x+3)(2x^2-7x-4)-(x+3)(2x+1)}{(x-5)(2x^2-7x-4)+(x-5)(2x+1)}$

82 Find the value of $\sqrt{1+\sqrt{21+12\sqrt{3}}}$

83 Find the values of b and c if the product of x^2+bx+c and x^2-2x+1 is $x^4-5x^3+5x^2+x-2$ for all values of x

84. Find the value of $x^4+x^2y^2+y^4$ when $x+y=2a$, $x-y=2b$

85. Find the cube root of

$$8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6,$$

by the method of Undetermined Coefficients

86 Form the equation whose roots are $\dfrac{\sqrt{m}}{\sqrt{m}+\sqrt{m-n}}$ and $\dfrac{\sqrt{m}}{\sqrt{m}-\sqrt{m-n}}$.

87. Solve the equations

$$24(y+z)=30(z+x)=40(x+y)=5xyz$$

88 Two variables are connected by the equation

$$\frac{1}{y}=\frac{x}{f}+\frac{1}{a},$$

where f and a are constants It is found that $x=2$ gives $y=6$, and that $x=7$ gives $y=1$ Draw a graph shewing the values of y corresponding to values of x from 2 to 7

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89. Simplify $\left\{\left(a-\frac{1}{x}\right)^a\left(a-\frac{1}{x}\right)^x\right\}-\left\{\left(x+\frac{1}{a}\right)^a\left(x-\frac{1}{a}\right)^x\right\}$.

90. If $5+4x+5x^2$ is written in the form $A+B(x-2)+C(x-2)^2$, find the values of A, B, and C.

91. Prove that

$$(c-b)(x-a)^2+(a-c)(x-b)^2+(b-a)(x-c)^2\equiv(b-c)(c-a)(a-b)$$

92. Prove that $(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$ is negative if a, b, c are real numbers not all equal.

93. If $\frac{x}{a}=\frac{b}{c}=\frac{d}{y}$, prove that

$$\frac{(x+a)(x+b)(x+d)}{(y+a)(y+c)(y+d)}=\frac{b^2x(c+a)}{c^2y(b+d)}$$

94. If the sums of the first p, q, and r terms of an A.P. are P, Q, R respectively, prove that $\frac{P}{p}(q-r)+\frac{Q}{q}(r-p)+\frac{R}{r}(p-q)=0$

95. Sixty-three solid leaden spheres of equal size are melted down and cast into a hollow sphere six inches thick. The outer radius of the hollow sphere is four times the radius of any one of the solid spheres. Find the radius of one of the solid spheres, assuming that the volume of a sphere varies as the cube of the radius.

96. With one inch as unit trace the graphs

(i) $y=1.4x-0.8$; (ii) $y=0.25x^3$

Hence find the roots of the equation $x^3=5.6x-3.2$

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97. Find what numerical values must be given to a and b in order that $x^3-(a-1)x^2-(b-3)x-2$ may be divisible by the square of $x+2$

98. Sum the series $(1+2)+(2+2^2)+(3+2^3)+(4+2^4)+\ldots$ to 15 terms

99. By the use of logarithms, find x from the equation

$$7^{2x-2}+4^{x+2}=7^{2x+1}-4^{x-3}$$

100. I think of an odd number; I multiply by 3 and divide by 2, the quotient being again odd; I multiply the quotient by 3 and divide by 2, obtaining a final quotient 64. What was the number originally thought of? Test your solution.

101. If $a-b-c=0$, prove that $a^4+b^4+c^4=2a^2b^2+2b^2c^2+2c^2a^2$.

102. Sum the series $1(3^2-2^2)+2(4^2-3^2)+\ldots+n\{(n+2)^2-(n+1)^2\}$

103. The reciprocal of a number is multiplied by 2·25 and the product is added to the number. Find graphically what the number must be if the resulting expression has the least possible value.

104. Draw the graphs of $y=x^3$ and $y=3x^2-4$ on the same axes, and find the roots of the equation $x^3-3x^2+4=0$

Shew that the function x^3-3x^2+4 is negative for values of x less than -1, and positive for all other values of x

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105 Prove that $(b+c)(c+a)(a+b)+abc\equiv(a+b+c)(bc+ca+ab)$

106 Simplify $\dfrac{1}{(a+b)(b+c)}\left\{\dfrac{(a+b)^3-(b+c)^3}{a-c}-\dfrac{(a+b)^3+(b+c)^3}{a+2b+c}\right\}$

107. Shew that $\dfrac{3}{4}+\dfrac{4}{8}+\dfrac{5}{16}+\cdots+\dfrac{m}{2^{m-1}}=2-\dfrac{m+2}{2^{m-1}}$.

108 If p, q, and r are respectively the q^{th}, r^{th}, and p^{th} terms of an A.P prove that $p^3+q^3+r^3=3pqr$

109 The following formula gives G the number of gallons of water delivered per hour by a pipe of diameter D inches and of length L yards under a head of water of H feet

$$G=\sqrt{\frac{(15D)^5\times H}{L}}.$$

Use it to find the diameter of a pipe 2 miles long, which will deliver 140,000 gallons of water an hour under a head of 30 feet

110 A train usually does a run of 70 miles at 30 miles an hour One day it is stopped and delayed 12 minutes, but by doing the remainder of the run at 40 miles an hour it arrives at the proper time Where did the stoppage take place? Verify the solution graphically

111 *A*'s present wages are 13*s* a week, and each year he is to receive a rise of 2*s* a week *B* starts with the same wages, but receives a half-yearly rise of 1*s* a week Find the total amounts received by each during the first n years, taking a year as containing exactly 52 weeks

112. Calculate the values of $x(9-x)^2$ for the values 0, 1, 2, 3, 9 of x Draw the graph of $x(9-x)^2$ from $x=0$ to $x=9$

If a very thin elastic rod, 9 inches in length, fixed at one end, swings like a pendulum, the expression $x(9-x)^2$ measures the tendency of the rod to break at a place x inches from the point of suspension From the graph find where the rod is most likely to break

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113 Simplify $\dfrac{(a+b-c)(a^2+b^2+c^2+bc+ca-ab)+c(c^2-3ab)}{a^2-ab+b^2}$

114 Solve the equation $(2a-b-x)^2+9(a-b)^2=(a+b-2x)^2$

115 Prove that the roots of the equation $px^2+2qx+r=0$ are real if the roots of $r^2x^2+2q^2x+p^2=0$ are real

116. If $ax^3=by^3=cz^3$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{d}$, show that

$$ax^2+by^2+cz^2=d^2\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)^3.$$

117. If $\frac{a}{b+c-a}$, $\frac{b}{c+a-b}$, $\frac{c}{a+b-c}$ are in A P, prove that

$$\frac{a-c}{b-c}=\frac{2b}{a+b-c}$$

118. If $\frac{x}{1+y}=a$, $\frac{y}{1+z}=b$, $\frac{z}{1+x}=c$, prove that

$$\frac{x}{a(1+b+bc)}=\frac{y}{b(1+c+ca)}=\frac{z}{c(1+a+ab)}$$

119 If a man spends 22*s* a year on tea whatever the price of tea is, what amounts will he receive when the price is 12, 16, 18, 20, 24, 28, 33, and 36 pence respectively? Give your results to the nearest quarter of a pound Draw a curve to the scale of 4 lbs to the inch and 10 pence to the inch, to shew the number of pounds that he would receive at intermediate prices

120 A manufacturer finds that when he is employing W workmen his total weekly expenditure (including wages, material, coal, gas, insurance of premises, etc) amounts to E pounds, and his receipts amount to R pounds After carefully balancing his books for many weeks, the following table of average results was drawn up

W	25	30	35	45	50	60
E	30 2	35 4	40 1	48 0	53 1	59 8
R	27 5	39 1	49 8	75 0	84 8	109 2

From these data determine simple algebraical relations between E and W, R and W, P and W, where £P represents his weekly profits Also find graphically (i) the number of workmen necessary to ensure a weekly profit of £18 10*s*, (ii) the smallest number of men that will enable the manufacturer to pay expenses

PART III.

CHAPTER XXXIX

PERMUTATIONS AND COMBINATIONS.

496 EACH of the *groups* or *selections* which can be made by taking some or all of a number of things (without regard to the *order* of the things in each group) is called a combination.

Thus the combinations which can be made by taking the four letters a, b, c, d two at a time are six in number namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd,$$

each of these presenting a different *selection* of two letters

If the things in each selection are arranged in all possible orders, each of such *arrangements* is called a permutation

Thus each of the foregoing *selections* of the letters a, b, c, d, taken two at a time, admits of two *arrangements*, hence the *permutations* of these letters two at a time are twelve in number namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd,$$
$$ba, \quad ca, \quad da, \quad cb, \quad db, \quad dc;$$

each of these presenting a different *arrangement* of two letters

Again, a single *combination* of three letters, such as abc, admits of the following *arrangements*

$$abc, \quad acb, \quad bca, \quad bac, \quad cab, \quad cba,$$

and so gives rise to *six different permutations*

497 More generally · when r things are selected out of n, each *selection* is called an r-combination, and the number of ways in which such a selection can be made is called *the number of combinations of* n *things* r *at a time*, and is denoted by the symbol ${}^nC_r$

Each *arrangement* which can be made by taking r things out of n is called an r-permutation, and the number of ways in which such an arrangement can be made is called *the number of permutations of* n *things* r *at a time*, and is denoted by the symbol ${}^nP_r$

498 If we were required to write down all the 3-permutations of 4 letters a, b, c, d, we might proceed as follows Take a and write each of the other 3 letters after it, we thus obtain three 2-permutations in which a stands first Similarly, there are three in which b stands first, and so on Hence the total number of 2-permutations is 4×3, or 12 Next take *any one* of these, such as ab, and write each of the other letters after it We thus obtain abc, abd, that is from *any one* of the twelve 2-permutations we can obtain two 3-permutations Hence the total number of 3-permutations is 12×2, or 24

499 It is obvious that it would often be a laborious task to find the number of combinations or permutations in any given case by writing them all down exhaustively

For example, the number of 3-combinations that can be formed out of 10 things would be found to be 120, that is, ${}^{10}C_3 = 120$

And the number of 4-permutations that can be formed out of 8 things would be found to be 1680, that is, ${}^{8}P_4 = 1680$

Hence it is necessary to find general formulæ, in terms of n and r, from which the values of ${}^{n}C_r$ and ${}^{n}P_r$ can be readily calculated in every case

Further, since the order of thought suggests *selection* of groups followed by *arrangement* of the things in each group, it would seem natural to consider combinations first and then to deal with permutations But it happens that many cases dealing with permutations can be dealt with very simply, by common sense reasoning, by means of an important principle which we shall now explain and illustrate

500 *If one operation can be performed in* m *ways, and (when it has been performed in any one of these ways) a second operation can then be performed in* n *ways, the number of ways of performing the two operations will be* $m \times n$

If the first operation be performed in *any one* way, we can associate with this any of the n ways of performing the second operation and thus we shall have n ways of performing the two operations without considering more than *one* way of performing the first, and so, corresponding to *each* of the m ways of performing the first operation, we shall have n ways of performing the two, hence altogether the number of ways in which the two operations can be performed is represented by the product $m \times n$

EXAMPLE *There are 10 steamers plying between Liverpool and Dublin, in how many ways can a man go from Liverpool to Dublin and return by a different steamer?*

There are *ten* ways of making the first passage, and with each of these there is a choice of *nine* ways of returning (since the man is not to come back by the same steamer), hence the number of ways of making the two journeys is 10×9, or 90

501. The same principle can be used when there are more than two operations, each of which can be performed in a given number of ways.

EXAMPLE 1. *In how many ways can 3 of the letters* A, B, C, a, b, c, d *be arranged in a row, using only one capital and two small letters, so that the capital always stands first?*

The first place can be filled up in 3 ways since any one of the capitals may be used. And when the first place has been filled up in any one of these ways, the second place can be filled up in 4 ways, since any one of the letters *a*, *b*, *c*, *d* may be used. And since each way of filling up the first place can be associated with each way of filling up the second, the number of ways of filling up the first two places is given by the product 3×4. And when the first two places have been filled in any one of these 12 ways, the third place can be filled up by using any of the 3 remaining small letters. Hence, reasoning as before, the number of ways in which the 3 places can be filled up is 12×3, or 36.

EXAMPLE 2. *Four persons enter a railway carriage in which there are six vacant seats; in how many ways can they take their places?*

The first person may seat himself in 6 ways; and then the second person in 5; the third in 4; and the fourth in 3; and since each of these ways may be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3$, or 360.

EXAMPLES XXXIX. a.

1. A field has 4 gates; in how many ways is it possible to enter the field by one gate and come out at another?

2. In how many ways can one consonant and one vowel be chosen out of the letters of the word *Cambridge*?

3. In how many ways is it possible to take one apple, one orange, and one pear from a basket containing 6 apples, 4 oranges, and 5 pears?

4. In how many ways can two prizes be given to a class of 12 boys, (i) if one boy may receive both; (ii) if no boy can receive more than one prize?

5. There are 10 competitors in a race for 3 prizes; in how many ways can the prizes be given?

6. How many different signals can be made by hoisting 4 flags on one mast, with 7 flags to choose from?

7. How many integral numbers can be formed with the four digits 1, 2, 3, 4? How many if 0 is substituted for 1?

8. In how many ways can the letters of the word *minus* be arranged? How many of these will begin with *m*? How many will *not* begin with *m*? How many will begin with *m* and end with *s*?

9. A man lives within reach of 2 boys' schools and 3 girls' schools: in how many ways can he send his 3 sons and 2 daughters to school?

Permutations of Unlike Things.

502 *To find the number of* permutations *of* n *unlike things taken* r *at a time*

This is the same thing as finding the number of ways in which we can fill up r blank places when we have n unlike things at our disposal

The first place may be filled up in n ways, for any one of the n things may be taken, when it has been filled up in any one of these ways, the second place can then be filled up in $n-1$ ways, and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$ And when the first two places have been filled up in any one of these ways, the third place can be filled up in $n-2$ ways And reasoning as before, the number of ways in which three places can be filled up is $n(n-1)(n-2)$

Proceeding thus, and noticing that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to

$$n(n-1)(n-2) \quad \text{to } r \text{ factors},$$

and the r^{th} factor is $n-(r-1)$, or $n-r+1$

Therefore the number of permutations of n things taken r at a time is
$$n(n-1)(n-2) \quad (n-r+1)$$

COR. The number of permutations of n things *taken all at a time* is
$$n(n-1)(n-2) \quad \text{to } n \text{ factors},$$
or
$$n(n-1)(n-2) \quad 3\ 2\ 1.$$

503 The product of the first n consecutive numbers is denoted by the symbol $\underline{|n}$, or $n!$ Either symbol is read "**factorial n.**"

We have thus proved the two following formulæ

$$\text{(i) } {}^nP_r=n(n-1)(n-2) \quad (n-r+1),$$
$$\text{(ii) } {}^nP_n=\underline{|n}, \text{ or } n!$$

NOTE It should be noticed that the suffix r in the symbol ${}^nP_r$ always indicates the *number of factors* in the formula we are using

EXAMPLE 1 *How many different four-figure numbers can be formed* (i) *with the digits* 1, 3, 5, 7, 9, (ii) *with the digits* 0, 1, 3, 5, 7, 9, *no digit being used more than once in each number?*

(i) We have 5 different things and we have to find the number of permutations of them 4 at a time

$$\text{the required number}={}^5P_4=5\ 4.3.2=120$$

(ii) Since a number cannot begin with 0, the first place can only be filled up in 5 ways Then the number of ways of arranging 3 out of the remaining 5 digits is ${}^5P_3$

$$\text{the required number}=5\times{}^5P_3=5\times5\ 4\ 3=300$$

EXAMPLE 2 *In how many ways can the letters of the word* courage *be arranged if the vowels are always to occupy the odd places?*

The 4 vowels can occupy the 4 odd places in $\underline{|4}$ ways The 3 consonants can occupy the 3 even places in $\underline{|3}$ ways

Each arrangement of vowels can be associated with each arrangement of consonants,

$\therefore$ the required number $=\underline{|4}\times\underline{|3}=4\cdot 3\cdot 2\times 3\cdot 2=144$

EXAMPLES XXXIX. b.

1 Find the numerical values of $\underline{|7}$, ${}^8P_5$, ${}^7P_5$, $8!$, ${}^9P_4$, $\dfrac{\underline{|9}}{\underline{|3}}$

2 How many arrangements can be made by taking (i) five, (ii) all of the letters of the word *number*?

3 If ${}^nP_4=18\times{}^{n-1}P_2$, find n

4 How many changes can be rung with 5 bells? How many of these will begin with one particular bell?

5 Using each digit only once in each number, find how many numbers between 2000 and 3000 can be formed with the digits 1, 2, 3, 4 5, 6

*6 How many permutations are there of the letters of the word *orange*, (i) beginning with *o*, (ii) not beginning with *o*?

7 In how many ways can 4 boys and 3 girls be arranged alternately with a boy at each end of the row?

8 Of the permutations of the letters of the word *factor* taken all together, how many do not begin with *fa*?

9 How many arrangements of the letters of the word *fragile* can be made, if the vowels are always to occupy the first, the last, and the middle places?

10 An examination consists of six papers of which two are in mathematics In how many ways can the papers be given out so that the mathematical papers are not consecutive?

11 Shew by general reasoning that ${}^{n+1}P_{r+1}=(n+1)\times{}^nP_r$

12 If 8 men and 5 women apply for 5 different situations, 3 of which must be filled by men, and 2 by women, in how many ways can the situations be filled?

13 There are 8 different situations vacant of which 3 must be held by men, and 2 by women, the remaining 3 may be held by either men or women If 10 men and 5 women present themselves as candidates, in how many ways can the situations be filled?

14 Find the number of ways in which 6 different books can be arranged (i) if 3 specified books are always together, (ii) if the 3 specified are always separated.

Permutations of things not all different.

504 The foregoing formulæ apply only when the things considered are *unlike* When we speak of things being *dissimilar, different, unlike*, it is assumed that they are *visibly unlike*, so as to be easily distinguished from each other Things are considered to be *alike* when they cannot be so distinguished from each other.

The following example should be very carefully studied

EXAMPLE *How many different words can be formed with the letters* a, d, e, d, d, b, *using all the six letters in each word?*

Let x be the required number of words; then if in *any one* of these words (*e g* *debadd*) we were to replace the letters d by new unlike letters, different from any of the rest, *from this single word* we could form $\lfloor 3$ new words If a similar change were made in each of the x words, we should obtain $x \times \lfloor 3$ new words But since the 6 letters have now become all different the number of words must be equal to $\lfloor 6$,

$$x \times \lfloor 3 = \lfloor 6, \quad \text{or} \quad x = \frac{\lfloor 6}{\lfloor 3} = 6 \cdot 5 \cdot 4 = 120$$

505 *To find the number of ways in which* n *things may be arranged among themselves, taking them all at a time, when* p *of the things are alike of one kind,* q *of them alike of another kind,* r *of them alike of a third kind, and the rest all different*

Let there be n letters, suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike

Let x be the required number of permutations, then if the p letters a were replaced by p unlike letters different from any of the rest, from *any one* of the x permutations, without altering the position of any of the remaining letters, we could form $\lfloor p$ new permutations Hence if this change were made in each of the x permutations, we should obtain $x \times \lfloor p$ permutations

If in each of *these* permutations the q letters b were replaced by q unlike letters, the number of permutations would be $x \times \lfloor p \times \lfloor q$

In like manner, by now replacing the r letters c by r unlike letters, we should finally obtain $x \times \lfloor p \times \lfloor q \times \lfloor r$ permutations

But the things are now all different, and therefore admit of $\lfloor n$ permutations among themselves Hence

$$x \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n,$$

that is,
$$x = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}, \quad \text{or} \quad \frac{n!}{p!\,q!\,r!},$$

which is the required number of permutations.

Any case in which the things are not all different may be treated similarly

EXAMPLE 1 *How many different permutations can be made out of the letters of the word* assassination *taken all together?*

We have 13 letters, of which 4 are *s*, 3 are *a*, 2 are *i*, and 2 are *n*. Hence the number of permutations

$$= \frac{|\underline{13}}{|\underline{4}\,|\underline{3}\,|\underline{2}\,|\underline{2}} = 13 \quad 11 \quad 10 \quad 9 \quad 8 \quad 7 \quad 3 \quad 5$$

$$= 1001 \times 10800 = 10810800$$

EXAMPLE 2 *How many numbers can be formed by using the digits* 1, 2, 5, 4, 5, 5, 6, 1, 8, *so that the odd digits always occupy the odd places?*

(i) The odd digits 1, 5, 5, 5, 1 can be arranged in the five odd places in $\frac{|\underline{5}}{|\underline{3}\,|\underline{2}}$, or 10 ways

(ii) The even digits 2, 4, 6, 8 can be arranged in the four even places in $|\underline{4}$, or 24 ways

Each of the ways in (i) can be associated with each of the ways in (ii)

Hence the required number $= 10 \times 24$, or 240

[*Examples* XXXIX c 1–9, *page* 458, *may be taken here*]

Permutations of things which may be repeated.

506 *To find the total number of* r-permutations *of* n *different things when each thing may be repeated up to* r *times in any arrangement*

Consider n different kinds of letters $a, b, c,$, not less than r of each kind Then the required number of permutations will be equal to the number of ways in which r places can be filled up by using *any one* of these letters in each place

The first place may be filled up in n ways, and, when it has been filled up in any one way, the second place may also be filled up in n ways, since we may use the same thing again Therefore the number of ways in which the first two places can be filled up is $n \times n$ or n^2

The third place can also be filled up in n ways, and therefore the first three places in n^3 ways, and so on

Since at any stage the index of n is always the same as the number of places filled up, we shall have the number of ways in which the r places can be filled up equal to n^r

EXAMPLE *In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?*

Any one of the prizes can be given in 4 ways, and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize Thus two prizes can be given away in 4^2 ways, three prizes in 4^3 ways, and so on. Hence the 5 prizes can be given away in 4^5, or 1024 ways

EXAMPLES XXXIX. c.

[*In Examples* 1–9 *each thing occurs once only in any arrangement.*]

1. Find the number of permutations which can be made by using all the letters of the following words:

(i) *tobacco*, (ii) *alluvial*,

(iii) *tittle-tattle*, (iv) *appropriation*.

In (i) and (iii) how many of the permutations begin with *t*?

2. Among the permutations of the letters of the word *series* how many begin and end with *s*? In how many are the vowels and consonants placed alternately?

3. How many different numbers can be formed by using the seven digits 2, 3, 4, 3, 3, 1, 2? How many with the digits 2, 3, 4, 3, 3, 0, 2?

4. Without assuming the general formula, find the number of permutations of all the letters of the word *zoological*.

5. In how many ways can the letters of the word *cannon* be arranged (i) if the two vowels always come together, (ii) if the relative position of the vowels and consonants is not altered?

6. I have 2 exactly similar copies of Algebra, 3 of Geometry, and single copies of Arithmetic and Trigonometry. In how many ways can these books be distributed among 7 boys, one volume to each?

7. How many *even* numbers, each of 7 digits, can be formed with the digits 3, 2, 5, 4, 3, 5, 5?

8. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice?

9. How many numbers greater than 30,000 can be made by using all the digits 1, 4, 4, 3, 5?

(*Permutations with repetitions.*)

10. Find the total number of ways in which 5 sparrows can perch on 3 trees when there is no restriction as to the choice of tree. In how many of these ways will one particular sparrow be alone on a tree?

11. How many 4-permutations can be made out of the letters *a*, *b*, *e*, *f*, *g*, *k*, when repetitions are allowed?

12. In how many ways can I make 4 journeys with 3 conveyances to choose from?

13. A letter-lock consists of 4 rings each marked with 5 different letters; how many unsuccessful attempts can be made to open the lock?

14. In how many ways can 5 hats be divided between 2 men?

15. In how many ways can a man harness 3 beasts to a plough when he has horses, oxen, mules, and asses to choose from?

16. Shew that the total number of permutations (with repetitions) of n different things, not more than r being taken at a time, is $\dfrac{n(n^r-1)}{n-1}$.

Combinations of Unlike Things.

507 *To find the number of* r-combinations *of* n *unlike things* [See Art 497]

Let ${}^nC_r$ denote the required number of combinations, or *groups* If the r things in each group are arranged in all possible ways, each group will give rise to $\lfloor r$ *arrangements*

$\therefore$ ${}^nC_r \times \lfloor r$ is equal to the number of r*-permutations* of n things,

hence $${}^nC_r \times \lfloor r = {}^nP_r = n(n-1)(n-2) \ldots (n-r+1),$$

$$\therefore {}^nC_r = \frac{n(n-1)(n-2) \ldots (n-r+1)}{\lfloor r} \qquad (1)$$

This formula for ${}^nC_r$ may be written in a different form, for if we multiply above and below by $\lfloor n-r$, we obtain

$$\frac{n(n-1)(n-2) \ldots (n-r+1) \times \lfloor n-r}{\lfloor r \lfloor n-r},$$

and since $n(n-1)(n-2) \ldots (n-r+1) \times \lfloor n-r = \lfloor n$, we have

$${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}, \text{ or } \frac{n!}{r!(n-r)!} \qquad (2)$$

NOTE In using formula (1) it is useful to remember that the suffix in the symbol ${}^nC_r$ denotes the number of factors in both numerator and denominator

EXAMPLE *From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?*

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11

Hence the number of ways $= {}^{11}C_4 = \dfrac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} = 330$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11

Hence the number of ways $= {}^{11}C_5 = \dfrac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5} = 462$

508 *The number of combinations of* n *things* r *at a time is equal to the number of combinations of* n *things* n − r *at a time*

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n-r$ things, that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n-r$ at a time,

$${}^nC_r = {}^nC_{n-r}$$

Thus $${}^{15}C_{13} = {}^{15}C_2 = \frac{15 \cdot 14}{1 \cdot 2} = 105$$

EXAMPLE *To prove that* ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$

${}^nC_r$=the number of r-combinations of $(n+1)$ things when one specified thing is always *excluded*

${}^nC_{r-1}$=the number of r-combinations of $(n+1)$ things when the specified thing is always *included*

The sum of these=the total number of r-combinations of $(n+1)$ things,

$$\therefore\ {}^nC_r+{}^nC_{r-1}={}^{n+1}C_r.$$

509. In the following examples the first thing to decide is whether the conditions imply *arrangements* or *selections only*

When arrangements are involved a formula for *permutations* must not be used until suitable *selections* have been made according to the conditions of the question

EXAMPLE 1 *From 7 masters and 4 boys a committee of 6 is to be formed in how many ways can this be done,* (i) *when the committee contains exactly 2 boys,* (ii) *at least 2 boys?*

Here we are not concerned with the possible arrangements of the members of the committee amongst themselves Hence it is a case of *selections only*

(i) The number of ways in which the 2 boys can be chosen is ${}^4C_2$; and the number of ways in which the 4 masters can be chosen is ${}^7C_4$ Each group of boys can be combined with each group of masters;

$$\therefore \text{ the required number}={}^4C_2\times{}^7C_4=\frac{4\cdot 3}{1\cdot 2}\times\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}=210$$

(ii) All the suitable combinations will be found by forming all the groups containing 2 boys and 4 masters, then 3 boys and 3 masters, and lastly 4 boys and 2 masters

The *sum* of these results will give the answer Hence the required number of ways $={}^4C_2\times{}^7C_4+{}^4C_3\times{}^7C_3+{}^4C_4\times{}^7C_2$

$$=\frac{4\cdot 3}{1\cdot 2}\times\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}+4\times\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}+1\times\frac{7\cdot 6}{1\cdot 2}$$

$$=210+140+21=371$$

EXAMPLE 2 *Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?*

Here each "word" means a different *arrangement*; hence we must first select sets of 3 consonants and 2 vowels and then combine them into sets of 5 letters which may be arranged among themselves

The number of ways of choosing the 3 consonants is ${}^7C_3$, and the number of ways of choosing the 2 vowels is ${}^4C_2$, hence the number of combined groups, each containing 3 consonants and 2 vowels, is ${}^7C_3\times{}^4C_2$

Each of these groups contains 5 different letters which may be arranged among themselves in $\lfloor 5$ ways;

$$\therefore \text{ the required number of words}=\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\times\frac{4\cdot 3}{1\cdot 2}\times\lfloor 5$$

$$=5\times\lfloor 7=25200$$

510 *To find the total number of ways in which it is possible to make a selection by taking some or all of* n *things*

Each thing may be dealt with in two ways, for it may be either taken or left, and since either way of dealing with any one thing may be associated with either way of dealing with each of the others, the number of ways of dealing with the n things is

$$2 \times 2 \times 2 \times 2 \quad \text{to } n \text{ factors}$$

But this includes the case in which all the things are left, therefore rejecting this case, the total number of ways is $2^n - 1$

This is sometimes referred to as "the total number of combinations of n things"

EXAMPLE *A man has 6 friends in how many ways may he invite one or more of them to dinner?*

He has to select some or all of his 6 friends,

the required number of ways $= 2^6 - 1$, or 63

Or thus The guests may be invited singly, by twos, threes, therefore the number of selections $= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$$= 6 + 15 + 20 + 15 + 6 + 1 = 63$$

511 *To find the number of ways in which* m+n *things can be subdivided into two groups containing* m *and* n *things respectively*

The number of ways of choosing a group of m things is ${}^{m+n}C_m$, and with each of such selections the n things make up the second group

$$\text{the required number} = \frac{\lfloor m+n}{\lfloor m \lfloor n}$$

NOTE If $n = m$, the groups are equal, and it is possible to interchange the two groups without obtaining a new subdivision, hence the number of *different* ways of subdivision is $\dfrac{\lfloor 2m}{\lfloor m \lfloor m \lfloor 2}$

EXAMPLE *In how many ways can 12 books be divided into 3 sets each containing 4 books?*

The books can be divided into sets of 4 and 8 in ${}^{12}C_4$ ways

Then each set of 8 can be divided into 2 sets of 4 in ${}^8C_4$ ways

Thus the number of ways of making up the 3 sets

$$= \frac{\lfloor 12}{\lfloor 4 \lfloor 8} \times \frac{\lfloor 8}{\lfloor 4 \lfloor 4} = \frac{\lfloor 12}{\lfloor 4 \lfloor 4 \lfloor 4}, \quad \text{or} \quad \frac{\lfloor 12}{(\lfloor 4)^3}$$

But these are not all *different* modes of subdivision, for since the sets are equal there are $\lfloor 3$ different orders in which the sets can be arranged without giving a different subdivision

$$\text{Hence the required number} = \frac{\lfloor 12}{(\lfloor 4)^3 \times \lfloor 3}$$

[*Examples* XXXIX d 1-20, *page* 464, *may be taken here.*]

512 *For a given value of* n *to find the value of* r *which makes* ^nC_r *greatest*

We have $^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}$, since ^nC_r has one more factor than $^nC_{r-1}$ both in numerator and denominator.

The multiplying factor $\frac{n-r+1}{r}$ may be written $\frac{n+1}{r}-1$, which shews that it decreases as r increases Hence by giving to r the values 1, 2, 3, . in succession, ^nC_r is continually increased until $\frac{n+1}{r}-1$ becomes equal to 1 or less than 1

Hence $^nC_r >$, or $= {}^nC_{r-1}$ according as $\frac{n+1}{r}-1 >$, or $=1$;

$$\text{that is, according as } \frac{n+1}{r} >, \text{ or } =2;$$

$$\text{that is, according as } \frac{n+1}{2} >, \text{ or } =r.$$

Also r can only have integral values

(i) If n is *even*, then $\frac{n+1}{2}$ is a fraction, and the greatest value r can have is $\frac{n}{2}$

Hence ^nC_r is greatest when $r=\frac{n}{2}$

(ii) If n is *odd*, then $\frac{n+1}{2}$ is integral, and when $r=\frac{n+1}{2}$ we have $^nC_r = {}^nC_{r-1}$

Hence ^nC_r is greatest when $r=\frac{n+1}{2}$, or $\frac{n-1}{2}$; the result being the same in the two cases

513 *To find the total number of ways in which it is possible to make a selection by taking some or all out of* p+q+r+ *things, whereof* p *are alike of one kind,* q *alike of a second kind,* r *alike of a third kind, and so on*

The p things may be disposed of in $p+1$ ways; for we may take 0, 1, 2, 3, p of them Similarly the q things may be disposed of in $q+1$ ways, the r things in $r+1$ ways, and so on

Hence the number of ways in which all the things may be disposed of is $(p+1)(q+1)(r+1)$.

But this includes the case in which none of the things are taken; therefore, rejecting this case, the total number of ways is

$$(p+1)(q+1)(r+1) \ldots -1$$

514 Miscellaneous Examples in Permutations and Combinations

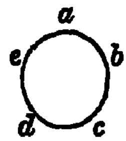
EXAMPLE 1 *In how many ways can 5 things, such as* a, b, c, d, e, *be arranged in a ring?*

If we call an arrangement *linear* or *circular* according as the things are placed in a *row* or in a *ring*, it will be seen that while any linear arrangement depends upon the *absolute* position in which the things stand, a circular arrangement depends only on the position of the things *relatively to each other*

Thus the 5 arrangements

$$abcde, \quad bcdea, \quad cdeab, \quad deabc, \quad eabcd,$$

which are all different *linear* arrangements, have no essential difference when regarded as *circular* arrangements For they are obtained by merely reading the 5 letters in cyclic order, starting from each letter successively Hence in the case of 5 different things each circular arrangement gives 5 linear arrangements



Hence the number of circular arrangements

$$= \tfrac{1}{5} \text{ of the number of linear arrangements}$$

$$= \tfrac{1}{5} |\underline{5} = |\underline{4}$$

Similarly n different things can be arranged round a circle in $|\underline{n-1}$ ways

We may arrive at the same result briefly as follows since in each arrangement we are concerned only with the position of a thing relatively to the others, let one thing be placed in *any one* position, then the remaining $n-1$ things can fill the remaining places in $|\underline{n-1}$ ways

EXAMPLE 2 *I have 6 sorts of books and 3 of each sort In how many ways can a selection be made from them?*

In the case of each book we may take 0, 1, 2, 3, that is, we may deal with each book in 4 ways, and therefore with the 6 sorts of books in 4^6 ways But this includes the case in which no selection is made,

hence the required number $= 4^6 - 1 = 4095$

EXAMPLE 3 *A railway carriage will accommodate 5 passengers on each side in how many ways can 10 persons take their seats when two of them decline to face the engine, and a third cannot travel backwards?*

Divide the 7 persons who can sit on either side into two groups containing 3 and 4 respectively This can be done in ${}^7C_3$ ways

Now each side admits of $|\underline{5}$ arrangements,

$$\text{the required number} = {}^7C_3 \times |\underline{5} \times |\underline{5}$$

$$= \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \times 120 \times 120$$

$$= 504000$$

EXAMPLE 4 *Find the number of arrangements that can be made from the letters* a, a, a, b, b, c, d, e, *taking them 4 at a time*

Here we have 8 letters of 5 different kinds

In finding *groups* of 4, these may be classified as follows

(i) 3 alike, 1 different, (ii) 2 alike, 2 others alike;

(iii) 2 alike, 2 different; (iv) all 4 different

(i) The *selection* can be made in 4 ways, for each of the letters b, c, d, e may be taken with the group aaa

the number of *arrangements* $=4\times\dfrac{\underline{|4}}{\underline{|3}}=16$

(ii) The *selection* can be made in 1 way only Hence the number of *arrangements* $=\dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}}=6$

(iii) The *selection* can be made in $2\times{}^4C_2$ ways, for we have a choice of 2 pairs of like letters, and 2 out of the remaining 4 letters

the number of *arrangements* $=2\times\dfrac{4\cdot 3}{1\cdot 2}\times\dfrac{\underline{|4}}{\underline{|2}}=12\times 12=144$

(iv) The *selection* can be made in ${}^5C_4$, or 5 ways Hence the number of *arrangements* $=5\times\underline{|4}=120$

the total number of arrangements $=16+6+144+120=286$

EXAMPLES XXXIX. d.

1. Find the numerical values of ${}^6C_3$, ${}^9C_6$, ${}^{11}C_8$, ${}^{50}C_{48}$, $\dfrac{15!}{13!\,2!}$.

2 In how many ways can 4 boys be chosen out of a form of 21, so as always to include the head of the form?

3 There are 8 bay, 7 black, and 5 roan horses for sale, in how many ways is it possible to buy 12 horses, 4 of each colour?

4 How many parcels of 6 books may be made out of 7 Latin and 3 Greek books, (i) when there is no restriction, (ii) when each parcel holds 1 Greek book, (iii) when each parcel holds all the Greek books?

5 A committee of 6 is to be chosen from 7 Englishmen, 1 Frenchman, 1 German, and 1 Italian In how many ways can the choice be made, (i) to include the Frenchman, (ii) to include exactly one foreigner, (iii) to include at least one foreigner?

6. If $2\times{}^nC_4=35\times{}^{\frac{n}{2}}C_3$, find n 7. If ${}^{28}C_{r+4}={}^{28}C_{r-2}$, find r

8 In the formula ${}^nC_r=\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ put $r=n$, and hence find a meaning for the symbol $\underline{|0}$

9. Out of the 26 letters of the alphabet how many words can be made consisting of 4 letters one of which must be a?

10. How many words can be made by taking 3 consonants and 2 vowels out of 15 consonants and 4 vowels?

11. In how many ways can 5 chairs be occupied by 3 men and 2 boys taken from 6 men and 5 boys? Explain clearly why the formula ${}^6P_3 \times {}^5P_2$ does not give the correct answer.

12. Of 15 men, 3 can steer and cannot row, and the rest can row but cannot steer; in how many ways can the crew of an eight-oar, with a coxswain, be made up?

13. There are 4 candidates for 2 vacancies. There are 3 electors, each of whom can vote for two candidates or plump for one: in how many ways can the votes be given?

14. From 5 ladies and 7 gentlemen how many different parties can be made up to travel in a railway carriage which has 4 seats on each side, supposing all the ladies to be included in each party? In how many different ways can the seats be occupied if the corner seats are always to be reserved for ladies?

15. Find the total number of selections that can be made out of 8 things.

16. How many parcels, each containing not more than 6, can be made with 8 books to choose from?

17. How many choices has a purchaser from 10 things exposed for sale?

18. A house has 9 windows in the front: how many different signals can be made by leaving one or more of the windows open?

19. At an election three districts are to be canvassed by 10, 12, and 8 men respectively. If 30 men volunteer, in how many ways can they be allotted to the different districts? [*Give the answer in factorials.*]

20. In how many ways can 52 cards be divided, (i) into four sets of 13 each; (ii) equally among four players?

(*Miscellaneous.*)

21. In how many ways can 6 letters be placed in 6 envelopes, one in each, if 2 of the letters are too large for one of the envelopes?

22. A telegraph has 4 arms, and each arm has 3 positions, including the position of rest; find the total number of signals that can be made.

23. In how many ways can 10 examination papers be arranged so that the best and worst papers never come together?

24. Find the number of ways in which 5 ladies and 5 gentlemen can be placed alternately in a ring.

25. Find the number of ways in which mn things can be divided equally among n persons

26. From 4 pears, 2 apples, and 3 oranges how many selections of fruit can be made, taking at least one of each kind

27. There are 10 points in a plane, 4 of which are collinear find the number (i) of straight lines, (ii) of triangles, which result from joining them.

28. In how many ways can a crew of 8 be made up when 2 of the men can only row on stroke side, and 1 only on bow side?

29. Find the number of ways in which n books can be arranged on a shelf so that two specified books are not together

30. Find the number of ways in which the letters of the word *abstemious* may be arranged, (i) without altering the place of any vowel (ii) without changing the order of the vowels

31. Out of 4 ladies and 5 gentlemen how many sets of two pairs for lawn-tennis can be arranged, (i) when there is no restriction as to the players on each side, (ii) when each pair consists of one lady and one gentleman?

32. Find the number of ways in which a selection can be made from m sorts of things, and n things of each sort

33. Find the number (i) of selections, (ii) of arrangements that can be made by taking 4 letters from the word *zoology*

34. A man has 8 bachelor friends, and he wishes to invite r of them to dine with him on successive evenings as long as he can have a different selection each time. For how many evenings is it possible to continue these parties, and how often will each of the 8 friends form one of the party?

35 In how many ways can n men be arranged in a row if two specified men are neither of them to be at either extremity of the row?

36. If there are 10 things of which 2 are alike, find the number of permutations of them taken 5 at a time

37. Find the number (i) of combinations, (ii) of permutations that can be made from the letters of the word *expression*, taken 4 at a time

38. How many choices are there in buying books from a book-stall where 5 copies of one book, 6 copies of another book, and single copies of 5 other different books are offered for sale?

39. Twenty-two men arrange to play a cricket match If two of the men are brothers, shew that the number of ways in which the teams can be made up so that the brothers do not play on the same side is $2\,\underline{|19} \div \underline{|10}\,\underline{|9}$

CHAPTER XL

MATHEMATICAL INDUCTION

515 In proving the truth of certain mathematical results or formulæ, it is often convenient to use an indirect method known as **Mathematical Induction.** We shall explain this method of proof by examples

EXAMPLE. *To prove that* $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots$ *to* n *terms* $=\frac{n}{n+1}$

We can easily shew that this formula is true in simple cases, such as when $n=1$, or 2, or 3 We wish, however, to prove it true in *all* cases

Assume that it is true when n terms are taken,

that is,
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

To each side add $\frac{1}{(n+1)(n+2)}$, which is the $(n+1)^{\text{th}}$ or *next* term in the series, then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots \text{ to } (n+1) \text{ terms}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}\left(n+\frac{1}{n+2}\right)$$
$$=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{n+1}{n+2}$$
$$=\frac{n+1}{(n+1)+1}$$

Now this last result is of the same form as that assumed for n terms, with $n+1$ written in the place of n In other words, if the result is true when we take a certain number of terms, whatever that number may be, it is true when we increase that number by one, but by trial it *is* true when 3 terms are taken, hence it is true for 4 terms, therefore for 5 terms, and so on Thus the result is true universally

The method of proof involves the following steps

(i) Assuming the truth of the formula in *any one case* (say the n^{th}), we shew that it must be true in the *next* case, viz the $(n+1)^{\text{th}}$

(ii) By trial we shew that it *is* true in a certain simple case, hence it is true in the case next after that verified, hence it is true in the next case, and so on Hence it is true in all cases

516. As another example we will prove one of the theorems relating to divisibility given in Art 468

EXAMPLE *Shew that* $x^n - y^n$ *is divisible by* $x - y$ *for all positive integral values of* n.

By going through one step of division we have

$$\frac{x^n - y^n}{x - y} = x^{n-1} + \frac{y(x^{n-1} - y^{n-1})}{x - y}$$

If therefore $x^{n-1} - y^{n-1}$ is divisible by $x - y$, then $x^n - y^n$ is also divisible by $x - y$

But $x^2 - y^2$ is divisible by $x - y$; therefore $x^3 - y^3$ is divisible by $x - y$; therefore $x^4 - y^4$ is divisible by $x - y$, and so on. Hence the proposition is universally true

517 Speaking generally, it will be seen that proof by induction can be conveniently applied to all theorems which admit of successive cases corresponding to the order of the natural numbers 1, 2, 3, ..., n But if the truth of a general theorem depends upon whether n is odd or even, it must be remembered that in considering *any one* case (say the n^{th}), the *next* case will be the $(n+2)^{\text{th}}$

EXAMPLES XL.

Prove by Induction

1. $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

2. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$.

4. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

5. $3 + 3^2 + 3^3 + \ldots + 3^n = \dfrac{3(3^n - 1)}{2}$

6. $\dfrac{1}{1.4} + \dfrac{1}{4 \cdot 7} + \dfrac{1}{7 \cdot 10} + \ldots$ to n terms $= \dfrac{n}{3n+1}$

7. Prove by Induction.

(i) that $x^n + y^n$ is divisible by $x + y$ when n is any odd positive integer,

(ii) that $x^n - y^n$ is divisible by $x + y$ when n is any even positive integer

CHAPTER XLI

THE BINOMIAL THEOREM

518 IT may be shewn by actual multiplication that

$$(x+a)(x+b)(x+c)(x+d)$$
$$=x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$+(abc+abd+acd+bcd)x+abcd \quad . \quad . \quad . (1)$$

We may, however, write down this result by inspection, for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors If we examine the way in which *the various partial products are formed*, we see *that*

(i) the term x^4 is formed by taking the letter x out of *each* of the factors

(ii) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor

(iii) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and *two* of the letters a, b, c, d out of the remaining factors.

(iv) the terms involving x are formed by taking the letter x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors

(v) the term independent of x is the product of all the letters a, b, c, d

EXAMPLE *Find the value of* $(x-2)(x+3)(x-5)(x+9)$

The product

$$=x^4+(-2+3-5+9)x^3+(-6+10-18-15+27-45)x^2$$
$$+(30-54+90-135)x+270$$
$$=x^4-5x^3-47x^2-69x+270$$

519 If in equation (1) of the preceding article we suppose $b=c=d=a$, we obtain

$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4$$

We shall now employ the same method to prove a formula known as the Binomial Theorem, by which any binomial of the form $x+a$ can be raised to any assigned positive integral power

520 *Expansion of* $(x+a)^n$ *when* n *is a positive integer.*

Consider the product of the n binomial factors

$$x+a, \quad x+b, \quad x+c, \quad \ldots \quad x+k.$$

In the continued product of these factors every term is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of these factors

The highest power of x is x^n, and is formed by taking the letter x from each of the n factors

The terms involving x^{n-1} are formed by taking the letter x from *any* $n-1$ of the factors, and one of the letters $a, b, c, \ldots k$ from the remaining factor, thus in the final product

the coefficient of $x^{n-1}=a+b+c+\ldots+k=S_1$, where S_1 stands for the *sum* of the letters $a, b, c, \ldots k$ taken *one* at a time

The terms involving x^{n-2} are formed by taking the letter x from *any* $n-2$ of the factors, and *two* of the letters $a, b, c, \ldots k$ from the two remaining factors, thus in the final product

the coefficient of $x^{n-2}=ab+ac+bc+\ldots=S_2$, where S_2 stands for the *sum of the products* of the letters $a, b, c, \ldots k$ taken *two* at a time

And, generally, the terms involving x^{n-r} are formed by taking the letter x from *any* $n-r$ of the factors, and r of the letters $a, b, c, \ldots k$ from the r remaining factors, thus in the final product

the coefficient of $x^{n-r}=S_r$, where S_r stands for the *sum of the products* of the letters $a, b, c, \ldots k$ taken r at a time

The last term in the product is $abc\ldots k$, denote it by S_n. Then

$$(x+a)(x+b)(x+c)\ldots(x+k)$$
$$=x^n+S_1x^{n-1}+S_2x^{n-2}+\ldots+S_rx^{n-r}+\ldots+S_{n-1}x+S_n$$

Here *the number of terms* in S_1 is n, that is, ${}^nC_1$, *the number of terms* in S_2 is the same as the number of combinations of n things 2 at a time, that is, ${}^nC_2$, *the number of terms* in S_3 is ${}^nC_3$, and so on

Now suppose $b, c, \ldots k$ each equal to a, then the product on the left becomes $(x+a)^n$ Also on the right

S_1 becomes $a+a+a+\ldots$ to ${}^nC_1$ terms,
S_2 " $a^2+a^2+a^2+\ldots$ to ${}^nC_2$ terms,
S_3 " $a^3+a^3+a^3+\ldots$ to ${}^nC_3$ terms,
.. ..
S_r " $a^r+a^r+a^r+\ldots$ to ${}^nC_r$ terms,
. .

Also S_n becomes a^n, that is, ${}^nC_na^n$, since ${}^nC_n=1$

Thus we obtain finally

$$(x+a)^n=x^n+{}^nC_1ax^{n-1}+{}^nC_2a^2x^{n-2}+{}^nC_3a^3x^{n-3}+\ldots$$
$$+{}^nC_ra^rx^{n-r}+\ldots+{}^nC_na^n.$$

Substituting in full for ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, ... we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3x^{n-3} + \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}a^rx^{n-r} + \ldots + a^n$$

This is the *Binomial Theorem*, and the expression on the right is said to be the expansion of $(x+a)^n$.

521. **Proof by Induction.** We have to prove that

$$(x+a)^n = x^n + {}^nC_1ax^{n-1} + {}^nC_2a^2x^{n-2} + \ldots + {}^nC_ra^rx^{n-r} + \ldots + a^n.$$

By actual multiplication or involution, we have

$$(x+a)^2 = x^2 + 2ax + a^2 \qquad = x^2 + {}^2C_1ax + a^2,$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 = x^3 + {}^3C_1ax^2 + {}^3C_2a^2x + a^3.$$

Assume that the formula is true for the n^{th} power of $(x+a)$, that is,

$$(x+a)^n = x^n + {}^nC_1ax^{n-1} + {}^nC_2a^2x^{n-2} + \ldots {}^nC_ra^rx^{n-r} + \ldots + a^n.$$

Multiply both sides by $x+a$, then we have

$$x(x+a)^n = x^{n+1} + {}^nC_1ax^n + {}^nC_2a^2x^{n-1} + \ldots + {}^nC_ra^rx^{n-r+1} + \ldots + a^nx,$$
$$a(x+a)^n = \qquad ax^n + {}^nC_1a^2x^{n-1} + \ldots + {}^nC_{r-1}a^rx^{n-r+1} + \ldots + a^{n+1}.$$

Therefore by addition, after collecting like terms on the right,

$$(x+a)^{n+1} = x^{n+1} + ({}^nC_1+1)ax^n + ({}^nC_2+{}^nC_1)a^2x^{n-1} + \ldots$$
$$+ ({}^nC_r+{}^nC_{r-1})a^rx^{n-r+1} + \ldots + a^{n+1}.$$

Now
$$ {}^nC_r + {}^nC_{r-1} = \frac{\lfloor n}{\lfloor r \lfloor n-r} + \frac{\lfloor n}{\lfloor r-1 \lfloor n-r+1}$$
$$= \frac{\lfloor n}{\lfloor r-1 \lfloor n-r}\left(\frac{1}{r} + \frac{1}{n-r+1}\right)$$
$$= \frac{\lfloor n}{\lfloor r-1 \lfloor n-r} \cdot \frac{n+1}{r(n+1-r)} = \frac{\lfloor n+1}{\lfloor r \lfloor n+1-r}$$
$$= {}^{n+1}C_r.$$

$\therefore$ also ${}^nC_1 + 1 = {}^{n+1}C_1$, ${}^nC_2 + {}^nC_1 = {}^{n+1}C_2$, and so on.

$$(x+a)^{n+1} = x^{n+1} + {}^{n+1}C_1ax^n + {}^{n+1}C_2a^2x^{n-1} + \ldots$$
$$+ {}^{n+1}C_ra^rx^{n+1-r} + \ldots + a^{n+1}.$$

Thus the terms of $(x+a)^{n+1}$ are of the same form as those assumed for the expansion of $(x+a)^n$, with $n+1$ taking the place of n. Hence, if the theorem is true for the n^{th} power of $x+a$, it is also true for the $(n+1)^{\text{th}}$ power. But we have seen that it is true for $(x+a)^3$, therefore it is true for $(x+a)^4$, therefore for $(x+a)^5$, and so on. Thus the theorem is true for any positive integral value of n.

522. In the formula

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2}a^2x^{n-2} +$$
$$+ \frac{n(n-1)(n-2) \ldots (n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}a^r x^{n-r} + \ldots + a^n,$$

the following points should be noted:

(i) The number of terms on the right is $n+1$, that is, one more than the index of $x+a$ on the left.

(ii) The index of a in any term is the last factor in the denominator of the coefficient of that term.

(iii) In every term the sum of the indices of x and a is n.

523. We shall often quote the theorem in the form

$$(x+a)^n = x^n + {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} + \ldots + {}^nC_r a^r x^{n-r} + \ldots + {}^nC_n a^n.$$

If we write $-a$ for a, we obtain

$$(x-a)^n = x^n + {}^nC_1(-a)x^{n-1} + {}^nC_2(-a)^2x^{n-2} + {}^nC_3(-a)^3x^{n-3} + \ldots$$
$$+ {}^nC_r(-a)^r x^{n-r} + \ldots + {}^nC_n(-a)^n$$
$$= x^n - {}^nC_1 ax^{n-1} + {}^nC_2 a^2x^{n-2} - {}^nC_3 a^3 x^{n-3} + \ldots + {}^nC_n(-a)^n.$$

Thus the terms of $(x+a)^n$ and $(x-a)^n$ are *numerically* the same, but in $(x-a)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

NOTE. For the sake of uniformity we may use ${}^nC_0$ as the coefficient of x^n. This is justified by the consideration that there is *only one* way of making *no* selection (i.e. rejecting all) out of n things. Hence ${}^nC_0 = 1$.

EXAMPLE 1. *Find the expansion of* $(x+3y)^6$.

Putting $3y$ for a in the formula, we have

$$\text{the expansion} = x^6 + {}^6C_1x^5(3y) + {}^6C_2x^4(3y)^2 + {}^6C_3x^3(3y)^3 + {}^6C_4x^2(3y)^4$$
$$+ {}^6C_5x(3y)^5 + {}^6C_6(3y)^6.$$

Hence by calculating the values of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$ we have

$$(x+3y)^6 = x^6 + 6 \cdot 3x^5y + 15 \cdot 9x^4y^2 + 20 \cdot 27x^3y^3 + 15 \cdot 81x^2y^4 + 6 \cdot 243xy^5 + 729y^6$$
$$= x^6 + 18x^5y + 135x^4y^2 + 540x^3y^3 + 1215x^2y^4 + 1458xy^5 + 729y^6.$$

EXAMPLE 2. *Find the expansion of* $(a-2x)^7$.

$$(a-2x)^7 = a^7 - {}^7C_1a^6(2x) + {}^7C_2a^5(2x)^2 - {}^7C_3a^4(2x)^3 + \ldots \text{ to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$, after calculating the coefficients up to ${}^7C_3$, the rest may be written down at once; for ${}^7C_4 = {}^7C_3$, ${}^7C_5 = {}^7C_2$, and so on. Hence

$$(a-2x)^7 = a^7 - 7a^6(2x) + \frac{7 \cdot 6}{1 \cdot 2}a^5(2x)^2 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}a^4(2x)^3 + \ldots$$
$$= a^7 - 7a^6(2x) + 21a^5(2x)^2 - 35a^4(2x)^3 + 35a^3(2x)^4$$
$$- 21a^2(2x)^5 + 7a(2x)^6 - (2x)^7$$
$$= a^7 - 14a^6x + 84a^5x^2 - 280a^4x^3 + 560a^3x^4$$
$$- 672a^2x^5 + 448ax^6 - 128x^7.$$

524 The symbols ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, are often referred to as the Binomial Coefficients of the n^{th} order, and it is important to be able to calculate their numerical values quickly We shall now explain a rule by which this may be done

In $(x+a)^1$ the coefficients are 1 1,
" $(x+a)^2$ " " 1 2 1,
" $(x+a)^3$ " " 1 3 3 1,
" $(x+a)^4$ " " 1 4 6 4 1

Here the first and last coefficient in each line is 1, and for the rest we notice that in any line each coefficient is equal to the coefficient immediately above it added to the preceding coefficient in the same line

Thus in the 3rd line $3=2+1$, $3=1+2$,

and in the 4th line $4=3+1$, $6=3+3$, $4=1+3$

The general rule is given in the Example of Art 508, where it is shewn that ${}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$, a result which may be verbally expressed by saying that *the* r^{th} *binomial coefficient of any order is equal to the sum of the* r^{th} *and* $(r-1)^{\text{th}}$ *coefficients of the preceding order*

By the use of this rule it is easy to write down consecutively the binomial coefficients for different values of n, as in the following Table

n	Coefficients								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

525 The Binomial Theorem may be applied to expand expressions which contain more than two terms

EXAMPLE *Find the expansion of* $(x^2+2x-1)^3$

Regarding $2x-1$ as a single term,

$$\begin{aligned} \text{the expansion} &= (x^2+\overline{2x-1})^3 \\ &= (x^2)^3+3(x^2)^2(2x-1)+3x^2(2x-1)^2+(2x-1)^3 \\ &= x^6+3x^4(2x-1)+3x^2(4x^2-4x+1)+8x^3-12x^2+6x-1 \\ &= x^6+6x^5+9x^4-4x^3-9x^2+6x-1 \end{aligned}$$

526 The simplest form of the Binomial Theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Art 523, by writing 1 for x, and x for a.

$$\text{Thus } (1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

$$= 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r + \ldots + x^n$$

The expansion of a binomial may always be made to depend upon the case in which the first term is unity,

$$\text{thus} \qquad (x+y)^n = \left\{x\left(1+\frac{y}{x}\right)\right\}^n = x^n(1+z)^n, \quad \text{where } z = \frac{y}{x}$$

EXAMPLE 1 *Expand* $(1+2x)^4(1-x^2)^3$ *as far as the term which involves* x^5.

$$(1+2x)^4 = 1 + 4(2x) + 6(2x)^2 + 4(2x)^3 + (2x)^4$$

$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4, \qquad (1)$$

$$(1-x^2)^3 = 1 - 3x^2 + 3x^4 - x^6 \qquad (2)$$

The product of the expressions (1) and (2) may now be obtained by using detached coefficients In (2) we must write 0 for the coefficients of any missing powers of x, and we may omit all terms which would give rise in the product to powers of x higher than the fifth

$$\begin{array}{r|}
1+8+24+32+16 \\
\underline{1+0-\ 3+\ 0+\ 3+\ 0} \\
1+8+24+32+16 \\
-\ 3-24-72-96 \\
\underline{+\ 3+24} \\
1+8+21+\ 8-53-72
\end{array}$$

Thus the required result is $1 + 8x + 21x^2 + 8x^3 - 53x^4 - 72x^5$

EXAMPLE 2 *Find the value of* $(1+\sqrt{1-x^2})^5 + (1-\sqrt{1-x^2})^5$.

Put y for $\sqrt{1-x^2}$, then we have to find the sum of the expansions of $(1+y)^5$ and $(1-y)^5$ In these the terms are numerically equal, but in the second expansion the second, fourth, and sixth terms are negative, and therefore destroy the corresponding terms of the first expansion

$$\text{Hence the required value} = 2\{1 + {}^5C_2y^2 + {}^5C_4y^4\}$$

$$= 2\{1 + 10(1-x^2) + 5(1-x^2)^2\}$$

$$= 2\{1 + 10 - 10x^2 + 5 - 10x^2 + 5x^4\}$$

$$= 32 - 40x^2 + 10x^4$$

[*Examples* XLI a 1-22, *page* 476, *may be taken here*]

527 General Term In the expansion of $(x+a)^n$, the coefficient of the second term is ${}^nC_1$ of the third term is ${}^nC_2$, of the fourth term is ${}^nC_3$; and so on, *the suffix of C in each term being one less than the number of the term to which it applies*, hence ${}^nC_r$ is the coefficient of the $(r+1)^{\text{th}}$ term This is called the **general term**, because by giving to r different numerical values any of the coefficients may be found from ${}^nC_r$, and by giving to x and a their appropriate indices the value of any assigned term may be obtained.

Thus the $(r+1)^{\text{th}}$ term $={}^nC_r x^{n-r}a^r$, and may be written

$$\frac{n(n-1)(n-2)\ \ (n-r+1)}{\lfloor r}x^{n-r}a^r, \quad \text{or} \quad \frac{\lfloor n}{\lfloor n-r \lfloor r}x^{n-r}a^r$$

In applying the form ${}^nC_r x^{n-r}a^r$ to any particular case, it should be observed that *the index of* a *is the same as the suffix of C, and that the sum of the indices of* x *and* a *is* n

The symbol T_{r+1} is often used to denote the $(r+1)^{\text{th}}$ term.

EXAMPLE 1 *Find the fifth term of* $(x+2y^3)^{17}$.

Here $$T_5=T_{4+1}={}^{17}C_4x^{13}(2y^3)^4$$
$$=\frac{17\ 16\ 15\ 14}{1\ 2\ 3\ 4}\times 16x^{13}y^{12}$$
$$=38080x^{13}y^{12}$$

EXAMPLE 2 *Find the fourteenth term of* $(3-a)^{15}$.

Here $$T_{14}={}^{15}C_{13}(3)^2(-a)^{13}$$
$$={}^{15}C_2\times(-9a^{13})$$
$$=-945a^{13}.$$

EXAMPLE 3 *Find the term containing* x^{16} *in the expansion of* $(x^3-2x)^{10}$

We have $$(x^3-2x)^{10}=\left\{x^3\left(1-\frac{2}{x^2}\right)\right\}^{10}=x^{30}\left(1-\frac{2}{x^2}\right)^{10}$$

Hence the term containing x^{16} will be obtained from the product of x^{30} and the term which contains $\frac{1}{x^{14}}$ in $\left(1-\frac{2}{x^2}\right)^{10}$ This is the term which contains $\left(\frac{2}{x^2}\right)^7$

Hence

$$\text{the required term}=x^{30}\times{}^{10}C_7\left(-\frac{2}{x^2}\right)^7=-\frac{10\ 9\ 8}{1\ 2\ 3}\times 2^7x^{16}=15360x^{16}$$

EXAMPLE 4 *Find the coefficient of* x^9 *in the expansion of* $\left(x^2+\frac{1}{x^3}\right)^{12}$.

The general term $={}^{12}C_r(x^2)^{12-r}\left(\frac{1}{x^3}\right)^r={}^{12}C_rx^{24-5r}$

Now if $\qquad 24-5r=9$, then $r=3$.

Thus the required coefficient $={}^{12}C_3=\frac{12\ 11\ 10}{1\ 2.3}=220$

EXAMPLES XLI. a.

1. As in Art. 519, find the expansions of

(i) $(x+2)(x-1)(x+4)$; (ii) $(x-3)(x+7)(x-2)(x+4)$;

(iii) $(x-4)(x+5)(x+4)(x-5)$; (iv) $(a+2b)(a-5b)(a+9b)$

Verify (iii) by an independent method.

Expand the following binomials:

2. $(x+2)^4$. 3. $(x+3)^5$. 4. $(a-x)^5$. 5. $(a+x)^7$.

6. $(1+b)^8$. 7. $(1-2y)^5$. 8. $\left(1+\frac{x}{2}\right)^6$. 9. $\left(2x+\frac{y}{2}\right)^4$.

10. $\left(2-\frac{x}{2}\right)^6$. 11. $\left(a-\frac{3}{b}\right)^7$. 12. $\left(x-\frac{1}{2x}\right)^5$. 13. $\left(ax+\frac{y}{a}\right)^9$.

Expand and simplify

14. $(a+b)^5-(a-b)^5$. 15. $(3-2x)^6+(3+2x)^6$.

16. $(x-\sqrt{3})^4+(x+\sqrt{3})^4$. 17. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.

18. Shew that $(x-\sqrt{1-x^2})^4+(x+\sqrt{1-x^2})^4=2+8x^2-8x^4$.

Expand the following trinomials

19. $(x^2-x-2)^3$. 20. $(1+x+x^2)^4$. 21. $(1-2a+3a^2)^3$.

22. Expand (i) $(a+x)^6(a-x)^2$, (ii) $(1-x)^5(1+2x)^3$.

Write down and simplify

23. The 4^{th} term of $(1+2x)^7$. 24. The 6^{th} term of $(2-y)^8$.

25. The 5^{th} term of $(a-5b)^8$. 26. The 15^{th} term of $(2x-1)^{17}$.

27. The 7^{th} term of $\left(1-\frac{1}{x}\right)^{10}$. 28. The 6^{th} term of $\left(3x+\frac{a}{2}\right)^9$.

29. The middle term of (i) $\left(x^2+\frac{1}{x}\right)^6$; (ii) $\left(3a-\frac{1}{2a}\right)^8$.

30. The two middle terms of $\left(x-\frac{1}{x}\right)^9$.

31. The 6^{th} term of $\left(\frac{2a}{3}-\frac{3}{2a}\right)^{10}$. 32. The 23^{rd} term of $\left(x^2+\frac{b}{x}\right)^{25}$.

33. Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

34. Find the coefficient of x in the expansion of $\left(x^2-\frac{a}{2x}\right)^{14}$.

35. Find the coefficients of x^4 and x^{-1} in $\left(x^3-\frac{1}{x^2}\right)^8$.

36. Find the term independent of x in $\left(2x^2+\frac{1}{x}\right)^{12}$.

37. Shew that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

38. If a_r denotes the coefficient of x^r in the expansion of $(1-x)^{2m-1}$, prove that $a_{r-1}+a_{2m-r}=0$

528 *In the expansion of* $(1+x)^n$ *the coefficients of terms equidistant from the beginning and end are equal*

The coefficient of the $(r+1)^{\text{th}}$ term from the beginning is ${}^nC_r$

The $(r+1)^{\text{th}}$ term from the end has $(n+1)-(r+1)$, or $n-r$ terms before it, therefore counting from the beginning it is the $(n-r+1)^{\text{th}}$ term, and its coefficient is ${}^nC_{n-r}$, which has been shewn to be equal to ${}^nC_r$ [Art 508] Hence the proposition follows

529 *To find the greatest term in the expansion of* $(1+x)^n$

Here
$$T_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\ldots r}x^r,$$

$$T_r=\frac{n(n-1)(n-2)\ldots(n-r+2)}{1\cdot 2\cdot 3\ldots(r-1)}x^{r-1}$$

$$T_{r+1}=T_r\times\frac{n-r+1}{r}x$$

Hence the $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r}x$, that is, by $\left(\frac{n+1}{r}-1\right)x$

The factor $\frac{n+1}{r}-1$ decreases as r increases, hence the $(r+1)^{\text{th}}$ term is not always greater than the r^{th} term, but only until $\left(\frac{n+1}{r}-1\right)x$ becomes equal to 1, or less than 1

Now
$$\left(\frac{n+1}{r}-1\right)x>1, \text{ so long as } \frac{n+1}{r}-1>\frac{1}{x},$$

that is,
$$\frac{n+1}{r}>\frac{1}{x}+1, \quad\text{or}\quad \frac{(n+1)x}{1+x}>r \qquad (1)$$

If $\frac{(n+1)x}{1+x}$ is an integer, denote it by p, then if $r=p$ the multiplying factor becomes equal to 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term

If $\frac{(n+1)x}{1+x}$ is not an integer, denote its integral part by q, then the greatest value of r consistent with (1) is q; hence the $(q+1)^{\text{th}}$ term is the greatest

Since we are only concerned with the *numerically* greatest term, the investigation will be the same for $(1-x)^n$, therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial

NOTE To find the greatest *coefficient* in $(1+x)^n$ we have only to find, as in Art 512, the value of r which makes ${}^nC_r$ greatest

530 If a binomial is given in the form $(x+a)^n$,

we have $$(x+a)^n = x^n\left(1+\frac{a}{x}\right)^n,$$

therefore, since x^n multiplies every term in $\left(1+\frac{a}{x}\right)^n$, it will be sufficient to find which is the greatest term in this latter expansion

EXAMPLE *Find the greatest term in the expansion of* $(x-4a)^8$, *when* $x=\frac{1}{2}$ *and* $a=\frac{1}{3}$.

Since $(x-4a)^8 = x^8\left(1-\frac{4a}{x}\right)^8$, it will be sufficient to find which is the greatest term of $\left(1-\frac{4a}{x}\right)^8$

Here $$T_{r+1} = T_r \times \frac{8-r+1}{r} \cdot \frac{4a}{x}, \text{ numerically,}$$

$$= T_r \times \frac{9-r}{r} \cdot \frac{8}{3}.$$

Hence $$T_{r+1} > T_r \text{ so long as } \frac{(9-r)8}{3r} > 1;$$

that is, $$72 - 8r > 3r, \quad \text{or} \quad 72 > 11r$$

The greatest value of r consistent with this is 6, hence the greatest term is the 7th

The 7th term of $(x-4a)^8 = {}^8C_6 x^2(-4a)^6 = {}^8C_2 \times \frac{1}{2^2} \times \left(\frac{4}{3}\right)^6 = \frac{28672}{729}$

Properties of the Binomial Coefficients.

531 *To find the sum of the coefficients in the expansion of* $(1+x)^n$

For the sake of brevity we shall now express the coefficients by the symbols $c_0, c_1, c_2, c_3, \ldots c_n$, so that

$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots + c_nx^n$$

Put $x=1$, then $2^n = c_0 + c_1 + c_2 + c_3 + \ldots + c_n$

$$= \text{the sum of the coefficients}$$

NOTE Since $c_0 = 1$, we have $c_1 + c_2 + c_3 + \ldots + c_n = 2^n - 1$, which is the result proved in Art 510

532 *In the expansion of* $(1+x)^n$ *the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

We have $(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n$

Put $x=-1$, then $0 = c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + \ldots$

$$c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots$$

By Art 531, each of these equal expressions $= \frac{1}{2}\, 2^n = 2^{n-1}$.

533 *To find the sum of the squares of the coefficients in the expansion of* $(1+x)^n$

We have $\quad (1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n,$

and $\quad (x+1)^n = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n$

If we multiply together the two series on the right, the coefficient of x^n is $\quad c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2$

Therefore this expression must be equal to the coefficient of x^n in the product $(1+x)^n(x+1)^n$, or $(1+x)^{2n}$ [See Art 474]

$$c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = \frac{\lfloor 2n}{\lfloor n \lfloor n}$$

NOTE The following method shews a device often useful

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

Writing $\frac{1}{x}$ for x, we have

$$\left(1+\frac{1}{x}\right)^n, \quad \text{or} \quad \frac{1}{x^n}(1+x)^n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots + \frac{c_n}{x_n}$$

by multiplying these two results together,

$$\frac{1}{x^n}(1+x)^{2n} = (c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2) + \text{other terms } \textit{all containing } x.$$

$$c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = \text{the coefficient of the term independent of } x$$
$$\text{in the expansion of } \frac{1}{x^n}(1+x)^{2n}$$
$$= \text{the coefficient of } x^n \text{ in } (1+x)^{2n} = \frac{\lfloor 2n}{\lfloor n \lfloor n}$$

EXAMPLE *If* $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, *prove that*

(i) $c_0 + 2c_1 + 3c_2 + 4c_3 + \ldots + (n+1)c_n = 2^n + n \cdot 2^{n-1}$;

(ii) $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \frac{1}{4}c_3 + \ldots + \frac{1}{n+1} c_n = \frac{2^{n+1}-1}{n+1}$

(i) The series $= (c_0 + c_1 + c_2 + \ldots + c_n) + (c_1 + 2c_2 + 3c_3 + \ldots + nc_n)$

$$= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \ldots + 1\right\}$$
$$= 2^n + n(1+1)^{n-1} = 2^n + n \cdot 2^{n-1}$$

(ii) Here $(n+1)\left\{c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \frac{1}{4}c_3 + \ldots + \frac{1}{n+1} c_n\right\}$

$$= (n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{\lfloor 3} + \ldots \text{ to } n+1 \text{ terms}$$
$$= \left\{1 + (n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{\lfloor 3} + \ldots \text{ to } n+2 \text{ terms}\right\} - 1$$
$$= (1+1)^{n+1} - 1 = 2^{n+1} - 1$$

Dividing by $n+1$ we obtain the required result

EXAMPLES XLI. b.

In the following expansions find which is the greatest term

1. $(1-x)^{25}$ when $x=3$ 2. $(x+y)^{15}$ when $x=5$, $y=2$.

3. $(a+7b)^n$ when $n=19$, $a=14$, $b=3$

4. $(2x-3a)^n$ when $n=13$, $a=4$, $x=9$

5. $\left(5a-\dfrac{b}{5}\right)^n$ when $n=16$, and $b=10a$

In the following expansions find the value of the greatest term.

6. $(1+2x)^9$ when $x=\dfrac{1}{3}$ 7. $(2+3x)^8$ when $x=\dfrac{1}{2}$

8. Shew that in the expansion of $\left(\dfrac{1}{5}+\dfrac{5x}{16}\right)^{12}$ there are two greatest terms, each equal to $\dfrac{99}{5^4\times 5^7}$, when $x=\dfrac{2}{5}$

9. Find the numerically greatest *coefficient* in the expansion of

$$\text{(i)}\ \left(1+\frac{2x}{3}\right)^{13}, \qquad \text{(ii)}\ (3-5x)^8$$

10. In the expansion of $(1+x)^{15}$ the coefficients of the $(r-1)^{\text{th}}$ and $(2r+3)^{\text{th}}$ terms are equal, find r

11. Find n when the coefficients of the 9^{th} and 15^{th} terms of $(1+x)^n$ are equal

12. Find the sum of the coefficients of $(x+y)^{12}$

13. Find the sum of the coefficients of $(2x+3y)^5$

If $(1+x)^n=c_0+c_1x+c_2x^2+ \quad +c_nx^n$, prove that

14. $c_1+2c_2+3c_3+ \quad +nc_n=n\ 2^{n-1}$

15. $c_2+2c_3+3c_4+ \quad +(n-1)c_n=1+(n-2)2^{n-1}$

16. $c_0+c_1x+2c_2x^2+ \quad +nc_nx^n=1+nx(1+x)^{n-1}$

17. $c_1-2c_2+3c_3- \quad +(-1)^{n-1}nc_n=0$

18. $c_0-\dfrac{c_1}{2}+\dfrac{c_2}{3}- \quad +(-1)^n\dfrac{c_n}{n+1}=\dfrac{1}{n+1}$

19. $\dfrac{c_1}{c_0}+\dfrac{2c_2}{c_1}+\dfrac{3c_3}{c_2}+ \quad +\dfrac{nc_n}{c_{n-1}}=\dfrac{n(n+1)}{2}$

20. $2c_0+\dfrac{2^2c_1}{2}+\dfrac{2^3c_2}{3}+ \quad +\dfrac{2^{n+1}c_n}{n+1}=\dfrac{3^{n+1}-1}{n+1}$

21. $c_0c_1+c_1c_2+c_2c_3+ \quad +c_{n-1}c_n=\dfrac{\underline{|2n}}{\underline{|n+1}\,\underline{|n-1}}$

(*Miscellaneous.*)

22. Find the coefficient of x^3 in the expansion of $\left(32x - \frac{1}{2}\right)^{20}$.

23. Expand $(3-2x)^6$.

24. Find the term containing x^7 in the expansion of $\left(x^3 - \frac{1}{x}\right)^{20}$.

25. Find the first four terms in the expansion of $\left(1+\frac{1}{x}\right)^x$, and write down the general term in the neatest form.
If $x=100$, find the sum of the first three terms.

26. Find the r^{th} term from the beginning and the r^{th} term from the end of $(a+2x)^n$.

27. Find the coefficient of x^4 in the expansion of $(1-x-x^2)^{10}$.

28. Shew that the term independent of x in the expansion of $\left(2x-\frac{3}{x^2}\right)^9$ is $-2^3 \times 3^4 \times 7$.

29. Find the value of $(1.012)^5$ to 3 places of decimals.

30. Find the coefficient of x^4 in the expansion of $(1+x+2x^2)^4$.

31. If $\qquad (1+x)^3 = 1 + a_1x + a_2x^2 + \ldots,$
and $\qquad (1+x)^5 = 1 + b_1x + b_2x^2 + \ldots,$
find the value of $\quad 1 + a_1b_1 + a_2b_2 + a_3b_3$.

32. Find the term independent of x in the expansion of $\left(2x^2 - \frac{a}{2x^3}\right)^{10}$.

33. Shew that $(1-x^2)^n$ may be put in the form

$$(1+x)^{2n} - 2nx(1+x)^{2n-1} + \frac{2n(2n-2)}{1.2}x^2(1+x)^{2n-2} - \ldots$$

34. Prove that the coefficient of x^n in $(1+x)^{2n}$ is equal to twice the coefficient of x^n in $(1+x)^{2n-1}$.

35. Find the coefficient of x^r in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$. Shew that $2n-r$ and $3n+r$ must each be a multiple of 5.

36. If A stands for the sum of the odd terms and B for the sum of the even terms in the expansion of $(x+a)^n$, prove that

$$A^2 - B^2 = (x^2-a^2)^n.$$

37. Shew that the middle term in the expansion of $(1+x)^{2n}$ can be expressed in the form $\frac{1.3.5\ldots(2n-1)}{\lfloor n}2^n x^n$.

38. Shew that the difference between the coefficients of x^{r+1} and x^r in the expansion of $(1+x)^{n+1}$ is equal to the difference between the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1+x)^n$.

Binomial Theorem for Negative and Fractional Indices.

534. We have now to consider whether the series

$$1+nx+\frac{n(n-1)}{1 \cdot 2}x^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}x^r+\ldots$$

is a true equivalent of $(1+x)^n$ when n is negative or fractional. When n is a positive integer the series ends with the $(n+1)^{\text{th}}$ term (Art 522), but if n is negative or fractional none of the factors in the numerator of the general term can ever vanish; thus the series becomes infinite since none of its terms can become zero.

By actual division we can shew that

$$(1-x)^{-2}\equiv\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\ldots,$$

and as the process of division is unending, the number of terms in the series is unlimited.

Now let us assume for the moment that the Binomial Theorem is true for a negative index, then by putting $n=-2$ and writing $-x$ for x in the formula for $(1+x)^n$, we have

$$(1-x)^{-2}=1+(-2)(-x)+\frac{(-2)(-3)}{1 \cdot 2}(-x)^2+\frac{(-2)(-3)(-4)}{1 \cdot 2 \cdot 3}(-x)^3+\ldots$$

$$=1+2x+3x^2+4x^3+\ldots$$

Thus the Binomial Theorem appears to hold good in this case. But if we were to make trial of a particular value of x, such as $x=2$, we should have

$$(-1)^{-2}=1+2 \cdot 2+3 \cdot 2^2+4 \cdot 2^3+\ldots\ldots$$

This contradictory result is sufficient to shew that we cannot take the series

$$1+nx+\frac{n(n-1)}{1 \cdot 2}x^2+\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3+\ldots$$

as the true arithmetical equivalent of $(1+x)^n$ in *all* cases.

Again, by putting $n=-1$, and writing $-x$ for x in the Binomial Formula, we should get

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots.$$

The series on the right is a G P and the sum of the first r terms

$$=\frac{1-x^r}{1-x}=\frac{1}{1-x}-\frac{x^r}{1-x}.$$

If x is numerically less than 1, we can make x^r as small as we please by taking r large enough. Thus the sum to infinity

$$=\frac{1}{1-x}=(1-x)^{-1}.$$

From this we may infer that the expansion of $(1-x)^{-1}$ by the Binomial Theorem is true when x is numerically less than 1.

535 If the sum of n terms of a series tends to a finite limit, which it cannot exceed, when n is made infinitely great, the series is said to be **convergent**, and the finite limit to which it converges is called the **sum to infinity**.

Thus the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ cannot exceed the limit 2 (Art 330), and is therefore convergent.

If by increasing n sufficiently, the sum of n terms of a series can be made to exceed any quantity, however large, the series is said to be **divergent**.

Thus the series $1+2+4+8+\ldots$ can be made greater than any quantity that can be named, by taking a sufficient number of terms, and is therefore divergent.

Again, the sum of the first n terms of the series

$$a-ar+ar^2+ar^3+\ldots=\frac{a(1-r^n)}{1-r}$$

$$=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

If r is numerically less than 1, the sum approaches to the finite limit $\frac{a}{1-r}$ when n is infinitely great, and the series is convergent.

If r is numerically greater than 1, the sum of the first n terms is $\frac{ar^n}{r-1}-\frac{a}{r-1}$, which can be made greater than any finite quantity by taking n large enough, thus the series is divergent.

The consideration of the convergency or divergency of series is beyond the elementary scope of this book. The more advanced reader will find the subject treated in Hall and Knight's *Higher Algebra*, Chap XXI. For a much fuller discussion he may consult Chrystal's *Algebra*, Part II, Chap XXVI. It will be sufficient here to say generally that divergent series are practically of no importance in algebraical work, but that convergent series may be introduced into mathematical reasoning as freely as any other functions to which the laws of Algebra are applicable.

It can be proved that in the Binomial Formula

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\ldots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}x^r+\ldots,$$

the series on the right is convergent for all values of n, *provided that x is numerically less than* 1, and that with this restriction the expansion is a true arithmetical equivalent of $(1+x)^n$.

536. Henceforth we shall assume that the formula

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ \dots$$

is true for any value of n so long as x is numerically less than 1. We may again remark that the series on the right is infinite when n is negative or fractional, but in any particular case we may write down as many terms as we please, or we may find the coefficient of any assigned term.

If we have to expand $(a+x)^n$ we may write the expression

$$a^n\left(1+\frac{x}{a}\right)^n, \quad \text{or} \quad x^n\left(1+\frac{a}{x}\right)^n,$$

we must then use the first or second of these forms according as x is less or greater than a.

In the following examples we shall assume that the values of the symbols are such as to make the expansions possible.

EXAMPLE 1. *Expand* $(1+x)^{-3}$ *to four terms.*

$$(1+x)^{-3}=1+(-3)x+\frac{(-3)(-3-1)}{1\cdot 2}x^2+\frac{(-3)(-3-1)(-3-2)}{1\cdot 2\cdot 3}x^3+\ \dots$$

$$=1-3x+\frac{3\cdot 4}{1\cdot 2}x^2-\frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}x^3+\dots$$

$$=1-3x+6x^2-10x^3+\dots$$

EXAMPLE 2. *Expand* $(4+3x)^{\frac{3}{2}}$ *to four terms.*

$$(4+3x)^{\frac{3}{2}}=4^{\frac{3}{2}}\left(1+\frac{3x}{4}\right)^{\frac{3}{2}}=8\left(1+\frac{3x}{4}\right)^{\frac{3}{2}}$$

$$=8\left[1+\frac{3}{2}\cdot\frac{3x}{4}+\frac{\frac{3}{2}(\frac{3}{2}-1)}{1\cdot 2}\left(\frac{3x}{4}\right)^2+\frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{1\cdot 2\cdot 3}\left(\frac{3x}{4}\right)^3+\dots\right]$$

$$=8\left[1+\frac{3}{2}\cdot\frac{3x}{4}+\frac{3}{8}\cdot\frac{9x^2}{16}-\frac{1}{16}\cdot\frac{27x^3}{64}+\dots\right]$$

$$=8+9x+\frac{27}{16}x^2-\frac{27}{128}x^3+\dots$$

EXAMPLE 3. *Find the general term in the expansion of* $(1+x)^{\frac{1}{2}}$.

$$T_{r+1}=\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-r+1)}{\lfloor r}x^r$$

$$=\frac{1(-1)(-3)(-5)\dots(-2r+3)}{2^r\lfloor r}x^r$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1}\frac{1\cdot 3\cdot 5\dots(2r-3)}{2^r\lfloor r}x^r$$

EXAMPLE 4. *Find the general term in the expansion of* $(1-x)^{-3}$.

$$T_{r+1}=\frac{(-3)(-4)(-5)\dots(-3-r+1)}{|\underline{r}}(-x)^r$$

$$=(-1)^r\frac{3\cdot4\cdot5\dots(r+2)}{|\underline{r}}(-1)^r x^r$$

$$=(-1)^{2r}\frac{3\cdot4\cdot5\dots(r+2)}{1\cdot2\cdot3\dots r}x^r$$

$$=\frac{(r+1)(r+2)}{1\cdot2}x^r,$$

by removing like factors from the numerator and denominator.

[*Examples* XLI. c. 1–18, *page* 486, *may be taken here.*]

537. *To find in its simplest form the general term in the expansion of* $(1-x)^{-n}$.

$$T_{r+1}=\frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{|\underline{r}}(-x)^r$$

$$=(-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{|\underline{r}}(-1)^r x^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\dots(n+r-1)}{|\underline{r}}x^r$$

$$=\frac{n(n+1)(n+2)\dots(n+r-1)}{|\underline{r}}x^r.$$

From this it appears that every term in the expansion of $(1-x)^{-n}$ is positive, and this form may be conveniently used in all cases when the index is negative.

EXAMPLE. *Find the general term of* $\dfrac{1}{\sqrt[3]{1-3x}}$.

We require the $(r+1)^{\text{th}}$ term of $(1-3x)^{-\frac{1}{3}}$.

$$T_{r+1}=\frac{\frac{1}{3}(\frac{1}{3}+1)(\frac{1}{3}+2)\dots(\frac{1}{3}+r-1)}{|\underline{r}}(3x)^r$$

$$=\frac{1\cdot4\cdot7\dots(3r-2)}{3^r\,|\underline{r}}3^r x^r$$

$$=\frac{1\cdot4\cdot7\dots(3r-2)}{|\underline{r}}x^r.$$

If the given expression had been $(1+3x)^{-\frac{1}{3}}$ we should have used the same formula for the general term, replacing $3x$ by $-3x$.

538. The following results should be verified and noted:

$$(1-x)^{-1}=1+x+x^2+x^3+\dots+x^r+\dots,$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3+\dots+(r+1)x^r+\dots,$$

$$(1-x)^{-3}=1+3x+6x^2+10x^3+\dots+\frac{(r+1)(r+2)}{1\cdot2}x^r+\dots.$$

EXAMPLES XLI. c.

Expand to 4 terms the following expressions

1. $(1+x)^{\frac{1}{3}}$ 2. $(1+x)^{\frac{3}{2}}$. 3. $(1-x)^{\frac{2}{5}}$ 4. $(1+3x)^{-2}$.

5. $(1+x^2)^{-3}$ 6 $(1+3x)^{-4}$ 7 $(2+x)^{-3}$ 8 $(1+2x)^{-\frac{1}{2}}$

9. $(a-2x)^{-\frac{3}{2}}$ 10. $(1-x)^{\frac{5}{2}}$ 11. $(9+2a)^{\frac{1}{2}}$ 12. $(8+12a)^{\frac{2}{3}}$

Write down and simplify

13. The 5^{th} term and the 10^{th} term of $(1+x)^{-\frac{3}{2}}$

14. The 3^{rd} term and the 11^{th} term of $(1+2x)^{\frac{11}{2}}$

15. The 4^{th} term and the $(r+1)^{th}$ term of $(1+x)^{-2}$

16. The 7^{th} term and the $(r+1)^{th}$ term of $(1-x)^{\frac{1}{2}}$

17. The $(r+1)^{th}$ term of $(a-bx)^{-1}$, and of $(1-nx)^{\frac{1}{n}}$.

18. The $(r+1)^{th}$ term of $(1+x)^{\frac{4}{3}}$, and of $(1+2x)^{-\frac{3}{2}}$

Find the general term in each of the following expansions

19 $(1-x)^{-5}$. 20 $(1+x^2)^{-3}$ 21. $(1+x)^{-\frac{1}{3}}$ 22. $(1+x)^{-\frac{2}{3}}$.

23. $(2-x)^{-10}$ 24. $(1+x)^{-\frac{p}{q}}$ 25. $\dfrac{1}{\sqrt{1+2a}}$ 26. $\dfrac{1}{\sqrt[3]{(1-3x)^2}}$

27. Find the general term in the expansion of $(1-x)^{\frac{2}{3}}$, and shew that all the terms after the first are negative

28. Find the first negative term in each of the following expansions

(i) $(1+2x)^{\frac{9}{2}}$, (ii) $(1+x)^{\frac{4}{3}}$; (iii) $(1+3x)^{\frac{11}{3}}$

539 To find the greatest term in the expansion of $(1+x)^n$ when n is unrestricted in value, we may proceed exactly as in Art 530 When the index is negative, we may conveniently use the form for the general term given in Art 537

EXAMPLE *Find which is the greatest term in the expansion of* $(3+7x)^{-n}$, *when* $x=\frac{3}{8}$, $n=\frac{11}{4}$

Since $(3+7x)^{-\frac{11}{4}}=3^{-\frac{11}{4}}\left(1+\frac{7x}{3}\right)^{-\frac{11}{4}}$, it will be sufficient to find the greatest term of $\left(1+\frac{7x}{3}\right)^{-\frac{11}{4}}$.

Here $$\mathrm{T}_{r+1}=\mathrm{T}_r\times\frac{\frac{11}{4}+r-1}{r}\cdot\frac{7x}{3}, \text{ numerically,}$$

$$=\mathrm{T}_r\times\frac{7+4r}{4r}\cdot\frac{7}{8}.$$

Hence $$\mathrm{T}_{r+1}>\mathrm{T}_r, \text{ so long as } \frac{(7+4r)7}{32r}>1;$$

that is, $$49+28r>32r, \text{ or } 49>4r$$

The greatest value of r consistent with this is 12, hence the greatest term is the 13^{th}

540 The methods used in the following examples deserve attention

EXAMPLE 1 *Find the coefficient of* x^r *in the expansion of* $\dfrac{2+x}{(1-x)^3}$

The expression $=(2+x)(1+x)^{-3}$

$$=(2+x)(1+p_1x+p_2x^2+\ldots+p_rx^r+\ldots), \text{ suppose}$$

the coefficient of $x^r = p_{r-1}+2p_r$

Now $$p_r=(-1)^r\frac{(r+1)(r+2)}{2} \qquad \text{[Art 538]}$$

$\therefore$ the required coefficient $=(-1)^{r-1}\dfrac{r(r+1)}{2}+(-1)^r(r+1)(r+2)$

$$=(-1)^r(r+1)\left\{r+2-\frac{r}{2}\right\}$$

$$=(-1)^r\frac{(r+1)(r+4)}{2}$$

EXAMPLE 2 *Find the value of the infinite series*

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{2^2}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{2^3}+\ldots$$

$$\text{The series}=1-\frac{1}{2}\cdot\frac{1}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot2}\cdot\frac{1}{2^2}-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot2\cdot3}\cdot\frac{1}{2^3}+\ldots$$

$$=1-\frac{1}{2}\cdot\frac{1}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{1\cdot2}\cdot\frac{1}{2^2}-\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{1\cdot2\cdot3}\cdot\frac{1}{2^3}+\ldots$$

$$=\left(1+\frac{1}{2}\right)^{-\frac{1}{2}}=\left(\frac{3}{2}\right)^{-\frac{1}{2}}=\left(\frac{2}{3}\right)^{\frac{1}{2}}=\sqrt{\frac{2}{3}}$$

EXAMPLES XLI d.

Find which is the greatest term in the following expansions:

1. $(1+x)^{-5}$ when $x=\frac{5}{8}$

2. $(1+x)^{\frac{11}{2}}$ when $x=\frac{2}{3}$

3. $(1-x)^{-20}$ when $x=\frac{2}{3}$

4. $(9+3x)^{\frac{3}{2}}$ when $x=2$

5. $(6+5x)^{\frac{21}{5}}$ when $x=\frac{1}{3}$

6. $\left(\frac{2a}{9}-\frac{x^2}{4}\right)^{-\frac{3}{2}}$ when $x=\frac{1}{12}$.

7. Find the numerically greatest *coefficient* in the expansion of

$$\text{(i)}\ \left(1+\frac{5x}{6}\right)^{\frac{3}{2}}; \qquad \text{(ii)}\ (3-4x)^{\frac{11}{4}}$$

8. Find the coefficient of x^{100} in the following expansions

$$\text{(i)}\ \frac{(1+x)^2}{1-x}, \qquad \text{(ii)}\ \frac{1-x}{(1+x)^2}, \qquad \text{(iii)}\ \frac{x}{(1+x)^3}.$$

9. Find the coefficient of x^r in the expansions of

$$\text{(i)}\ \frac{1+x+x^2}{(1-x)^3}, \qquad \text{(ii)}\ \frac{1-x+x^3}{(1-x)^3}, \qquad \text{(iii)}\ \frac{1+3x}{(1+x)^2}$$

Verify each case when $r=3$

10. Shew that

$$(4+2x+3x^2)/(1-x)^3=4+10x+19x^2+28x^3+\dots+(9r+1)x^r+\dots.$$

11. Find the coefficient of x^r in the expansion of $(2-3x)/(1+x)^4$

Find the sum to infinity of the following series

12. $1-\frac{1}{6}+\frac{1\cdot 3}{6\cdot 12}-\frac{1\cdot 3\cdot 5}{6\cdot 12\cdot 18}+\dots$ 13. $1+\frac{1}{6}+\frac{1\cdot 4}{3\cdot 6}\cdot\frac{1}{4}+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\cdot\frac{1}{8}+\dots$

Shew that

14. $1-\frac{1}{4}\cdot\frac{1}{2}+\frac{1\cdot 3}{4\cdot 8}\cdot\frac{1}{2^2}-\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}\cdot\frac{1}{2^3}+\dots$ to infinity $=\frac{2}{5}\sqrt{5}$

15. $1+\frac{3}{4}+\frac{3\cdot 5}{4\cdot 8}+\frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\dots$ to infinity $=\sqrt{8}$

16. $1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1\cdot 4}{3\cdot 6}\cdot\frac{1}{4^2}+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\cdot\frac{1}{4^3}+\dots$ to infinity $=\sqrt[3]{\frac{4}{3}}$

17. $1+\frac{7}{3\cdot 8}+\frac{1\cdot 4}{3^2}\cdot\frac{7^2}{8^2\,\underline{|2}}+\frac{1\cdot 4\cdot 7}{3^3}\cdot\frac{7^3}{8^3\,\underline{|3}}+\dots$ to infinity $=2$

18. Find the sum of the infinite series whose $(n+1)^{\text{th}}$ term is

$$\frac{(2n+2)(2n+4)(2n+6)(2n+8)}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{1}{3^n}$$

19. Prove that

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1\cdot 2}\left(\frac{x}{1+2x}\right)^2+\dots.$$

$$\left[\textit{Note that } \left(\frac{1+2x}{1+x}\right)^n=\left(\frac{1+x}{1+2x}\right)^{-n}=\left(\frac{1+2x-x}{1+2x}\right)^{-n}\right]$$

20. Prove that

$$\left(\frac{1+x}{1-x}\right)^{\frac{n}{2}}=1+n\left(\frac{x}{1+x}\right)+\frac{n(n+2)}{1\cdot 2}\left(\frac{x}{1+x}\right)^2+\frac{n(n+2)(n+4)}{1\cdot 2\cdot 3}\cdot\left(\frac{x}{1+x}\right)^3+\dots.$$

21. Prove that $1+\frac{3n}{4}+\frac{3n(3n+3)}{1\cdot 2}\cdot\frac{1}{4^2}+\frac{3n(3n+3)(3n+6)}{1\cdot 2\cdot 3}\cdot\frac{1}{4^3}+\dots$

$$=3^n\left\{1+\frac{n}{4}+\frac{n(n+1)}{1\cdot 2}\cdot\frac{1}{4^2}+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cdot\frac{1}{4^3}+\dots\right\}$$

$$=1+3n+\frac{n(n-1)}{1\cdot 2}\,3^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\,3^3+\dots$$

22. Find the coefficient of x^r in

(i) $\sqrt{1+x+x^2+x^3+\dots \text{ to inf}}$, (ii) $(1+2x+3x^2+4x^3+\dots \text{ to inf})^{\frac{1}{2}}$,

(iii) $(1-3x+6x^2-10x^3+\dots \text{ to inf})^{\frac{2}{3}}$; (iv) $(1-2x+3x^2-4x^3+\dots \text{ to inf})^{-n}$

Application of the Binomial Theorem to Approximations

541 If x is a small quantity, so that the successive powers x^2, x^3, x^4, ... are rapidly diminishing, a few terms of the series

$$1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ldots$$

may be taken as an approximation to the value of $(1+x)^n$ The number of terms to be taken in each case will depend on the value of x, and on the degree of accuracy required

EXAMPLE 1 *Find the value of* $(1\cdot 04)^5$ *by the Binomial Theorem*

$$(1\cdot 04)^5=\left(1+\frac{4}{100}\right)^5=\left(1+\frac{4}{10^2}\right)^5$$

$$=1+5\cdot\frac{4}{10^2}+10\cdot\frac{4^2}{10^4}+10\cdot\frac{4^3}{10^6}+5\cdot\frac{4^4}{10^8}+\frac{4^5}{10^{10}}$$

$$=1+\frac{2}{10}+\frac{16}{10^3}+\frac{64}{10^5}+\frac{128}{10^7}+\frac{1024}{10^{10}}$$

$$=1+0\cdot 2+0\cdot 016+0\cdot 00064+0\cdot 0000128+0\cdot 0000001024.$$

Here the successive terms diminish very rapidly, and if we only want an approximate result, it is obvious that all the terms need not be calculated For example, to obtain a result true to 3 decimal figures we may omit the last two terms, since the fourth begins with 4 ciphers

Thus $(1\cdot 04)^5=1\cdot 2166\ldots$ $=1\cdot 217$, correct to 3 decimal places

EXAMPLE 2 *Find the fourth root of* 621 *to four decimal places*

The complete fourth power nearest to 621 is 625, or 5^4, hence we proceed as follows

$$\sqrt[4]{621}=(5^4-4)^{\frac{1}{4}}=5\left(1-\frac{4}{5^4}\right)^{\frac{1}{4}}$$

$$=5\left\{1-\frac{1}{4}\cdot\frac{4}{5^4}-\frac{3}{32}\cdot\frac{4^2}{5^8}-\frac{7}{128}\cdot\frac{4^3}{5^{12}}-\ldots\right\}$$

$$=5-\frac{1}{5^3}-\frac{3}{2}\cdot\frac{1}{5^7}-\frac{7}{2}\cdot\frac{1}{5^{11}}-\ldots$$

$$=5-\frac{2^3}{10^3}-3\cdot\frac{2^6}{10^7}-7\cdot\frac{2^{10}}{10^{11}}-\ldots$$

$$=5-0\cdot 008-0\cdot 0000192-\ldots$$

The 4^{th} term begins with 7 ciphers, hence this and subsequent terms may be neglected

Thus $\sqrt[4]{621}=4\cdot 99198\ldots$ $=4\cdot 9920$, to 4 decimal places

In these examples the calculation has been made easy by expressing the fractions with powers of 10 in the denominator Sometimes it will be necessary to proceed as in the next example

EXAMPLE 3 *Find the value of $\frac{1}{\sqrt{50}}$ to five decimal places*

$$\frac{1}{\sqrt{50}}=(50)^{-\frac{1}{2}}=(49+1)^{-\frac{1}{2}}=\frac{1}{7}\left(1+\frac{1}{7^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{7}\left\{1-\frac{1}{2}\cdot\frac{1}{7^2}+\frac{3}{8}\cdot\frac{1}{7^4}-\frac{5}{16}\cdot\frac{1}{7^6}+\quad\right\}$$

$$=\frac{1}{7}-\frac{1}{2}\cdot\frac{1}{7^3}+\frac{3}{8}\cdot\frac{1}{7^5}-\frac{5}{16}\cdot\frac{1}{7^7}+$$

$$=0\ 1428571-0\ 0014577+0\cdot0000223-$$

$$=0\ 14142,\text{ to 5 decimal places}$$

$$\begin{array}{r|l}
7 & 1\ 00 \\ \hline
7 & 0\ 1428571=\frac{1}{7} \\ \hline
7 & 0\ 0204081 \\ \hline
7 & 0\ 0029154=\frac{1}{7^3} \\ \hline
7 & 0\ 0004165 \\ \hline
 & 0\ 0000595=\frac{1}{7^5}
\end{array}$$

The 4th term is neglected since it will not affect the first 5 decimal figures

EXAMPLE 4 *When* x *is so small that its square and higher powers may be neglected find the value of*

$$\frac{\sqrt[3]{8+3x}-\sqrt[5]{1-x}}{(1+5x)^{\frac{3}{5}}}$$

$$\text{The expression}=\frac{(8+3x)^{\frac{1}{3}}-(1-x)^{\frac{1}{5}}}{(1+5x)^{\frac{3}{5}}}=\frac{2\left(1+\frac{3x}{8}\right)^{\frac{1}{3}}-(1-x)^{\frac{1}{5}}}{(1+5x)^{\frac{3}{5}}}$$

$$=\frac{2\left(1+\frac{1}{8}x+\quad\right)-\left(1-\frac{1}{5}x+\quad\right)}{(1+3x+\quad)}=\frac{1+\frac{9}{20}x}{1+3x},\text{ neglecting } x^2,$$

$$=\left(1+\tfrac{9}{20}x\right)(1+3x)^{-1}=\left(1+\tfrac{9}{20}x\right)(1-3x)$$

$$=1-\tfrac{51}{20}x$$

[*Examples* XLI e 1–22, *page* 492, *may be taken here*]

542 Suppose $(1+x)^n=1+ax+bx^2+cx^3+\quad$, where x is small, then $1+ax$ is called a *first approximation*, $1+ax+bx^2$ a *second approximation*, and so on In a large number of numerical calculations the required accuracy is secured by using the following first approximations which should now be sufficiently obvious

When r, s, t are small quantities such that their squares, and the products of any two of them may be neglected,

(i) $(1+r)(1+s)=1+(r+s)$, $\quad(1+r)(1-s)=1+(r-s)$

Hence $\quad(1+r)^2=1+2r \qquad$ Also $(1-r)^2=1-2r$

(ii) $(1+r)(1+s)(1+t)=1+(r+s+t)$

Hence $\quad(1+r)^3=1+3r \qquad$ Also $(1-r)^3=1-3r$

(iii) $\sqrt{1+r}=1+\frac{1}{2}r$, $\quad\sqrt{1-r}=1-\frac{1}{2}r$

$\frac{1}{1+r}=1-r$, $\quad\frac{1}{1-r}=1+r$

EXAMPLE 1. *Find approximately the values of*

$$\text{(i) } 1\ 0065\times 993, \quad \text{(ii) } \frac{793}{10004}; \quad \text{(iii) } \frac{(1\ 00027)^3}{(9991)^2}.$$

(i) $1\ 0065\times 993=(1+0065)(1-007)=1+0065-007$, approximately,

$$=1-0005=9995$$

The error here is the neglected product 0065×007, or 0000455, thus the result correct to 4 significant figures is 9995

(ii) Here $10004=10^4+4=10^4(1+0004)$

$$\frac{793}{10004}=\frac{793}{10^4(1+0004)}=0793(1+0004)^{-1}=0793(1-0004), \text{ approx },$$

$$=0793-0004 \text{ of } 0793=0793-00003172=07927$$

From this example it may be seen that to divide a number by $10000+x$, where x is very small, we have only to divide the number by 10000 and subtract x ten-thousandths of the result. Similarly for any divisor of the form 10^n+x

(iii) $(1\ 00027)^3=(1+00027)^3=1+00081$, approximately,

$$(9991)^2=(1-0009)^2=1-0018, \qquad ,,$$

$$\text{the required quotient}=\frac{1+00081}{1-0018}=(1+00081)(1-0018)^{-1}$$

$$=(1+00081)(1+0018)=1+00081+0018$$

$$=1+00261=1\ 00261$$

EXAMPLE 2 *L and R are the lengths in inches of the arms of an untrue balance In an experiment by "double-weighing" it was found that* $\left(\frac{L}{R}\right)^2=\frac{49\ 998}{49\ 982}$ *Find by how much the length of L exceeds that of R*

We have $\frac{L}{R}=\sqrt{\frac{49\ 998}{49\ 982}}=\left(1+\frac{016}{49\ 982}\right)^{\frac{1}{2}}=\left(1+\frac{016}{50}\right)^{\frac{1}{2}}$, approximately,

$$=1+\tfrac{1}{2}(00032)=1+00016$$

Thus L is longer than R by about 00016 of either

EXAMPLE 3. *The time* t *of the beat of a pendulum* x *centimetres long is* $\pi\sqrt{x/981}$, *if the pendulum makes* n *beats in a day, find how many beats it will make if its length is changed to* x+h *centimetres, where* h *is very small compared with* x

Let t' be the time of a beat of the lengthened pendulum, and n' the number of beats in a day,

then $$t'=\pi\sqrt{(x+h)/981}, \quad \text{and} \quad nt=n't'$$

$$\frac{n'}{n}=\frac{t}{t'}=\left(\frac{x}{x+h}\right)^{\frac{1}{2}}=\left(1+\frac{h}{x}\right)^{-\frac{1}{2}}=1-\frac{1}{2}\frac{h}{x}, \text{ approximately}$$

Thus $n'=n-\frac{n}{2}\cdot\frac{h}{x}$, so that the longer pendulum *loses* $\frac{n}{2}\frac{h}{x}$ beats per day.

EXAMPLE 4 *The specific heat s of a metal is found from the equation $ms\theta = Mt$, where m, θ, M, and t are quantities which are determined experimentally If there may be an error of one per cent either way in the measurements of m, θ, and t, while the error in M is negligible, find the largest possible percentage error in the value obtained for s.*

Since $s = \dfrac{Mt}{m\theta}$, the percentage error in s will obviously be greatest when the error in t is in *either* way and the errors in m and θ are in the *opposite* way

Let the true values of t, m, θ, and s be represented by t', m', θ', and s', then we may put

$$t' = t(1-.01), \quad m' = m(1+.01), \quad \theta' = \theta(1+.01)$$

$$s' = \frac{Mt'}{m'\theta'} = \frac{Mt(1-.01)}{m\theta(1+.01)^2} = s(1-.01)(1+.01)^{-2}$$

$$= s(1-.01)(1-2\times .01), \text{ approximately,}$$

$$= s(1-3\times .01) = s(1-.03), \text{ neglecting the square of } .01$$

Thus the greatest possible percentage error in s is 3

EXAMPLES XLI. e.

Find to 5 places of decimals the value of

1. $(1.02)^4$ 2. $(0.97)^5$ 3 $\sqrt{50}$ 4. $\sqrt[3]{1003}$

Find to 4 places of decimals the value of

5. $\sqrt[3]{217}$ 6. $\sqrt[5]{30}$ 7. $\sqrt[4]{620}$ 8 $\sqrt[3]{998}$.

9. $\dfrac{1}{\sqrt{47}}$ 10. $(126)^{-\frac{1}{3}}$ 11. $(626)^{\frac{3}{4}}$ 12. $\sqrt[3]{\frac{251}{250}}$

13 Find the square root of 2 to 6 places of decimals

[*Note that* $\sqrt{2} = \sqrt{\frac{100}{49} \cdot \frac{98}{100}} = \frac{10}{7}\left(1-\frac{1}{50}\right)^{\frac{1}{2}}$]

When x is so small that its square and higher powers may be neglected, find the value of

14. $(1+3x)^{\frac{1}{3}}\,(1-2x)^{-\frac{1}{3}}$ 15. $(8+4x)^{\frac{1}{3}}\,(16-x)^{\frac{1}{4}}$

16. $\dfrac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$. 17. $\dfrac{\sqrt[3]{1+x}+\sqrt[5]{(1-x)^2}}{\sqrt{1+x}+\sqrt[3]{1-x}}$.

18. $\dfrac{(1+4x)^{\frac{1}{2}}+(1-9x)^{\frac{2}{3}}}{\sqrt{1+7x}+\sqrt[3]{1-3x}}$ 19 $\dfrac{(1+\frac{3}{2}x)^{-7}+(1-2x)^{-1}}{(1+\frac{7}{2}x)^{-3}+(1+x)^{-2}}$

20. $\dfrac{(4+x)^{\frac{3}{2}}\times(8-6x)^{\frac{1}{3}}}{\sqrt[5]{(1+x)^2}}$ 21 $\dfrac{\sqrt[3]{(8+4x)^2}\times\sqrt[6]{64-36x}}{\sqrt{16-5x}+4\sqrt{1+2x}}$

22. Find the first 3 terms in the expansion of $\dfrac{4}{(2+x)^3\times\sqrt{1+4x}}$.

Find by the formulæ of Art 542 the approximate values of

23 $1\cdot0078\times0\;9954$ 24. $(0\;9987)^2$ 25 $(1\;005)\times(1\;0002)^2$

26 $1\cdot00039\times0\cdot99978$ 27 $1\;00064-1\;0078$

28 $(1\;00018)^2\times(1\;00004)^3$ 29 $1\;0058-0\;9982$

30 $\dfrac{678}{1003}$ 31 $\dfrac{1}{0\;9998}$ 32 $\dfrac{1\;007}{(0\;995)^2}$

33 $\dfrac{1}{(0\;98)^3}$ 34. $\dfrac{1}{\sqrt{0\;997}}$ 35 $\dfrac{\sqrt{0\;9987}}{(1\;0003)^2}$

36 In three consecutive years the population of a town was increased by 0 8%, 1 2%, 2 5% Find approximately the total percentage increase in the three years

37 A bar of copper increases by 0 0000168 of its length for a rise of temperature of 1 degree Centigrade Find the expansion in area of a rectangular copper plate 45 cm long and 33 cm wide when the temperature is raised 10 degrees, the metal being assumed to expand equally in all directions [The decimal 0 0000168 is called the coefficient of linear expansion for copper]

38 The coefficient of linear expansion for zinc is 0 0000298 If the area of a zinc roof is 1000 square feet at 0° C, find its area to the nearest square foot at 50° C

39 If $\left(\dfrac{x}{y}\right)^2=\dfrac{9\;98}{9\cdot92}$, shew that x exceeds y by about 0 3 %

40 If $t^2=\dfrac{\tau^2 l}{g}$, and $T^2=\dfrac{\tau^2 L}{g}$, and L is very nearly equal to l, shew that $\dfrac{T-t}{t}=\dfrac{L-l}{2l}$, very nearly Find the error in this equation to the first significant figure when $l=36$ and $L-l=0\;1$

41 The time of one beat of a pendulum, l feet in length, is given by the formula $\tau\sqrt{\dfrac{l}{32\cdot2}}$ How many beats will a "seconds pendulum" gain in 10 hours if its length is shortened by 2 %?

42 The value of P has to be found from the formula $P=\dfrac{\lambda\times t^2}{d\times\sqrt{l}}$, where λ is a constant, while t, d, and l are found by experiment Find the percentage error in the value of P, supposing there is an error of 0 4 % in the value of t, and an error of 1 0 % in the value of l, the former error being in excess and the latter in defect

43 If in the formula of Ex 42, there is an error of 0 3 % too little in the observed value of t, an error of 1 0 % too much in the value of d, and of 2 0 % too much in the value of l, find the percentage error in the value of P

44. The modulus of torsion of a wire is found from the formula $n=\frac{2\pi Il}{t^2r^4}$. If there are errors of 0.2% too little in the value of l, 1.0% too much in the value of t, and 2.5% too little in the value of r, find the resulting error per cent in the value of n.

(*Miscellaneous.*)

45. Express the 10^{th} term of $(1+x)^{\frac{1}{2}}$ with a numerator containing the product of odd numbers, and a denominator a power of 2.

46. Find the coefficient of x^3 in the expansion of $(1+x-6x^2)^n$.

47. Find the first 3 terms in the expansion of $\frac{1}{(2-x)(2-3x)^{\frac{1}{3}}}$.

48. In the expansion of $\frac{3+x^2}{(1+x)^3}$ find the coefficient (i) of x^5, (ii) of x^{2r+1}, also deduce the first result from the second.

49. Find the coefficient of x^n in the expansion of $(1-4x)^{-\frac{1}{2}}$, and shew that it is equal to the term independent of x in $\left(x+\frac{1}{x}\right)^{2n}$.

50. Shew that $\sqrt{x^2+4}-\sqrt{x^2+1}$ is equal to

$$1-\frac{1}{4}x^2+\frac{7}{64}x^4-\ldots \quad \text{or} \quad \frac{3}{2x}\left\{1-\frac{5}{4x^2}+\frac{21}{8x^4}-\ldots\right\},$$

according as x is less or greater than 1.

51. If b is so small compared with a that b^2, b^3, ... may be neglected, find the sum of n terms of the H. P.

$$\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\ldots$$

52. If $\sqrt{N}=a+x$, where x is very small compared with a, shew that

$$\sqrt{N}=a\,\frac{3N+a^2}{N+3a^2}, \text{ very nearly.}$$

Test the formula when $N=10, 26, 123$.

[*We have* $N=a^2+2ax+x^2$. *As a first approximation* $N=a^2+2ax$, *whence* $x=\frac{N-a^2}{2a}$, *and* $2a+x=\frac{N+3a^2}{2a}$. *Substitute this value in* $N=a^2+(2a+x)x$, *thence find* x *and substitute in* $\sqrt{N}=a+x$.]

53. If $\sqrt[n]{N}=a+x$, where x is very small compared with a, show by taking three terms of the expansion of $(a+x)^n$ that

$$\sqrt[n]{N}=a\,\frac{(n+1)N+(n-1)a^n}{(n-1)N+(n+1)a^n}, \text{ approximately.}$$

CHAPTER XLII

PARTIAL FRACTIONS

543 THE algebraic sum of two or more algebraic proper fractions can always be expressed in the form of a single proper fraction

Thus $$\frac{3}{x-1}-\frac{2}{x-3}=\frac{x-7}{(x-1)(x-3)},$$

$$\frac{1}{1-x}+\frac{2}{1+x}+\frac{2}{1-3x}=\frac{5-10x+x^2}{(1-x)(1+x)(1-3x)}$$

We shall now explain the converse process, namely that of resolving a fraction into a group of simpler fractions, each having for its denominator one of the factors of the denominator of the given fraction Such fractions are known as **partial fractions** The resolution is effected by the method of Undetermined Coefficients, and some easy cases have already occurred See Art 475, Ex 2

In each of the above examples we notice that the single fraction on the right has a numerator of lower dimensions than the denominator We may assume this to be the case in the fractions given for resolution into partial fractions. For if the numerator of a fraction is not of lower dimensions than the denominator, by division the fraction can be expressed in a form partly integral and partly fractional, the latter part having a numerator of lower dimensions than the denominator

EXAMPLE 1 *Resolve* $\frac{x+16}{(2x-3)(x+2)}$ *into partial fractions*

The denominators $2x-3$ and $x+2$ being of the first degree, we may assume

$$\frac{x+16}{(2x-3)(x+2)}\equiv\frac{A}{2x-3}+\frac{B}{x+2},$$ where A and B are constants

Multiplying both sides by $(2x-3)(x+2)$, we have

$$x+16\equiv A(x+2)+B(2x-3), \qquad (1)$$
$$\equiv(A+2B)x+(2A-3B)$$

Since this is identically true, by equating coefficients we have

$$A+2B=1, \quad 2A-3B=16,$$

whence $$A=5, \quad B=-2,$$

$$\frac{x+16}{(2x-3)(x+2)}=\frac{5}{2x-3}-\frac{2}{x+2}$$

NOTE Since (1) is true for all values of x, we may also find A and B by giving different numerical values to x in this identity

EXAMPLE 2 *Resolve* $\dfrac{2x^3+7x^2-2x-2}{2x^2+x-6}$ *into partial fractions*

Here the numerator is not of lower dimensions than the denominator; hence we proceed as follows

by division,
$$\frac{2x^3+7x^2-2x-2}{2x^2+x-6}=x+3+\frac{x+16}{(2x-3)(x+2)}$$
$$=x+3+\frac{5}{2x-3}-\frac{2}{x+2},\quad \text{by Ex. 1.}$$

EXAMPLE 3 *Resolve* $\dfrac{3x^2-10x-2}{(x-1)(x-2)(2x+1)}$ *into partial fractions*

Assume
$$\frac{3x^2-10x-2}{(x-1)(x-2)(2x+1)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{2x+1};$$
then $3x^2-10x-2\equiv A(x-2)(2x+1)+B(x-1)(2x+1)+C(x-1)(x-2)$

Put $x=1$, then $\quad 3-10-2=-3A, \quad$ or $\quad A=3$

Put $x=2$, then $\quad 12-20-2=\ 5B, \quad$ or $\quad B=-2$

To find C, equate the coefficients of x^2, then $3=2A+2B+C$,

whence $\qquad 3=6-4+C, \quad$ or $\quad C=1$

Thus
$$\frac{3x^2-10x-2}{(x-1)(x-2)(2x+1)}=\frac{3}{x-1}-\frac{2}{x-2}+\frac{1}{2x+1}$$

EXAMPLE 4 *Express* $\dfrac{2x^2-5x+10}{(x+2)(x^2+x+5)}$ *in partial fractions*

Since x^2+x+5 has no rational factors, we shall have two fractions with denominators $x+2$ and x^2+x+5 The numerator of the first will be constant as before, but the numerator of the second may be of the first degree in x Hence we assume

$$\frac{2x^2-5x+10}{(x+2)(x^2+x+5)}\equiv\frac{A}{x+2}+\frac{Bx+C}{x^2+x+5},\quad \text{where A, B, C are constants}$$
$$2x^2-5x+10\equiv A(x^2+x+5)+(x+2)(Bx+C)$$
$$\equiv(A+B)x^2+(A+2B+C)x+(5A+2C)$$

By equating coefficients, we have
$$A+B=2,\quad A+2B+C=-5,\quad 5A+2C=10$$

These three equations give $A=4$, $B=-2$, $C=-5$
$$\frac{2x^2-5x+10}{(x+2)(x^2+x+5)}=\frac{4}{x+2}-\frac{2x+5}{x^2+x+5}$$

NOTE If we had assumed $\dfrac{2x^2-5x+10}{(x+2)(x^2+x+5)}\equiv\dfrac{A}{x+2}+\dfrac{C}{x^2+x+5}$, leaving out the term Bx, the next step would have given
$$2x^2-5x+10\equiv Ax^2+(A+C)x+(5A+2C),$$
and on equating coefficients we should have had
$$A=2,\quad A+C=-5,\quad 5A+2C=10,$$
and these three equations will be found to be inconsistent

EXAMPLE 5. *Resolve* $\dfrac{7x-10}{(3x-4)(x-1)^2}$ *into partial fractions.*

As before, we may assume the fraction $\equiv \dfrac{A}{3x-4}+\dfrac{Bx+B'}{(x-1)^2}$.

Now $$\frac{Bx+B'}{(x-1)^2}=\frac{B(x-1)+(B-B')}{(x-1)^2}=\frac{B}{x-1}+\frac{B-B'}{(x-1)^2},$$

we may at once assume

$$\frac{7x-10}{(3x-4)(x-1)^2}\equiv\frac{A}{3x-4}+\frac{B}{x-1}+\frac{C}{(x-1)^2}, \text{ where A, B, C are constants.}$$

$$7x-10\equiv A(x-1)^2+B(3x-4)(x-1)+C(3x-4).$$

Put $x-1=0$, then $7-10=-C$, or $C=3$.

By equating coefficients, $A+3B=0$ and $-10=A+4B-4C$;

whence $$A=-6,\quad B=2.$$

$$\frac{7x-10}{(3x-4)(x-1)^2}=-\frac{6}{3x-4}+\frac{2}{x-1}+\frac{3}{(x-1)^2}.$$

Similarly, corresponding to a factor of the form $(x-a)^r$ in the denominator, we may assume fractions with constant numerators, and denominators $(x-a)$, $(x-a)^2$, ... $(x-a)^r$.

[*Examples* XLII. 1–12, *page* 498, *may be taken here.*]

544. The following example shews a useful application of partial fractions.

EXAMPLE. *Find the general term in the expansion of* $\dfrac{7x-10}{(3x-4)(x-1)^2}$ *in ascending powers of* x.

We have $$\frac{7x-10}{(3x-4)(x-1)^2}=-\frac{6}{3x-4}+\frac{2}{x-1}+\frac{3}{(x-1)^2} \quad \text{[Ex. 5, Art. 543.]}$$

$$=\frac{6}{4\left(1-\frac{3x}{4}\right)}-\frac{2}{1-x}+\frac{3}{(1-x)^2}$$

$$=\frac{3}{2}\left(1-\frac{3x}{4}\right)^{-1}-2(1-x)^{-1}+3(1-x)^{-2}$$

$$=\frac{3}{2}\left\{1+\frac{3x}{4}+\left(\frac{3x}{4}\right)^2+\dots+\left(\frac{3x}{4}\right)^r+\dots\right\}$$

$$-2\{1+x+x^2+\dots+x^r+\dots\}$$

$$+3\{1+2x+3x^2+\dots+(r+1)x^r+\dots\}.$$

Thus the general term $$=\left\{\frac{3}{2}\cdot\frac{3^r}{4^r}-2+3(r+1)\right\}x^r$$

$$=\left(\frac{3^{r+1}}{2^{2r+1}}+3r+1\right)x^r.$$

545. In some special cases a fraction may be resolved into partial fractions very readily without the use of Undetermined Coefficients

EXAMPLE *Separate* $\dfrac{x^3+8}{(x-2)^4}$ *into partial fractions*

Put $x-2=y$, so that $x=y+2$,

then
$$\frac{x^3+8}{(x-2)^4}=\frac{(y+2)^3+8}{y^4}=\frac{y^3+6y^2+12y+16}{y^4}$$
$$=\frac{1}{y}+\frac{6}{y^2}+\frac{12}{y^3}+\frac{16}{y^4}$$
$$=\frac{1}{x-2}+\frac{6}{(x-2)^2}+\frac{12}{(x-2)^3}+\frac{16}{(x-2)^4}.$$

EXAMPLES XLII.

Resolve into partial fractions

1. $\dfrac{5x+1}{(x+5)(x-3)}$ 2. $\dfrac{4x-19}{(x-1)(x-2)}$. 3. $\dfrac{14x}{x^2+x-12}$

4. $\dfrac{8-x}{1+x-6x^2}$ 5. $\dfrac{x^2-6x-7}{(x-1)(x-2)(x+3)}$ 6. $\dfrac{x^2+8x+1}{(2-x)(1+x+x^2)}$.

7. $\dfrac{3x^2+x+1}{x(x+1)^2}$ 8. $\dfrac{x^2-2x+10}{(x+2)(x-1)^2}$ 9. $\dfrac{x^3-2x-13}{x^2-2x-3}$

10. $\dfrac{x^2+4x+7}{(x+2)(x+3)^2}$ 11. $\dfrac{3x^2-x-2}{(1+2x)(x+2)^2}$. 12. $\dfrac{3x^2+92x}{(x^2+1)(x+6)}$

Find the general term when the following expressions are expanded in ascending powers of x

13. $\dfrac{3-4x}{(1-x)(1-2x)}$ 14. $\dfrac{2}{x^2-8x+15}$ 15. $\dfrac{7+6x}{2+7x+3x^2}$

16. $\dfrac{2+13x}{2x^2-x-1}$ 17. $\dfrac{2x-4}{(1-x^2)(1-2x)}$ 18. $\dfrac{x^2}{(x^2-1)(x-2)}$

19. $\dfrac{3-2x-x^2}{(1-x)(1+4x)^2}$ 20. $\dfrac{4+7x}{(2+3x)(1+x)^2}$. 21. $\dfrac{x^2}{(x-1)^2(x-2)(x-3)}$

22. Express $\dfrac{x^3-6x+5}{(x-4)^4}$ and $\dfrac{x^3+3x^2-2x-4}{(x+2)^4}$ in partial fractions

23. Express $\dfrac{1}{(3n-2)(3n+1)}$ in partial fractions, thence find the sum of

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots \text{ to } n \text{ terms}$$

MISCELLANEOUS EXAMPLES IX.

EXAMPLES FOR REVISION.

A.

1 A horse bought for £x, is sold for £39 at a profit of x per cent Find x

2 Given that $y=\dfrac{3-2z}{4z+1}$ and $z=\dfrac{5+4x}{3x-2}$, find the value of y in terms of x, and of x in terms of y

3 If $x-2$ is a factor of $120x^3-167x^2-ax+56$, find the numerical value of a, and hence find the two remaining factors of the expression

4. The first and last terms of an A P are -3 and 25, and the sum of the series is 1837 Find the number of terms and the common difference

5 Find the coefficient of x in the expansion of $\left(x^{13}-\dfrac{3}{x^5}\right)^7$.

6. Solve the equations

$$2(1-x)+(2-y)=3(1-x)(2-y),$$
$$4(1-x)+(2-y)=9(1-x)(2-y)$$

7 Draw the graph of $y=\frac{8}{9}+\frac{4}{3}x-x^2$, and from the graph, or otherwise, determine the value of x for which y is a maximum

Shew that the line whose equation is $y=2x+1$ touches the above curve, and write down the coordinates of the point of contact

B

8. Find the factors of

(i) $(m^2-n^2)t-mn(t^2-1)$, (ii) $x^4-7x^2y^2+y^4$

9 A man has £a in one bank and £b in another If he has £c more to deposit, how should he divide it between the two banks so that (i) the two accounts may be equal, (ii) the first account may be double the second?

10 From the statement $ax^2+bx=cx-\dfrac{1}{d}$, find

(i) d when $a=1$, $b=2$, $c=-1$, $x=-2$,

(ii) a and c when $x=0{\cdot}4$, $b=3$, $d=-1$, and $a+c=2{\cdot}25$

11 A gallon of water weighs 10 lbs, and a gallon of milk 10·32 lbs; if a milkman sells a mixture of 10 gallons weighing 102·5 lbs, how much milk is there in the mixture?

12 If l and m are real quantities, shew that the expression

$$x^2-(l+m)x+l^2-lm+m^2$$

will be positive for all real values of x

13. If $A-24y+10y^2+8y^3+y^4$ is an exact square, what is the value of A?

14. Find the number of arrangements of the expanded form of the expression $a^m b^3$, such that the two b's come together in each arrangement. Prove also that the number of arrangements in which the two b's never come together is $\frac{m(m+1)}{2}$.

C

15. If $x+\frac{1}{x}=y$, prove that $x^5+\frac{1}{x^5}=y^5-5y^3+5y$.

16. If p dozen oranges worth b pence per score are selected from a case containing q gross of average value 23 for a shilling, what is the average value per dozen of the oranges which are left in the case?

17. The time of swing of a pendulum is equal to $k\sqrt{l}$ seconds, where l is the length of the pendulum in inches and k a constant. The length of a pendulum beating seconds is 39.14 inches; hence find the time of swing of a pendulum 60 inches long.

18. With the same axes of x and y draw the graphs of

$$\text{(i) } y=2x-\tfrac{1}{3}x^2, \qquad \text{(ii) } y=\tfrac{1}{2}x-2.$$

Write down the quadratic equation whose roots are the values of x at the points of intersection of the graphs.

19. Find the coefficient of x^r in the expansion of $\frac{5+4x}{(1-x)^2}$.

20. Shew that x^2+qx+1 and x^3+px^2+qx+1 have a common factor of the form $x+a$ when

$$(p-1)^2-q(p-1)+1=0.$$

21. If $\frac{x^2+15x-22}{(x-1)(x-2)(x+2)} \equiv \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+2}$, find A, B, and C.

D

22. An alloy, whose value is £a per lb., is composed of two metals; one of the metals, worth £b per lb. is extracted; this metal was contained in the alloy in the ratio x : 1. What is the value of the other metal per lb.?

23. Solve the equations

$$\text{(i) } \frac{2x+3}{4x+3}+\frac{3x+2}{4x+2}=\frac{5(x+2)}{4x+5}, \qquad \text{(ii) } x^2+x\left(\frac{p^2}{q}-\frac{q^2}{p}\right)-pq=0.$$

24. Find the sum of n terms of the geometric progression of which the third term is -24 and the sixth term is 3.

How many terms must be taken so that this sum may differ from the sum to infinity by less than 0.001?

25. If x is so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-4x)^{\frac{1}{2}}+(1-3x)^{\frac{1}{3}}}{(1-2x)^{\frac{1}{4}}}=2-2x-\frac{13x^2}{4}.$$

26. If a, b, c, d, x are positive, and $\frac{a}{b}>\frac{c}{d}$, prove that

$$\frac{a}{b}>\frac{a+xc}{b+xd}>\frac{c}{d}.$$

27. Calculate the values of $(\sqrt{2})^x$ for $x=0, 1, 2, 3, 4, 5, 6$, and plot the results in a graph.

28. If $a^2=q+r$, $b^2=r+p$, $c^2=p+q$, and $2s=a+b+c$, prove that

$$4s(s-a)(s-b)(s-c)=qr+rp+pq.$$

E

29. Simplify (i) $\dfrac{3^{2n}-3\cdot 9^{n-1}}{6^{2n-1}}$, (ii) $\dfrac{1}{\sqrt{11-2\sqrt{30}}}-\dfrac{3}{\sqrt{7-\sqrt{40}}}$.

30. If $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are the roots of the equation $px^2-2x+p^2=0$, prove that $\alpha=\pm\beta$.

31. If $ax-by=0$, $x+y=xy$, and $x^2+y^2=1$, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}.$$

32. If n straight lines of unlimited length, p of which are parallel, are drawn in a plane, prove that the number of triangles so formed is

$$\tfrac{1}{6}(n-p)(n-\overline{p+1})\{n+2(p-1)\},$$

supposing that no three of the lines are concurrent.

33. What is the error made in taking the sum of the infinite series $1, 0·2, 0·04, 0·008, \ldots$ as being $1·248$?

Find, to three decimal places, the sum of the square roots of the terms of this series (i) taken as all positive, (ii) taken as alternately positive and negative.

34. Running a certain course, starting level, A finishes 176 yards in front of B. With a minute's start, B would have finished 160 yards in front of A, and with 160 yards start he would have finished three seconds later than A. What is the length of the course, and what are the times of A and B for the full course?

35. Draw a graph of the expression $\left(\frac{x}{3}\right)^2(6-x)$, using the values $x=0, 0·5, 1·0, 1·5, \ldots 5·5, 6·0$.

F

36. Solve the equation $\sqrt{x-a}+\sqrt{x-b}=\sqrt{a+b}$, and verify the solution

37 Find p so that the sum of the squares of the roots of the equation $(p-6)x^2+(12-p)x+3p=0$ may be equal to 4

38. Find $\sqrt{97+56\sqrt{3}}$ in the form $x+\sqrt{y}$, and find the square root of the last surd

39. If $2ax=a^2+1$, $2by=b^2+1$, and $xy-\sqrt{(x^2-1)(y^2-1)}=c$,

prove that
$$\sqrt{\frac{c+1}{c-1}}=\frac{a+b}{a-b}$$

40. The two middle terms of an A P of $2n$ terms are a and b Find the difference between the sum of the last n terms and that of the first n terms

41 The coefficients of the 5^{th}, 6^{th}, and 7^{th} terms in the expansion of $(1+x)^n$ are in arithmetic progression Find n

42. Trace, with the same axes, the graphs of
$$y=0\ 3x^2-1\ 2, \qquad y=x^3$$
Hence solve the equation $x^3=0\ 3x^2-1\ 2$

G

43. Solve the equations
$$ax-by=\tfrac{1}{2}(b-a), \qquad ax+by=c(1+z), \qquad by-cz=\tfrac{1}{2}(c-b)$$

44. Find a geometric series such that each term exceeds by unity the sum of all the terms before it

45. Find a quantity such that when it is subtracted from each of the quantities a, b, c, the remainders are in continued proportion

46. If $a+b+c=0$, prove that $\dfrac{a^2}{2a^2+bc}+\dfrac{b^2}{2b^2+ac}+\dfrac{c^2}{2c^2+ab}=1$

47 Find the number of different permutations that can be made out of the letters a, b, c, d, b, a, taken three at a time

48. If the increase in a tree's girth in one year is proportional to its girth at the beginning of the year, and its girth is doubled in 11 years, in how many years will its girth be trebled?

49. Find the value of the infinite series
$$1+\frac{1}{6}+\frac{1\ 3}{6\ 12}+\frac{1\ 3\ 5}{6\ 12\ 18}+\dots$$

CHAPTER XLIII

THE USE OF EXPONENTIAL AND LOGARITHMIC SERIES

546 IN this chapter we shall deal with the use of certain expansions known as Exponential and Logarithmic Series Rigorous proofs of these expansions do not fall within the scope of an elementary text-book, but the student may conveniently here learn some of the applications of such series, postponing their formal discussion to a later stage of his reading

547 The infinite series

$$1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+ \quad +\frac{1}{\lfloor r}+$$

is always denoted by the symbol e

The numerical value of e is obviously greater than 2

Also
$$e<1+\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \quad \text{to inf}\right)$$
$$<1+\frac{1}{1-\frac{1}{2}}$$
$$<3$$

Thus the value of e lies between 2 and 3. By taking a sufficient number of terms of the expansion, and expressing them in decimal form, the value of e can be obtained to any required degree of accuracy To 6 places of decimals it is found to be 2 718282

548 The series which we have denoted by e is very important as it is the base to which logarithms are first calculated. Logarithms to this base are known as the Napierian system, so named after Napier their discoverer They are also called *natural* logarithms from the fact that they are the first logarithms which naturally come into consideration in algebraical investigations

When logarithms are used in theoretical work it is to be remembered that the base e is always understood, just as in arithmetical work *common* logarithms to the base 10 are invariably employed

The connection between natural and common logarithms will be explained later

Exponential Series.

549. When x has any finite value, it can be proved that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \quad + \frac{x^r}{\underline{|r}} + \quad \text{to inf}$$

and that the series on the right is convergent [See Art 535.]

This is known as the **Exponential Theorem**

This theorem can be put in another form, as follows

Write cx in the place of x, then

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \quad .$$

Now let $e^c = a$, so that $c = \log_e a$, then we obtain

$$a^x = 1 + x \log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \quad \ldots$$

EXAMPLE 1 *Find the coefficient of* x^r *in the expansion of* $\dfrac{a - bx}{e^x}$

$$\frac{a - bx}{e^x} = (a - bx)e^{-x}$$

$$= (a - bx)\left\{1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \quad + \frac{(-1)^r x^r}{\underline{|r}} + \quad \right\};$$

the coefficient of $x^r = \dfrac{(-1)^r}{\underline{|r}}\, a - \dfrac{(-1)^{r-1}}{\underline{|r-1}}\, b = \dfrac{(-1)^r}{\underline{|r}}(a + rb).$

EXAMPLE 2 *Find the sum of the infinite series*

$$\frac{1 \cdot 2}{\underline{|1}} + \frac{2 \cdot 3}{\underline{|2}} + \frac{3 \cdot 4}{\underline{|3}} + \frac{4 \cdot 5}{\underline{|4}} +$$

If we denote the successive terms by u_1, u_2, u_3 , we have

$$u_n = \frac{n(n+1)}{\underline{|n}} = \frac{n+1}{\underline{|n-1}} \qquad \ldots\ldots(1)$$

$$= \frac{(n-1)+2}{\underline{|n-1}} = \frac{1}{\underline{|n-2}} + \frac{2}{\underline{|n-1}}, \qquad (2)$$

Putting $n = 1, 2, 3,$ successively, from (1) we have $u_1 = 2$ And from (2) we have

$$u_2 = 1 + \frac{2}{\underline{|1}},\quad u_3 = \frac{1}{\underline{|1}} + \frac{2}{\underline{|2}},\quad u_4 = \frac{1}{\underline{|2}} + \frac{2}{\underline{|3}},\quad u_5 = \frac{1}{\underline{|3}} + \frac{2}{\underline{|4}}, \text{ and so on.}$$

Hence, by collecting terms suitably,

$$\text{the series} = \left(2 + \frac{2}{\underline{|1}} + \frac{2}{\underline{|2}} + \frac{2}{\underline{|3}} + \frac{2}{\underline{|4}} + \quad \right) + \left(1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \quad \right)$$

$$= 2e + e = 3e$$

Logarithmic Series

550 From the Exponential Theorem the following formula can be deduced

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\quad +(-1)^{r-1}\frac{x^r}{r}+$$

This is known as the **Logarithmic Series**

In this formula the number of terms on the right is infinite, and the series is convergent when x is *greater than* -1 *and not greater than* $+1$ Hence within this range of values the series may be legitimately used for arithmetical calculation

The form of the series should be carefully noted and compared with that of the exponential series

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\quad +\frac{x^r}{\underline{|r}}+$$

In the latter the first term is 1, all the terms are positive, all the denominators are factorials, and x^r occurs in the $(r+1)^{\text{th}}$ term.

In the logarithmic series the first term is x, the terms are alternately positive and negative, there are no factorials in the denominators, and x^r occurs in the r^{th} term

The following examples are given to enforce these points In each case it is assumed that the symbols are of such value as to make the expansions legitimate

EXAMPLE 1 *If* $a=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\quad$, *express* b *in ascending powers of* a

From the given result we have $a=\log_e(1+b)$,

$$e^a=1+b,\quad \text{or}\quad b=e^a-1$$

Hence $$b=a+\frac{a^2}{\underline{|2}}+\frac{a^3}{\underline{|3}}+\frac{a^4}{\underline{|4}}+$$

EXAMPLE 2 *Shew that*

$$\log_e(1+3x+2x^2)=3x-\frac{5x^2}{2}+3x^3-\frac{17x^4}{4}+\quad ,$$

and find the general term of the series

$$\log_e(1+3x+2x^2)=\log_e(1+x)(1+2x)=\log_e(1+x)+\log_e(1+2x)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\quad\right)+\left(2x-\frac{4x^2}{2}+\frac{8x^3}{3}-\frac{16x^4}{4}+\quad\right)$$

$$=3x-\frac{5x^2}{2}+3x^3-\frac{17x^4}{4}+$$

The general term $=(-1)^{r-1}\frac{x^r}{r}+(-1)^{r-1}\frac{2^rx^r}{r}=\frac{(-1)^{r-1}}{r}(2^r+1)x^r$

Construction of Logarithmic Tables.

551 We shall now give some account of the way in which logarithmic series are used in the calculation of Napierian Logarithms, leading up to the construction of Tables of Common Logarithms

The series for $\log_e(1+x)$ cannot be used for values of $x>1$, moreover, it converges so slowly that it is of little use for numerical calculations We can, however, deduce from it other series by the aid of which Tables of Logarithms may be constructed

We have $$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots;$$

replacing x by $-x$, we have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots.$$

$$\log_e\frac{1+x}{1-x}=\log_e(1+x)-\log_e(1-x)$$

$$=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ \right\}\ . \qquad \ldots\ldots\ldots\ldots(1)$$

In this result put $\frac{1+x}{1-x}=\frac{n+1}{n}$, so that $x=\frac{1}{2n+1}$, then

$$\log_e\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ .\right\};$$

that is,

$$\log_e(n+1)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\} \quad \ldots\ldots(2)$$

552 In Art 407 it was proved that to transform logarithms from any base a to a new base b, we have to multiply them by the *modulus* $\frac{1}{\log_a b}$ Hence logarithms of numbers to base 10 can be obtained by multiplying the Napierian logarithms of these numbers by the modulus $\frac{1}{\log_e 10}$

From the series (2) of the preceding article, we can obtain $\log_e 2$ by putting $n=1$ Again, by putting $n=2$, we obtain $\log_e 3-\log_e 2$; whence $\log_e 3$ is found, and therefore also $2\log_e 3$ or $\log_e 9$ is known

Now by putting $n=9$ in series (2), we can obtain $\log_e 10-\log_e 9$; whence the value of $\log_e 10$ is found to be 2 30258509

Thus the modulus for the system of common logarithms is $\frac{1}{2\ 30258509\ldots}$, or 0 43429448. . ; we shall denote this modulus by μ.

By multiplying the series (2) of Art 551 throughout by μ we obtain a formula adopted to the calculation of common logarithms Thus

$$\mu\log_e(n+1)-\mu\log_e n=2\mu\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ \ldots\right\},$$

that is,

$$\log_{10}(n+1)-\log_{10}n=2\left\{\frac{\mu}{2n+1}+\frac{\mu}{3(2n+1)^3}+\frac{\mu}{5(2n+1)^5}+\ \ldots\right\}$$

From this result we see that if the logarithm of one of two consecutive numbers is known, the logarithm of the other may be found, and thus a table of logarithms can be constructed

It should be noticed that the above formula is only needed to calculate the logarithms of *prime* numbers, for the logarithm of a *composite* number may be obtained by adding together the logarithms of its component factors

EXAMPLE *Calculate the value of* $\log_{10}2$ *to 6 decimal places*

Putting $n=1$ in the last series, we have

$$\log_{10}2=2\left\{\frac{\mu}{3}+\frac{\mu}{3\cdot 3^3}+\frac{\mu}{5\cdot 3^5}+\frac{\mu}{7\cdot 3^7}+\ \ldots\right\}$$

The calculation may be arranged as follows

$\mu=$	43429448,	$\cdot\ \mu/3$	$=$ 144764	83
$\mu/3^3=\mu/3-9=$	01608498,	$\mu/(3\cdot 3^3)$	$=$ 005361	66
$\mu/3^5$	$=$ 00178722,	$\mu/(5\cdot 3^5)$	$=$ 000357	44
$\mu/3^7$	$=$ 00019858,	$\mu/(7\cdot 3^7)$	$=$ 000028	37
$\mu/3^9$	$=$ 00002206,	$\mu/(9\cdot 3^9)$	$=$ 000002	45
$\mu/3^{11}$	$=$ 00000245,	$\mu/(11\cdot 3^{11})$	$=$ 000000	22
$\mu/3^{13}$	$=$ 00000027,	$\mu/(13\cdot 3^{13})$	$=$ 000000	02
			150515	

$\cdot\ \log_{10}2=150515\times 2=301030$

553 Theoretically the series for $\log_{10}(n+1)-\log_{10}n$ given in the last article is sufficient for the calculation of common logarithms It has the advantage of converging rapidly, so that (except for small values of n) only a few terms of the series need be taken to obtain the necessary approximation, but in practice the arithmetical work is often inconvenient

For example, when $n=16$, we get $\log_{10}17-\log_{10}16$,

that is, $$\log_{10}17=4\log_{10}2+2\left\{\frac{\mu}{33}+\frac{\mu}{3(33)^3}+\frac{\mu}{5(33)^5}+\ \ldots\right\},$$

and the calculation of the terms of the series would be very tedious.

We shall now give other series which will effect a saving of labour.

554 In the formula

$$\log_{10}(1+x)=\mu\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right),$$

by writing $\frac{1}{n}$ for x, we obtain $\log_{10}\frac{n+1}{n}$, hence

$$\log_{10}(n+1)-\log_{10}n=\frac{\mu}{n}-\frac{\mu}{2n^2}+\frac{\mu}{3n^3}-\dots \qquad (1)$$

Again, by writing $-\frac{1}{n}$ for x, we obtain $\log_{10}\frac{n-1}{n}$, hence by changing signs on both sides of the formula, we have

$$\log_{10}n-\log_{10}(n-1)=\frac{\mu}{n}+\frac{\mu}{2n^2}+\frac{\mu}{3n^3}+\dots \qquad (2)$$

The following example shews the use of these series in obtaining the logarithms of some of the smaller prime numbers It will be seen that the calculation is usually less laborious than in the example of Art 552 The numerical details are left as an exercise for the student

EXAMPLE *To explain how the common logarithms of 2, 3, 5, 7, 11 may be found*

(i) Putting $n=10$ in series (2), we get $\log 10-\log 9$, thus

$$1-2\log 3=\frac{\mu}{10}+\frac{\mu}{2\cdot 10^2}+\frac{\mu}{3\cdot 10^3}+\dots,$$

whence $\log 3$ is readily found to be $\cdot4771213$, to seven decimal places

(ii) By putting $n=3$ in series (2), we get $\log 3-\log 2$, thus

$$\log 3-\log 2=\frac{\mu}{3}+\frac{\mu}{2\cdot 3^2}+\frac{\mu}{3\cdot 3^3}+\dots,$$

whence $\log 2$ is found to be $\cdot3010300$

(iii) $\log 5=\log\frac{10}{2}=1-\log 2=\cdot6989700$

(iv) By putting $n=8$ in series (2), we get $\log 8-\log 7$, thus

$$3\log 2-\log 7=\frac{\mu}{8}+\frac{\mu}{2\cdot 8^2}+\frac{\mu}{3\cdot 8^3}+\dots,$$

whence $\log 7$ is found to be $\cdot8450980$

(v) By putting $n=10$ in series (1), we get $\log 11-\log 10$, thus

$$\log 11-1=\frac{\mu}{10}-\frac{\mu}{2\cdot 10^2}+\frac{\mu}{3\cdot 10^3}-\dots;$$

whence $\log 11$ is found to be $1\cdot0413927$

NOTE We may also find $\log 7$ quickly as follows

By putting $n=50$ in series (2), we get $\log 50-\log 49$, thus

$$2-\log 2-2\log 7=\frac{\mu}{50}+\frac{\mu}{2\cdot 50^2}+\frac{\mu}{3\cdot 50^3}+\dots.$$

555. We shall now give some further examples on the logarithmic series. In all cases it is assumed that the symbols are such as to make the expansions legitimate.

EXAMPLE 1. *To find* $\log 10008$ *to 7 decimal places.*

Since $\log 10008 = \log(10^4 \times 1.0008) = 4 + \log 1.0008$, it is only necessary to find $\log 1.0008$ by putting $x = .0008$ in the series

$$\log_{10}(1+x) = \mu\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right\}.$$

Thus to 7 places of decimals,

$$\begin{aligned}\log 10008 &= 4 + .43429448\{.0008 - \tfrac{1}{2}(.0008)^2\}\\ &= 4.0003473.\end{aligned}$$

EXAMPLE 2. *If* α, β *are the roots of the equation* $ax^2 - bx + c = 0$, *shew that*

$$\log(a - bx + cx^2) = \log a - (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 - \frac{\alpha^3+\beta^3}{3}x^3 - \dots$$

By the Theory of Quadratics, we have

$$\alpha+\beta = \frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Now
$$\begin{aligned}a - bx + cx^2 &= a\left(1 - \frac{b}{a}x + \frac{c}{a}x^2\right)\\ &= a\{1 - (\alpha+\beta)x + \alpha\beta x^2\}\\ &= a(1-\alpha x)(1-\beta x).\end{aligned}$$

$$\begin{aligned}\therefore \log(a - bx + cx^2) &= \log a + \log(1-\alpha x) + \log(1-\beta x)\\ &= \log a - \left(\alpha x + \frac{\alpha^2x^2}{2} + \frac{\alpha^3x^3}{3} + \dots\right) - \left(\beta x + \frac{\beta^2x^2}{2} + \frac{\beta^3x^3}{3} + \dots\right)\\ &= \log a - (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 - \frac{\alpha^3+\beta^3}{3}x^3 - \dots\end{aligned}$$

EXAMPLE 3. *If* $\log_e \dfrac{1}{1+x+x^2+x^3}$ *is expanded in ascending powers of* x, *shew that the coefficient of* x^n *is* $\dfrac{3}{n}$ *if* n *is a multiple of 4, and* $-\dfrac{1}{n}$ *if* n *is not a multiple of 4.*

$$\log_e \frac{1}{1+x+x^2+x^3} = \log_e \frac{1-x}{1-x^4} = \log_e(1-x) - \log_e(1-x^4).$$

(i) If n is not a multiple of 4, the term involving x^n comes only from $\log_e(1-x)$;

$$\therefore \text{ the required coefficient} = -\frac{1}{n}.$$

(ii) If n is a multiple of 4, put $n = 4m$;

$$\text{then the required coefficient} = -\frac{1}{4m} + \frac{1}{m} = \frac{3}{4m} = \frac{3}{n}.$$

EXAMPLES XLIII.

1. Find the coefficient of x^r in the expansions of

$$\text{(i)}\ \frac{1-x}{e^x}; \qquad \text{(ii)}\ \frac{ax+b}{e^x}.$$

2. Find the coefficient of x^n in the series

$$1+\frac{a+bx}{\underline{|1}}+\frac{(a+bx)^2}{\underline{|2}}+\ldots+\frac{(a+bx)^r}{\underline{|r}}+\ldots$$

3. Shew that

$$\text{(i)}\ e^{-2}=1-\frac{2^3}{\underline{|3}}+\frac{2^4}{\underline{|4}}-\frac{2^5}{\underline{|5}}+\ldots; \quad \text{(ii)}\ \frac{e^2-1}{2e}=1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\ldots\ldots$$

4. Find the coefficient of x^r in the expansion of $\dfrac{1-x-x^2}{e^x}$.

5. Shew that $1+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|4}}+\dfrac{1}{\underline{|6}}+\ldots=\dfrac{1}{2}(e+e^{-1})$.

6. Shew that $e^{-1}=2\left(\dfrac{1}{\underline{|3}}+\dfrac{2}{\underline{|5}}+\dfrac{3}{\underline{|7}}+\ldots\right)$.

7. Prove that

$$\left\{1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right\}\left\{1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\ldots\right\}=1.$$

Find the sum of the following infinite series:

8. $\dfrac{1}{\underline{|2}}+\dfrac{2}{\underline{|3}}+\dfrac{3}{\underline{|4}}+\dfrac{4}{\underline{|5}}+\ldots$

9. $\dfrac{1^2}{\underline{|2}}+\dfrac{2^2}{\underline{|3}}+\dfrac{3^2}{\underline{|4}}+\dfrac{4^2}{\underline{|5}}+\ldots\ldots$

10. $1+\dfrac{1+2}{1\cdot 2}+\dfrac{1+2+3}{1\cdot 2\cdot 3}+\dfrac{1+2+3+4}{1\cdot 2\cdot 3\cdot 4}+\ldots$

11. $a^2-b^2+\dfrac{1}{\underline{|2}}(a^4-b^4)+\dfrac{1}{\underline{|3}}(a^6-b^6)+\ldots$

12. Expand $\log\sqrt{1+x}$ in ascending powers of x.

13. Shew that $\log_{10}\left(\dfrac{1}{1-x}\right)=\dfrac{1}{\log_e 10}\left(x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\ldots\right)$.

14. If $y=-x+\dfrac{x^2}{\underline{|2}}-\dfrac{x^3}{\underline{|3}}+\ldots$, and y is less than 1, express x in a series of ascending powers of y.

15. Shew that

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\right\}$$

By putting $m=2$, and $n=1$, calculate the value of $\log_e 2$ to five decimal places.

16. Expand $\log_e(1+x-2x^2)$ in ascending powers of x to four terms, and find the general term.

17. Find the general term of the expansion of $\log_e(1+2x-8x^2)$. Hence write down the first four terms of the series.

18. Prove that

$$\log_e \frac{1+x}{1-3x} = 4x + 4x^2 + \frac{28}{3}x^3 + 20x^4 + \dots$$

and find the general term of the series.

19. Prove that the coefficient of x^n in the expansion of $\log_e(1+x+x^2)$ is $-\frac{2}{n}$ or $\frac{1}{n}$ according as n is or is not a multiple of 3.

20. If α and β are the roots of the equation $x^2-px+q=0$, shew that

$$\log_e(1+px+qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \dots .$$

Deduce the expansion of $\log_e(1+3x+2x^2)$.

21. Prove that

$$\log_e(n+1) - \log_e(n-1) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right).$$

Hence find the Napierian logarithm of $\frac{1001}{999}$ correct to 10 decimal places.

22. If $x<1$, find the sum of the infinite series

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \dots$$

23. Prove that

$$\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \dots = \frac{1}{17} + \frac{1}{19} + \frac{1}{3}\left(\frac{1}{17^3} + \frac{1}{19^3}\right) + \dots .$$

24. Assuming $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, and $\frac{1}{\log_e 10} = \cdot 43429448$, calculate the values of $\log 13$ and $\log 17$ to 5 places of decimals.

25. Prove that

$$\log a + \log\frac{a^2}{b} + \log\frac{a^3}{b^2} + \dots \text{ to } n \text{ terms} = \frac{n}{2}(\log a^{n+1} - \log b^{n-1}).$$

If $a=25$, $b=2$, and $n=100$, calculate the sum of the series to the nearest integer.

CHAPTER XLIV

COMPOUND INTEREST AND ANNUITIES

556 PROBLEMS connected with Compound Interest and Annuities give useful practice in logarithmic work We shall here prove and illustrate the necessary formulæ for the solution of such problems, using the terms and phraseology of the subject in their ordinary arithmetical sense

One difference of usage should be noted instead of taking as the rate of interest the interest on £100 for one year, it will be found more convenient to take the interest on £1 for one year Thus our formulæ will involve a symbol for the rate *per pound* instead of the usual "rate *per cent*"

557 *To find the interest and amount in* n *years of a given sum at compound interest*

Let P denote the principal, r the interest on £1 for one year, M the amount, all expressed in pounds

Let £R denote the *amount* of £1 in one year, then $R=1+r$

The amount of P at the end of the first year is PR, and, since this is the principal for the second year, the amount at the end of the second year is $PR\times R$ or PR^2 Similarly the amount at the end of the third year is $PR^2\times R$ or PR^3, and so on, hence the amount in n years is PR^n,

that is, $$M=PR^n=P(1+r)^n$$

Also the interest $=M-P=P(R^n-1)$

By the aid of logarithms any of the four quantities involved in the formula $M=PR^n$ may be found when the other three are known

558 *To find the present value and discount of a given sum due in a given time, allowing compound interest*

Let P be the given sum, V the present value, D the discount, R the amount of £1 for one year, n the number of years

Since V is the sum which, put out to interest at the present time, will in n years amount to P, we have

$$P=VR^n,$$

$$V=PR^{-n}$$

Also $$D=P-V=P(1-R^{-n})$$

559 If interest is paid more than once a year, and each instalment of interest, as it becomes due, is added to the principal, the formula for M requires modification

Thus if interest is paid q times a year, the interest of £1 for each period is $\frac{r}{q}$, and therefore in q years, or nq periods,

$$M=P\left(1+\frac{r}{q}\right)^{nq}$$

In this case the interest is said to be "converted into principal" q times a year

EXAMPLE 1 *Find to the nearest pound the amount of £100 in 15 years, allowing compound interest at 4 %, convertible half-yearly*

Here $R=1+\frac{1}{2}\cdot\frac{4}{100}=1\cdot02$, and the number of payments is 30

Hence $$M=100(1\cdot02)^{30},$$

$$\log M=2+30\log 1\cdot02=2\cdot258$$
$$=\log 181\cdot1, \text{ from the Tables}$$

the required amount=£181, to the nearest pound

With four-figure logarithms a more accurate result cannot be obtained In some of the examples which follow it will be necessary to use logarithms taken from seven-figure Tables

EXAMPLE 2 *Find in how many years £1130 will amount to £3000 at 5 % compound interest*

If n be the number of years, we have $3000=1130(1\cdot05)^n$

$$\log 3000=\log 1130+n\log 1\cdot05,$$

$$3\cdot4771=3\cdot0531+n(\cdot0212),$$

that is, $$n=\frac{3\cdot4771-3\cdot0531}{\cdot0212}=\frac{\cdot4240}{\cdot0212}=20$$

Thus the number of years is 20

EXAMPLE 3 *Find the present value of £6000 due in 20 years, allowing compound interest at 4 % per annum Given*

$$\log 6=\cdot7781513,\quad \log 104=2\cdot0170333,\quad \log 2\cdot73833=\cdot4374853$$

Let £V denote the present value, then

$$6000=V(1\cdot04)^{20},$$

$$3+\log 6=\log V+20(\cdot0170333),$$

that is, $$\log V=3+\cdot7781513-\cdot3406660$$
$$=3\cdot4374853,$$

whence $$V=2738\cdot33$$

Thus the present value=£2738·33, or £2738, to the nearest pound

[*Examples* XLIII 1–8, *page* 517, *may be taken here*]

Annuities.

560 An **annuity** is a fixed sum, paid under certain stated conditions, at regular intervals of time Unless it is otherwise stated we shall suppose the payments annual.

If the annuity is payable unconditionally for a fixed term of years it is called an **annuity certain.** If the annuity is to continue for ever it is called a **perpetuity.**

561. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest*

Let A be the annuity, R the amount of £1 for one year, n the number of years, M the amount

At the end of the first year A is due, and the amount of this sum for the remaining $n-1$ years is AR^{n-1}, at the end of the second year A is again due, and the amount of this sum in the remaining $n-2$ years is AR^{n-2}, and so on

$$M = AR^{n-1} + AR^{n-2} + \ldots + AR^2 + AR + A$$
$$= A(1 + R + R^2 + \ldots\ldots \text{ to } n \text{ terms})$$
$$= A\frac{R^n - 1}{R - 1}$$

562 Part of the business of Life Insurance Companies is to grant annuities, payable over a stated period, in return for a sum of money paid down This sum is the **purchase price** or **present value** of the annuity

563 *To find the present value of an annuity to continue for a given number of years, allowing compound interest*

Let A be the annuity, R the amount of £1 in one year, n the number of years, V the required present value

The present value of A due in 1 year is AR^{-1},
the present value of A due in 2 years is AR^{-2},
the present value of A due in 3 years is AR^{-3}, and so on [Art 558]

Now V is the sum of the present value of these different payments

$$V = AR^{-1} + AR^{-2} + AR^{-3} + \ldots \quad \text{to } n \text{ terms}$$
$$= AR^{-1}\frac{1 - R^{-n}}{1 - R^{-1}}$$
$$= A\frac{1 - R^{-n}}{R - 1}.$$

COR. Since $R > 1$, R^{-n} becomes indefinitely small when n is infinite

564 If mA is the present value of an annuity A, the annuity is said to be worth m **years' purchase**

In the case of a perpetual annuity $mA=\frac{A}{r}$,

hence $$m=\frac{1}{r}=\frac{100}{\text{rate per cent}},$$

that is, the number of years' purchase of a perpetual annuity is obtained by dividing 100 by the rate per cent

Irredeemable Stocks, such as some Government Securities, Corporation Stocks, Railway Debentures, are examples of perpetual annuities In applying the above formula to any given case it must be remembered that the numerator is the *current value of* £100 *stock* Thus when $2\frac{1}{2}$ p c. Consols were quoted at 80 they were worth 32 years' purchase

565 **A freehold estate** is an estate which yields a perpetual annuity called the *rent*, and thus the value of the estate is equal to the present value of a perpetuity equal to the rent Hence if we know the number of years' purchase of an estate, we can obtain the rate per cent at which interest is reckoned by dividing 100 by the number of years' purchase

EXAMPLE 1 *Find the amount of an annuity of* £100 *in* 15 *years, allowing compound interest at* 4 *per cent per annum Given*

$$\log 1\ 04=\cdot01703, \textit{ and } \log 180085=5\ 25545$$

We have $$M=100\frac{(1\ 04)^{15}-1}{04}=2500\{(1\ 04)^{15}-1\}$$

Now $$\log(1\ 04)^{15}=15\times\ 01703=\ 25545=\log 1\ 80075,$$

whence $$(1\ 04)^{15}=1\ 80075$$

$$M=2500\times\ 80075=2001\ 875$$

Thus the required amount=£2001 875, or £2001 17*s* 6*d*

EXAMPLE 2 *A man borrows* £20000 *at* 5 *per cent compound interest If the principal and interest are to be paid by* 20 *equal annual instalments, find approximately the amount of each of these*

Let £A be the value of each instalment, then £20000 is the present value of an annuity of £A payable for 20 years

Hence $20000=A\frac{1-(1\ 05)^{-20}}{05}$, whence $A\{1-(1\cdot05)^{-20}\}=1000$

Now $$\log(1\ 05)^{-20}=-20(\ 0212)=-\ 424=\bar{1}\ 576=\log\ 3767,$$

whence $$(1\ 05)^{-20}=\ 3767$$

$$A(1-\ 3767)=1000, \text{ whence } A=\frac{1000}{\cdot6233}=1604 \text{ nearly.}$$

Thus the value of each instalment is £1604

566. **A deferred annuity, or reversion,** is an annuity which does not begin until after the lapse of a certain number of years When the annuity is deferred for n years, it is said to begin *after* n years, and the first payment is made at the end of $n+1$ years

567. *To find the present value of a deferred annuity to begin at the end of* p *years and to continue for* n *years, allowing compound interest*

Let A be the annuity, R the amount of £1 in one year, V the present value

The first payment is made at the end of $p+1$ years

Hence the present values of the first, second, third, . payments are respectively

$$AR^{-(p+1)},\quad AR^{-(p+2)},\quad AR^{-(p+3)},\quad \ldots$$

$$\therefore\ V=AR^{-(p+1)}+AR^{-(p+2)}+AR^{-(p+3)}+\ldots\ldots \text{ to } n \text{ terms}$$

$$=AR^{-(p+1)}\ \frac{1-R^{-n}}{1-R^{-1}}$$

$$=\frac{AR^{-p}}{R-1}-\frac{AR^{-p-n}}{R-1}$$

COR. The present value of a *deferred perpetuity* to begin after p years is obtained by making n infinite

In this case $$V=\frac{AR^{-p}}{R-1}$$

EXAMPLE. *The reversion of an estate worth £450 per annum is bought for £5000 Within what time must the buyer take possession so as not to lose by his purchase, supposing interest to be at 5 per cent ?*

Let n be the number of years, then £5000 is the present value of a perpetuity of £450 deferred for n years

$$5000=\frac{450(1\cdot05)^{-n}}{\cdot05},\ \text{whence } 5=9(1\cdot05)^{-n}$$

$$\log 5=\log 9-n\log 1\cdot05,$$

that is, $$n=\frac{\log 9-\log 5}{\log 1\cdot05}=\frac{\cdot9542-\cdot6990}{\cdot0212}=\frac{\cdot2552}{\cdot0212}=12$$

Thus he must take possession in 12 years.

EXAMPLES XLIV.

[*Use four-figure Tables unless special logarithms are quoted*]

Find, to the nearest pound, the amount at compound interest of

1. £370 in 25 yrs at 4 % 2. £450 in 20 yrs. at $2\frac{1}{2}$ %

3. What sum will amount to £3000 in 15 years at $3\frac{1}{2}$ % ?

4. In what time will £P become £100P at $5\frac{1}{2}$ % ?

5. At 5 % for $6\frac{1}{2}$ years, prove the formula $M = P \times (1\ 05)^6 \times 1\ 025$ Hence find the present value of £3000 due in $6\frac{1}{2}$ years at 5 %

6. At what rate per cent will £50 become £5000 in 50 years?

7 Find, to the nearest pound, the amount at compound interest of £6000 in 10 years at 5 % per annum paid quarterly

Given log 1 0125 = 0054

8 If the rate of interest is such that a sum of money doubles itself in 10 years, shew that £1 will amount to £1000 in about 100 years

9 Find the amount of an annuity of £250 left unpaid for 12 years at 4 %

Given log 1 04 = 01703, log 1 6009 = 20436

10. Find the present value of an annuity of £900 to continue for 20 years at $4\frac{1}{2}$ %

Given log 1 045 = 01912, log 41458 = 4 6176

11 A Corporation borrows £5000 to be repaid with interest at 3 % in 10 equal annual instalments What sum (to the nearest pound) must be repaid each year?

12 If at the beginning of each year a man invested £50 at 4 % compound interest, find to the nearest shilling what his savings amounted to at the end of 20 years

Given log 1 04 = 0170333, log 2 19112 = 3406660

13 Calculate at 3 % the purchase price of an annuity of £150 to continue for 20 years, the first payment to be made one year from the date of purchase

Given log 1 03 = 0128372, log 5 53677 = 7432560

14. A freehold estate worth £180 a year is sold for £4000, find the rate of interest

15 How many years' purchase is a freehold estate worth at $6\frac{1}{4}$ %

16 If a perpetuity is worth 25 years' purchase, find at the same rate of interest the amount of an annuity of £500 to continue for 2 years

17. The reversion after 6 years of a freehold estate is bought for £20,000, at what rent should it be let so that the owner may receive 5 % on the purchase money?

Given log 1 05 = 0211893, log 1 340096 = 1271358

18 What is the present value of a perpetual annuity of £10 payable at the end of the first year, £20 at the end of the second, £30 at the end of the third, and so on, increasing £10 each year, interest being taken at 5 % per annum?

CHAPTER XLV.

SCALES OF NOTATION.

568 THE common or denary scale of notation is that in which ordinary arithmetical numbers are expressed by means of multiples of powers of 10; for instance

$$235 = 2 \times 10^2 + 3 \times 10 + 5,$$
$$9041 = 9 \times 10^3 + 0 \times 10^2 + 4 \times 10 + 1$$

In this system ten is said to be the **radix** of the scale, and the necessary symbols are the ten digits 0, 1, 2, 3, . . 9 Any number other than ten may be taken as the radix of a scale of notation, thus, if 7 is the radix, a number expressed by 2503 represents

$$2 \times 7^3 + 5 \times 7^2 + 0 \times 7 + 3$$

In this scale no digit higher than 6 can occur

More generally, a number in a scale whose radix is r may be expressed as follows

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0,$$

where $a_0, a_1, a_2, \ldots a_n$ represent the digits in order, beginning with that in the units' place Each of these digits is a positive integer or zero, and each must be less than r

It must be remembered that, except in the denary scale, a number expressed by 10 does not stand for *ten*, but for the radix itself.

569 The ordinary operations of Arithmetic may be performed in any scale; but as the powers of the radix are no longer powers of *ten*, in determining the *carrying figures* we must divide not by ten, but by the radix of the scale we are considering

EXAMPLE *Find the sum of* 3264, 5042, 1465 *in the scale of seven, and subtract* 4541 *from the result*

$$\begin{array}{lr} \text{(i)} & 3264 \\ & 5042 \\ & \underline{1465} \\ \text{(ii)} & 13134 \\ & \underline{4541} \\ & 5263 \end{array}$$

(i) Here 5, 2, 4 make *eleven*, or 1 *seven*+**4**, set down **4** and carry 1

7, 4, 6 make *seventeen*, or 2 *sevens*+**3**, set down **3** and carry 2, and so on

(ii) After the first step of subtraction, since we cannot take 4 from 3, we add *seven*; thus we have to take 4 from ten which leaves **6**; then 6 from eight, which leaves **2**; and finally 5 from ten, which leaves **5**

570 The names *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octonary*, *nonary*, *denary*, *undenary*, and *duodenary* (or *duodecimal*) are used to denote the scales corresponding to the values *two*, *three*, *twelve* of the radix We shall not consider any scale higher than these In the undenary and duodenary scales we shall use the symbols t and e as digits to denote *ten* and *eleven* respectively.

EXAMPLE *Divide* 15et20 *by* 9 *in the scale of twelve*

$$\begin{array}{l} 9\,\underline{|15et20} \\ \quad 1ee96 \quad 6 \end{array}$$

Here $15 = 1$ *twelve* $+ 5 =$ *seventeen* $= 1 \times 9 + 8$
we set down 1 and carry 8

Also $8 \times$ *twelve* $+ e =$ *one hundred and seven* $= e \times 9 + 8$

we set down e and carry 8, and so on.

571 *To express a given integer* N *in any new scale*

Let r be the radix of the new scale, and let $a_0, a_1, a_2, \quad a_n$ be the required digits by which N is to be expressed, beginning with that in the units' place, then

$$N = a_n r^n + a_{n-1} r^{n-1} + \ . \ + a_2 r^2 + a_1 r + a_0$$

We have now to find the values of $a_0, a_1, a_2, \quad a_n$

Divide N by r, then the remainder is a_0, and the quotient is

$$a_n r^{n-1} + a_{n-1} r^{n-2} + \qquad + a_2 r + a_1$$

If this quotient is divided by r, the remainder is a_1,
if the next quotient „ „ „ „ a_2,
and so on, until there is no further quotient divisible by r

Thus the required digits are the remainders found by successive divisions by *the radix of the new scale*

EXAMPLE 1 *Express the denary number* 4213 *in the scale of nine*

$$\begin{array}{l} 9\,\underline{|4213} \\ \ 9\,\underline{|468} \quad 1 \\ \quad 9\,\underline{|52} \quad 0 \\ \qquad 5 \quad 7 \end{array}$$

Here we divide successively by 9 (the new radix), performing the division in the scale of the given number

Thus $4213 = 5 \times 9^3 + 7 \times 9^2 + 0 \times 9 + 1$

the required number is 5701

EXAMPLE 2 *Transform* 21125 *from scale seven to scale eleven*

$$\begin{array}{l} e\,\underline{|21125} \\ \ e\,\underline{|1244} \quad t \\ \quad e\,\underline{|61} \quad 0 \\ \qquad 3 \quad t \end{array}$$

Here we work in the scale of seven, thus

$21 = 2 \times$ *seven* $+ 1 =$ *fifteen* $= 1 \times e + 4$

we set down 1 and carry 4

Next $4 \times 7 + 1 =$ *twenty nine* $= 2 \times e + 7$

we set down 2 and carry 7, and so on.

The successive remainders are t, 0, t, and the last quotient is 3.
Thus the required number is $3t0t$

[*Examples* XLV 1–12, *page* 524, *may be taken here*]

Radix Fractions.

572 Fractions may also be expressed in any scale of notation; thus,

just as $\cdot 253$ in scale ten denotes $\frac{2}{10}+\frac{5}{10^2}+\frac{3}{10^3}$,

so 253 in scale 6 denotes $\frac{2}{6}+\frac{5}{6^2}+\frac{3}{6^3}$,

and 253 in scale r denotes $\frac{2}{r}+\frac{5}{r^2}+\frac{3}{r^3}$

Fractions thus expressed are called **radix-fractions.** The general type of such fractions in scale r is

$$\frac{b_1}{r}+\frac{b_2}{r^2}+\frac{b_3}{r^3}+\ldots\ldots,$$

where $b_1, b_2, b_3, \ldots$ are integers, all less than r, of which any one or more may be zero

573 *To express a given radix-fraction* F *in any new scale.*

Let r be the radix of the new scale, and let $b_1, b_2, b_3, \ldots$ be the required digits by which F is to be expressed, beginning from the left, then

$$F=\frac{b_1}{r}+\frac{b_2}{r^2}+\frac{b_3}{r^3}+\ldots$$

We have now to find the values of $b_1, b_2, b_3, \ldots$

Multiply both sides of the equation by r, then

$$rF=b_1+\frac{b_2}{r}+\frac{b_3}{r^2}+\ldots\ldots$$

Hence b_1 is equal to the integral part of rF, and, if we denote the fractional part by F_1, we have

$$F_1=\frac{b_2}{r}+\frac{b_3}{r^2}+\ldots$$

Multiply again by r, then b_2 is the integral part of rF_1. In the same way by successive multiplications by r, each of the digits may be found, and the fraction expressed in the new scale

EXAMPLE 1. *Express* $\frac{7}{8}$ *(scale ten) as a radix fraction in scale six*

$\frac{7}{8}\times 6=\frac{7\times 3}{4}=5+\frac{1}{4}$;

$\frac{1}{4}\times 6=\frac{1\times 3}{2}=1+\frac{1}{2}$,

$\frac{1}{2}\times 6=3$

Here, as in Art 571, we multiply successively by the radix of the new scale, performing the work in the scale of the given fraction.

$$\therefore \text{ the required fraction}=\frac{5}{6}+\frac{1}{6^2}+\frac{3}{6^3}=\cdot 513.$$

EXAMPLE 2 *Transform* 606·7 *from scale eight to scale five*

Here we must treat the integral and fractional parts separately.

$$\begin{array}{r|ll} 5 & 606 & \\ 5 & 116 & 0 \\ 5 & 17 & 3 \\ & 3 & 0 \end{array} \qquad \begin{array}{r} \cdot 7 \\ \underline{5} \\ 4\,3 \\ \underline{5} \\ 1\,7 \end{array}$$

Here we divide or multiply by the new radix, performing the work in the scale of the given number

The digits of the radix-fraction recur; hence the required number is $3030\cdot\dot{4}\dot{1}$

Some Properties of Numbers.

574 *In any scale of notation of which the radix is* r, *the sum of the digits of any whole number when divided by* r − 1 *will leave the same remainder as the whole number when divided by* r − 1

Let N denote the number, $a_0, a_1, a_2, \ldots a_n$, the digits beginning with that in the units' place, and S the sum of the digits,

then
$$\begin{aligned} N &= a_0 + a_1 r + a_2 r^2 + \ldots + a_{n-1} r^{n-1} + a_n r^n, \\ S &= a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n \\ N - S &= a_1(r-1) + a_2(r^2-1) + \ldots + a_{n-1}(r^{n-1}-1) + a_n(r^n-1) \end{aligned}$$

Now every term on the right is divisible by $r-1$,

$$\frac{N-S}{r-1} = \text{an integer}, \text{ that is, } \frac{N}{r-1} = I + \frac{S}{r-1},$$

where I is some integer, which proves the proposition

Hence a number in scale r is divisible by $r-1$ when the sum of its digits is divisible by $r-1$

575 A denary number divided by 9 leaves the same remainder as the sum of its digits divided by 9 The rule known as "casting out the nines" for testing the accuracy of multiplication is founded on this property The rule may be explained as follows

Let two numbers be represented by $9a+b$ and $9c+d$, and their product by P, then $P = 81ac + 9bc + 9ad + bd$

Hence $P/9$ has the same remainder as $bd/9$, and therefore *the sum of the digits* of P, when divided by 9, gives the same remainder as *the sum of the digits of bd*, when divided by 9 If on trial this should not be the case, the multiplication must have been incorrectly performed In practice b and d are readily found from the sums of the digits of the two numbers

Thus, to test the accuracy of $4758 \times 827 = 3935866$

$4+7=11$, cast out 9, and 2 is left $2+5+8=15$, cast out 9, and the remainder is 6 Similarly the remainder from 827 is 8 The remainder from the product 6×8 is 3 Again the remainder from 3935866 is 4. Thus the result is not correct

576 Any number of n digits can be expressed by the formula

$$N=a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+\ldots+a_2r^2+a_1r+a_0$$

The smallest value N can have is when $a_{n-1}=1$ and all the other digits are zero In this case $N=r^{n-1}$

The greatest value of N is obtained by making each digit as large as possible, that is, by putting

$$a_{n-1}=a_{n-2}=\ldots=a_2=a_1=a_0=r-1.$$

In this case $N=(r-1)(r^{n-1}+r^{n-2}+\ldots+r^2+r+1)=r^n-1$

Thus in scale r a number of n digits cannot be less than r^{n-1}, nor greater than r^n-1, that is, it must be less than r^n

EXAMPLE *If* N *is a denary number of* n *digits, how many digits are there in the square of* N ?

N^2 is less than $10^n\times10^n$, or 10^{2n}, also it is not less than $10^{n-1}\times10^{n-1}$, or 10^{2n-2} Now 10^{2n} (expressed by 1 followed by $2n$ ciphers) is the smallest number with $2n+1$ digits Similarly, 10^{2n-2} is the smallest number with $2n-1$ digits

N^2 cannot have more than $2n$ digits, nor fewer than $2n-1$

577. *If the square root of a number consists of* $2n+1$ *figures, when the first* $n+1$ *of these have been obtained by the ordinary method, the remaining* n *may be obtained by division*

Let N denote the given number; a the part of the square root already found, that is the first $n+1$ digits found by the common rule, with n ciphers annexed, x the remaining part of the root

Then
$$\sqrt{N}=a+x,$$
$$N=a^2+2ax+x^2,$$
$$\frac{N-a^2}{2a}=x+\frac{x^2}{2a}. \qquad (1)$$

Now $N-a^2$ is the remainder after $n+1$ digits of the root, represented by a, have been found, and $2a$ is the divisor at the same stage of the work We see from (1) that $N-a^2$ divided by $2a$ gives x, the rest of the quotient required, increased by $\frac{x^2}{2a}$

We shall shew that $\frac{x^2}{2a}$ is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain x, the rest of the root

For x contains n digits, and therefore x^2 contains $2n$ digits at most, also a is a number of $2n+1$ digits (the last n of which are ciphers) and thus $2a$ contains $2n+1$ digits at least, and therefore $\frac{x^2}{2a}$ is a proper fraction.

EXAMPLE. *Find the first 7 figures of the square root of 2.*

```
                (i)
         2,00,00  (1·414213
         1
 24      100
          96
 281      400
          281
 2824     11900
          11296
 2828 2     604|00*
            565|64
 2828 41     38|3600
             28|2841
 2828 423    10|075900
              8|485269
              1|590631
```

```
               (ii)
          2,00,00,  (1·414
 24       100
 281       400
 2824      11900
 2,8,2,8     604 (213
              38
              10
               2
```

Thus the required square root $=1\cdot414213\ldots$.

In (i) the work is given in full; in (ii) the work is contracted.

At the stage marked * *four* digits of the root have been obtained, and the 'trial divisor' consists of *four* digits, viz. 2828. The remainder at this stage is 604, and if instead of bringing down a new period, and appending a new digit to the divisor, we divide 604 by 282(8), cutting off the last digit and using the contracted method, we can obtain *three* new digits of the root, as shewn in (ii), and it is clear that the work is merely the shortened form of that shewn to the left of the vertical line in (i).

578. We give two more examples.

EXAMPLE 1. *Shew that* 1·44 *is a square number in any scale whose radix is greater than* 4.

Let r be the radix; then $1\cdot44=1+\dfrac{4}{r}+\dfrac{4}{r^2}=\left(1+\dfrac{2}{r}\right)^2.$

Thus the given number is the square of 1·2.

EXAMPLE 2. *Express the senary radix-fraction* ·503 *as a vulgar fraction in the same scale.*

$$\cdot503=\frac{5}{6}+\frac{0}{6^2}+\left(\frac{3}{6^3}+\frac{3}{6^4}+\frac{3}{6^5}+\ldots\text{ to inf}\right),\text{ in scale ten,}$$

$$=\frac{5}{6}+\frac{3}{6^3}\cdot\frac{1}{1-\frac{1}{6}}=\frac{153}{180}=\frac{17}{20},\text{ in scale ten.}$$

Now $17=25$ in scale six, and $20=32$,, ,, $\Big\}$ $\therefore$ the required fraction $=\dfrac{25}{32}$.

EXAMPLES XLV.

Find the value of

1. 2341 + 1234 + 3412 + 4123 in the scale of five

2 437813 + 306218 + 534623 in the scale of nine.

3. $et06 + 795t + 856e + 3e7t$ in the duodenary scale.

4. 623005 − 341654 in the septenary scale.

5. (i) 31044 × 4302; (ii) $(3024)^2$ in the quinary scale.

6. Divide 22653 by 26, and 6435 by 222 in the scale of seven.

7. Find the square root of 222521 in the scale of six, and of 14320241 in the scale of five

8 Express the denary numbers 4532, 860 in the senary scale, and find their product in that scale

9. Express the septenary numbers 3625, 203116 in scale ten.

10. Transform 54321 from scale six to scale seven

11. Transform $112t3$ from the duodenary to the septenary scale.

12. Express the quinary number 30014 in powers of twelve

13. Express the denary fraction $\frac{5}{16}$ in the nonary scale.

14. Express the decimal 1375 as a radix-fraction (i) in the quaternary, (ii) in the octonary scale

15. Express the denary number 42 28 in the quinary scale

16. Transform 20213 from scale six to scale eight

17. Transform 20 73 from the nonary to the ternary scale

18. The radix-fraction $20\dot{2}$ is in the quinary scale; express it as a vulgar fraction (i) in the denary, (ii) in the quaternary scale

19. Express in the scale of eleven the greatest and least numbers that can be formed with 4 digits in the scale of seven

20 In what scale is a hundred denoted by 400? And in what scale is 647 the square of 25?

21. If 432, 565, 708 are in A P, find the radix of the scale

22 In what scale are the radix-fractions ·16, 20, 28 in G P?

23. Divide 264 734 by $3t$ 08 in the scale of twelve

24. Shew how to weigh 227 lbs using single weights of the series 1 lb, 2 lbs, 4 lbs, 8 lbs, 16 lbs,

25. Express the senary radix-fraction $31\dot{5}$ as a denary vulgar fraction.

26. Shew that in any scale greater than three 1 331 is a perfect cube.

27. N and N′ are two numbers expressed with the same digits but in different order Shew that N − N′ is divisible by $r - 1$

28. If N is a number in the scale of r, and D is the difference between the sums of the digits in the odd and even places, then N − D or N + D is a multiple of $r + 1$.

CHAPTER XLVI.

EASY INEQUALITIES

579 AN *inequality* is a statement that one expression is greater or less than another

Some easy cases of Inequalities have already been given in connection with Ratio and the Progressions [See Arts 416, 419, 488.]

For convenience we here repeat the necessary definitions

If $a-b$ is *positive*, a is said to be algebraically *greater* than b

If $a-b$ is *negative*, a is said to be algebraically *less* than b

The sign $>$ is used for the words "is *greater* than"

" $<$ " " "is *less* than"

Thus $4>-5$ because $4-(-5)$, or $4+5$ is positive,
and $-8<-3$ " $-8-(-3)$, or $-8+3$ is negative

In accordance with these definitions zero must be regarded as greater than any negative quantity

580 It will be found that inequalities sometimes reduce to equalities when the symbols involved have special values Accordingly the sign $\geqq$ is sometimes used as a equivalent for the words "is greater than or equal to" Similarly the sign $\leqq$ means "is less than or equal to"

Throughout this chapter we shall suppose that all the symbols denote real positive quantities unless the contrary is explicitly stated

581 If $a>b$, then if x is any positive quantity it is evident that

$$\text{(i)}\ a+x>b+x, \qquad \text{(ii)}\ a-x>b-x;$$
$$\text{(iii)}\ ax>bx, \qquad \text{(iv)}\ \frac{a}{x}>\frac{b}{x};$$

that is, *an inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same* positive *quantity*

582 If $$a-x>b,$$
by adding x to each side,
$$a>b+x;$$
which shews that *in an inequality any term may be transposed from one side to the other if its sign is changed*

583 If $a>b$, then evidently $b<a$,
that is, *if the sides of an inequality are transposed, the sign of inequality must be reversed*

584 If $a>b$, then $a-b$ is positive and $b-a$ is negative; that is, $\quad -a-(-b)$ is negative, and therefore $-a<-b$,
hence, *if the signs of all the terms of an inequality are changed, the sign of inequality must be reversed*

585 If $a>b$, then $-a<-b$, and therefore $-ax<-bx$,
that is, $$a(-x)<b(-x),$$
hence, *if the sides of an inequality are multiplied by the same* negative *quantity, the sign of inequality must be reversed*

586 *If* a *and* b *are any real positive quantities* $a^2+b^2\geq 2ab$

Since $(a-b)^2$ is always positive, or zero, $a^2-2ab+b^2\geq 0$,
$$a^2+b^2\geq 2ab$$

Thus, unless a and b are equal, $a^2+b^2>2ab$ Similarly, unless x and y are equal, $x+y>2\sqrt{xy}$.

A large number of inequalities depend upon these simple results.

EXAMPLE 1. *If* a, b, c *denote positive quantities, prove that*
$$\text{(i) } a^2+b^2+c^2\geq bc+ca+ab; \qquad \text{(ii) } a^3+b^3\geq a^2b+ab^2.$$

(i) We have $a^2+b^2\geq 2ab,\quad b^2+c^2\geq 2bc,\quad c^2+a^2\geq 2ca$

Adding these results, on dividing by 2, we have
$$a^2+b^2+c^2\geq bc+ca+ab$$

(ii) Since $a^2+b^2\geq 2ab$, we have $a^2-ab+b^2\geq ab$,
$$(a^2-ab+b^2)(a+b)\geq ab(a+b), \text{ that is } a^3+b^3\geq a^2b+ab^2.$$

EXAMPLE 2 *Shew that* $a^4+b^4>a^3b+ab^3$ *unless* $a=b$

The inequality holds if $a^4+b^4-a^3b-ab^3$ is positive

Now $\quad a^4+b^4-a^3b-ab^3=(a^3-b^3)(a-b)=(a-b)^2(a^2+ab+b^2)$,
and each of these factors is positive

EXAMPLE 3 *If* x *may have any real value, find which is the greater,* x^3+16x *or* $7x^2+10$

By the Remainder Theorem, $x^3+16x-(7x^2+10)$ has a factor $x-1$

Hence we find $\quad x^3-7x^2+16x-10=(x-1)(x^2-6x+10)$
$$=(x-1)\{(x-3)^2+1\}$$

The second factor is always positive; hence x^3+16x is greater or less than $7x^2+10$ according as x is greater or less than 1

EXAMPLE 4 *To find the maximum value of the product of two quantities whose sum is given*

Let a and b be the two quantities; then $4ab=(a+b)^2-(a-b)^2$

But since $a+b$ is constant, we see that the product ab will be greatest when $(a-b)^2$ is zero Thus the value of the product is greatest when the two quantities are equal.

EXAMPLES XLVI.

[*In the following examples the student should note each case in which an inequality reduces to an equality for special values of the symbols.*]

1. Prove that $(a+b)(b+c)(c+a) \geq 8abc$.

2. Prove that the sum of a real positive quantity and its reciprocal is never less than 2.

3. Prove that $(ab+cd)(ac+bd) \geq 4abcd$.

4. Shew that if $a>b$ and $x>y$, it does not necessarily follow that $ax>by$, if some of the symbols may denote negative quantities.

5. Prove the inequalities

$$\text{(i)}\ \frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}; \qquad \text{(ii)}\ m^2+\frac{1}{m^2}>m+\frac{1}{m}.$$

6. If $p^2+q^2=r^2+s^2=1$, prove that $pr+qs \leq 1$.

7. Prove that $(a+b+c)^2>3(bc+ca+ab)$.

8. Prove that $a^3-3b^3>3a^2b-5ab^2$, if $a>b$.

9. For what values of x, positive or negative, will x^4+x^3+2x be greater than 4?

10. Shew that $\dfrac{2x-1}{x^2+2}<\dfrac{1}{2}$, for real values of x.

11. If $a>b$, shew that $(\sqrt{a}+\sqrt{b})^2>4b$ and $<4a$.

Hence shew that the difference between $\frac{1}{2}(a+b)$ and $\sqrt{ab}$ is less than $\frac{1}{8}(a-b)^2/b$ and greater than $\frac{1}{8}(a-b)^2/a$.

12. If the product of two quantities is constant, shew that their sum is increased by increasing their difference.

13. Under what conditions is $a^3+b^3+c^3>3abc$?

14. Shew that $6abc \leq \Sigma bc(b+c)$.

15. Shew that $\dfrac{x+1}{x^2+3}$ lies between $-\dfrac{1}{6}$ and $\dfrac{1}{2}$, for real values of x.

16. If $a:b=c:d$, shew that $a+d>b+c$ provided that $a>b$ and $a>c$.

17. If a, b, c, d are unequal quantities, prove that

$$a^2+b^2+c^2+d^2>4\sqrt{abcd}.$$

18. If p, q, r, s are positive and arranged in order of magnitude, prove that if $q+r=p+s$, then $qr>ps$.

CHAPTER XLVII

Miscellaneous Equations

587 In previous chapters dealing with equations all the principal methods of solution have been explained and illustrated in the text We shall now give some further examples in most of which solution is effected by some special artifice.

Equations Involving One Unknown.

Example 1. *Solve the equation*

$$\frac{x-bc}{b+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}=a+b+c.$$

The equation may be written

$$\left(\frac{x-bc}{b+c}-a\right)+\left(\frac{x-ca}{c+a}-b\right)+\left(\frac{x-ab}{a+b}-c\right)=0,$$

or $$\frac{x-(bc+ca+ab)}{b+c}+\frac{x-(bc+ca+ab)}{c+a}+\frac{x-(bc+ca+ab)}{a+b}=0;$$

$$\{x-(bc+ca+ab)\}\left\{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right\}=0.$$

Since the second factor is not zero, we must have

$$x-(bc+ca+ab)=0, \quad \text{or} \quad x=bc+ca+ab$$

Example 2 *Solve the equation* $\left(\dfrac{2x+p+r}{2x+q+r}\right)^2=\dfrac{x+p}{x+q}$

The equation may be written

$$\left(1+\frac{p-q}{2x+q+r}\right)^2=1+\frac{p-q}{x+q};$$

$$\frac{2(p-q)}{2x+q+r}+\frac{(p-q)^2}{(2x+q+r)^2}=\frac{p-q}{x+q}.$$

Removing the factor $p-q$, and transposing, we have

$$\frac{p-q}{(2x+q+r)^2}=\frac{1}{x+q}-\frac{2}{2x+q+r}$$

$$=\frac{r-q}{(x+q)(2x+q+r)};$$

whence $$(p-q)(x+q)=(r-q)(2x+q+r),$$

or $$x\{p-q-2(r-q)\}=r^2-q^2-q(p-q);$$

that is, $$x(p+q-2r)=r^2-pq;$$

$$\therefore\ x=\frac{r^2-pq}{p+q-2r}.$$

EXAMPLE 3 *Solve the equation*

$$\frac{ax+b}{cx+b}+\frac{bx+a}{cx+a}=\frac{(a+b)(x-2)}{cx+a+b}$$

The right-hand side $=\dfrac{(ax+b)+(bx+a)+(a+b)}{cx+a+b}$,

$$\therefore\ (ax+b)\left\{\frac{1}{cx+b}-\frac{1}{cx+a+b}\right\}+(bx+a)\left\{\frac{1}{cx+a}-\frac{1}{cx+a+b}\right\}=\frac{a+b}{cx-a-b},$$

or
$$\frac{a(ax+b)}{(cx+b)(cx+a+b)}+\frac{b(bx+a)}{(cx+a)(cx+a+b)}=\frac{a+b}{cx+a+b}$$

By removing $cx+a+b$ from the denominators, and transposing, we have

$$\frac{a(ax+b)}{cx+b}-a+\frac{b(bx+a)}{cx+a}-b=0;$$

that is,
$$\frac{ax(a-c)}{cx+b}+\frac{bx(b-c)}{cx+a}=0;$$

$$\text{either}\quad x=0,\quad\text{or}\quad (a^2-ac)(cx+a)+(b^2-bc)(cx+b)=0$$

In the latter case we have

$$x(a^2c-ac^2+b^2c-bc^2)=-a^3+a^2c-b^3+b^2c,$$

or
$$x\{ac(a-c)+bc(b-c)\}=a^2(c-a)+b^2(c-b)$$

Thus
$$x=0,\quad\text{or}\quad \frac{a^2(c-a)+b^2(c-b)}{ac(a-c)+bc(b-c)}$$

EXAMPLE 4 *Solve the equation* $4^{2x+1}+16=65\cdot 4^x$

We have
$$4\cdot 4^{2x}-65\cdot 4^x+16=0$$

By writing y for 4^x, we obtain

$$4y^2-65y+16=0,\quad\text{or}\quad (4y-1)(y-16)=0,$$

whence
$$y=\tfrac{1}{4},\quad\text{or}\quad 16$$

Thus
$$4^x=\tfrac{1}{4}=4^{-1},\quad\text{or}\quad 4^x=4^2;$$
$$x=-1,\quad\text{or}\quad 2$$

EXAMPLE 5 *Solve the equation* $\dfrac{x^2}{3}+\dfrac{48}{x^2}=10\left(\dfrac{x}{3}-\dfrac{4}{x}\right)$.

Divide each side by 3, then we have

$$\frac{x^2}{9}+\frac{16}{x^2}=\frac{10}{3}\left(\frac{x}{3}-\frac{4}{x}\right)$$

Write y for $\dfrac{x}{3}-\dfrac{4}{x}$, then $\dfrac{x^2}{9}+\dfrac{16}{x^2}=y^2+\dfrac{8}{3}$

$$y^2+\frac{8}{3}=\frac{10}{3}y,\quad\text{whence}\quad y=\frac{4}{3}\quad\text{or}\quad 2$$

$$\therefore\ \frac{x}{3}-\frac{4}{x}=\frac{4}{3},\quad\text{or}\quad \frac{x}{3}-\frac{4}{x}=2$$

From these equations we obtain $x=6,\ -2,\ 3\pm\sqrt{21}$

EXAMPLE 6. *Solve* $\dfrac{\sqrt{x+48}+\sqrt{x}}{\sqrt{x+48}-\sqrt{x}}=\dfrac{\sqrt{x-4}+\sqrt{3}}{\sqrt{x-4}-\sqrt{3}}$

By Art 427 (*Componendo et dividendo*), we have

$$\frac{\sqrt{x+48}}{\sqrt{x}}=\frac{\sqrt{x-4}}{\sqrt{3}}$$

Squaring and simplifying, we have

$$3x+144=x^2-4x;$$

whence $\qquad x^2-7x-144=0, \quad \text{or} \quad (x-16)(x+9)=0;$

$$x=16, \quad \text{or} \quad -9$$

Both of these values will be found to satisfy the given equation

588 Before clearing an equation of radicals any common factor which contains the unknown should be removed by division

EXAMPLE *Solve the equation*

$$\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$$

We have $\sqrt{(x-3)(x+7)}+\sqrt{(x-3)(x+2)}=\sqrt{(x-3)(6x+13)}$.

The factor $x-3$ can now be removed from every term;

thus $\qquad \sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

This equation may now be solved in the usual way as explained on page 343 The solution gives $x=2$, or $-\frac{5}{3}$

On trial it will be found that the second of these values does not satisfy the given equation

By equating the factor $x-3$ to zero, we have $x=3$

Thus finally the required roots are 2 and 3

589. The artifice used in the following example is sometimes useful

EXAMPLE *Solve* $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$. (1)

Now it is evident that

$$3x^2-7x-30-(2x^2-7x-5)\equiv x^2-25 \qquad (2)$$

Divide each member of (2) by the corresponding member of (1),

thus $\qquad \sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5 \qquad (3)$

Now (2) is an *identity*; that is it is true for *all* values of x, whereas (1) is satisfied by the values we are seeking, hence also equation (3) is true for these values

From (1) and (3) we have, by subtraction,

$$\sqrt{2x^2-7x-5}=5;$$

whence we obtain $x=6$, or $-\frac{5}{2}$

Both of these values will be found to satisfy the given equation

590 Any equation which can be expressed in the form

$$ax^2+bx+c+p\sqrt{ax^2+bx+c}=q$$

can be solved by putting $y=\sqrt{ax^2+bx+c}$

The resulting equation $y^2+py=q$ will give two values of y, each of which will give two values of x We thus have four values of x ultimately Of these only those which make y positive are solutions of the original equation, the others satisfy the equation

$$ax^2+bx+c-p\sqrt{ax^2+bx+c}=q$$

Numerical examples of this type have already been given on page 344.

591. **Reciprocal Equations** When an equation has all its terms brought to one side and arranged in descending order, if the coefficients are the same when read from left to right or right to left, it is known as a **reciprocal equation**, and it is so called because it remains unaltered when x is replaced by its reciprocal $1/x$

EXAMPLE *Solve* $3x^4-16x^3+26x^2-16x+3=0$

Dividing throughout by x^2, and rearranging the terms, we have

$$3\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+26=0$$

Put $\quad x+\frac{1}{x}=y,\quad$ then $x^2+\frac{1}{x^2}=y^2-2,$

$$3(y^2-2)-16y+26=0,\quad \text{whence } y=2, \text{ or } \frac{10}{3}.$$

Thus we have $\quad x+\frac{1}{x}=2,\quad$ and $\quad x+\frac{1}{x}=\frac{10}{3}$

These equations give $x=1,\ 1,\ 3,\ \frac{1}{3}$.

EXAMPLES XLVII. a.

Solve the following equations

1. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$

2. $\dfrac{(x+a)(x+b)}{x+a+b}=\dfrac{(x+c)(x+d)}{x+c+d}$.

3. $\dfrac{x-a-1}{x-a-2}-\dfrac{x-a}{x-a-1}=\dfrac{x-b-1}{x-b-2}-\dfrac{x-b}{x-b-1}$.

4. $\dfrac{x+bc}{b-c}+\dfrac{x+ca}{a-c}+\dfrac{x-ab}{a+b}=a+b-c$

5. $\dfrac{x-2ab}{2a+b}+\dfrac{x+bc}{b-c}+\dfrac{x+2ac}{2a-c}=2a+b-c$.

Solve the equations:

6. $\dfrac{bc(ax-1)}{a(b+c)}+\dfrac{ca(bx-1)}{b(c+a)}+\dfrac{ab(cx-1)}{c(a+b)}=3.$

7. $\dfrac{x+p}{x-q}=\left(\dfrac{2x+p}{2x-q}\right)^2.$

8. $\dfrac{x+a}{x-b}=\left(\dfrac{2x+a+c}{2x-b+c}\right)^2.$

9. $\dfrac{b+c}{x+2a}+\dfrac{c+a}{x+2b}=\dfrac{a+b+2c}{x+a+b}$

10. $\dfrac{ax+b}{b-x}+\dfrac{bx+a}{a-x}=\dfrac{(a+b)(x+2)}{a+b-x}.$

11. $3^{2x+1}+9=28\cdot 3^x.$

12. $4^x-9\cdot 2^x+8=0$

13. $3\sqrt{x}-8=3x^{-\frac{1}{2}}.$

14. $6x^{\frac{3}{4}}=7x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$

15. $3^{2x+3}-55=28(3^x-2).$

16. $6(6^x+6^{-x})=37$

17. $x^2+\dfrac{4}{x^2}=15\left(\dfrac{x}{2}+\dfrac{1}{x}\right)-17\tfrac{1}{2}$

18. $\dfrac{x^2}{3}+\dfrac{3}{x^2}=5\left(\dfrac{x}{3}-\dfrac{1}{x}\right)$

19. $\dfrac{\sqrt{x}+\sqrt{x-15}-\sqrt{x-7}}{\sqrt{x}+\sqrt{x-15}+\sqrt{x-7}}=\dfrac{1}{4}$

20. $\dfrac{p\sqrt{a^2-x^2}+q(a-x)}{p\sqrt{a^2-x^2}-q(a-x)}=\dfrac{pb+qc}{pb-qc}.$

21. $\sqrt{x^2+14x+33}+\sqrt{x^2-6x-27}=10\sqrt{x+3}$

22. $\sqrt{x^2+4x-5}-\sqrt{x^2-12x+11}=\sqrt{x^2-17x+16}$

23. $\sqrt{2x^2+3x-2}-\sqrt{18x^2+5x-7}+\sqrt{8x^2-2x-1}=0.$

24. $\sqrt{x^2+6ax+8a^2}+\sqrt{x^2+3ax+2a^2}=2\sqrt{x^2-4a^2}.$

25. $\sqrt{4x^2-10x+19}-\sqrt{4x^2-10x+3}=2$

26. $\sqrt{2x^2-11x+69}+\sqrt{(2x-7)(x-2)}=11.$

27. $\sqrt{4x^2+8x-28}+\sqrt{3x^2+8x-24}=x+2$

28. $\sqrt{7x^2-11x+6}+\sqrt{6x^2-11x+15}=2(x+3)$

29. $\sqrt{3x-24}+\sqrt{x+7}=\sqrt{3x-14}+\sqrt{x-3}$

30. $8\sqrt{(3x+4)(x+2)}-3x^2-10x+97=0$

31. $4x^2+5x-2\sqrt{3x^2-5x+2}=x(15-2x)$

32. $6x^4-35x^3+62x^2-35x+6=0$

33. $2x^4-13x^3+24x^2-13x+2=0$

34. $12(x^4+1)+89x^2=56x(x^2+1)$

35. $6x^4+25x^3+12x^2-25x+6=0.$

36. $(x+9)(x-3)(x-7)(x+5)=385$

[*Multiply alternate factors together and form a quadratic in* x^2+2x.]

37. $(x+3a)(x-5a)(x^2-16a^2)=180a^4$

38. $(1-10x)\sqrt{1+2x}=(1+10x)\sqrt{1-2x}.$

39. $(x-4)^3+(x-5)^3=31\{(x-4)^2-(x-5)^2\}$

40 $4\{(x^2-16)^{\frac{3}{4}}+8\}=x^2+16(x^2-16)^{\frac{1}{4}}$ [*Write y^4 for x^2-16.*]

41. $(a+x)^{\frac{2}{3}}+4(a-x)^{\frac{2}{3}}=5(a^2-x^2)^{\frac{1}{3}}$

42. $\sqrt{x^2+ax-1}-\sqrt{x^2+bx-1}=\sqrt{a}-\sqrt{b}$

Equations in two or more Unknowns.

592 The principal methods of solving equations in two unknowns, when either or both of the equations is of higher degree than the first, have been given in Chapter XXVI Of this class we shall here only give a few additional examples

EXAMPLE 1 *Solve* $\qquad x^4+y^4=97, \qquad (1)$

$$x+y=5 \qquad (2)$$

First Method

From (2), we have $\qquad (x+y)^4=625,$

or $\qquad x^4+4x^3y+6x^2y^2+4xy^3+y^4=625 \qquad (3)$

Subtract (1) from (3), and divide the result by 2, then

$$2x^3y+3x^2y^2+2xy^3=264,$$

that is, $\qquad xy(2x^2+3xy+2y^2)=264,$

or $\qquad xy\{2(x+y)^2-xy\}=264$

Substituting for $x+y$ from (2), we have

$$x^2y^2-50xy+264=0, \quad \text{or} \quad (xy-6)(xy-44)=0$$

Hence we have the two pairs of equations

$$\left.\begin{aligned} x+y&=5, \\ xy&=6, \end{aligned}\right\} \qquad \left.\begin{aligned} x+y&=5, \\ xy&=44 \end{aligned}\right\}$$

From the first we obtain $x=2,\ y=3;\ \ x=3,\ y=2$

" second " $\quad x=\frac{1}{2}\{5\pm\sqrt{-151}\},\ \ y=\frac{1}{2}\{5\mp\sqrt{-151}\}.$

Second Method

Since $x+y$ is given, assume $x-y=z$; then

$$\left.\begin{aligned} x+y&=5, \\ x-y&=z, \end{aligned}\right\} \text{ whence } \ x=\tfrac{1}{2}(5+z),\ \ y=\tfrac{1}{2}(5-z). \qquad (4)$$

Substituting in (1), $\quad (5+z)^4+(5-z)^4=97\times 2^4$

Hence by expanding $(5+z)^4$ and $(5-z)^4$,

$$5^4+6\cdot 5^2z^2+z^4=97\times 2^3,$$

whence $\qquad z^4+150z^2-151=0, \quad \text{or} \quad (z^2-1)(z^2+151)=0$

Thus we have $z=\pm 1$, or $z=\pm\sqrt{-151}$, whence from equations (4) we obtain the same values of x and y as before.

EXAMPLE 2. *Solve* $\dfrac{x+y}{2x-y}+\dfrac{x+2y}{x-y}=8\frac{1}{4}$, ... (1)

$$2x^2+3y^2=270 \qquad . \qquad . \quad . \quad (2)$$

From (1) $4(x^2-y^2+2x^2+3xy-2y^2)=33(2x^2-3xy+y^2)$;

$$\therefore\ 18x^2-37xy+15y^2=0, \quad \text{or} \quad (2x-3y)(9x-5y)=0$$

Hence . $2x=3y$, ... (3)

or $9x=5y$. . .. . (4)

From (3), $\dfrac{x^2}{9}=\dfrac{y^2}{4}=\dfrac{2x^2+3y^2}{30}=9$, by equation (2).

$$\therefore\ x=\pm 9, \quad y=\pm 6.$$

From (4), $\dfrac{x^2}{25}=\dfrac{y^2}{81}=\dfrac{2x^2+3y^2}{293}=\dfrac{270}{293}$, by equation (2).

$$\therefore\ x=\pm\frac{15\sqrt{30}}{\sqrt{293}}, \quad y=\pm\frac{27\sqrt{30}}{\sqrt{293}}.$$

[*Examples* XLVII. b 1-13, *page* 536, *may be taken here*]

593 Equations of a higher degree than the first, involving more than two unknowns, can only be solved in particular cases. Such equations introduce no new principles, but they often afford scope for considerable skill and ingenuity.

EXAMPLE 1 *Solve the equations:*

$$x(ax+by+cz)=k, \quad y(ax+by+cz)=l, \quad z(ax+by+cz)=m.$$

Multiply these equations by a, b, c respectively, and add, then

$$(ax+by+cz)^2=ak+bl+cm;$$

$$\therefore\ ax+by+cz=\pm\sqrt{ak+bl+cm}$$

$$\therefore\ x=\pm\frac{k}{\sqrt{ak+bl+cm}}, \quad y=\pm\frac{l}{\sqrt{ak+bl+cm}}, \quad z=\pm\frac{m}{\sqrt{ak+bl+cm}}$$

EXAMPLE 2. *Solve the equations*

$$2x+5y-3z=0, \quad 3x-9y+z=0, \quad x^2-x+2yz-z=5$$

From the first two equations, by cross multiplication (Art 420),

$$\frac{x}{5-27}=\frac{y}{-9-2}=\frac{z}{-18-15};$$

whence $\dfrac{x}{2}=\dfrac{y}{1}=\dfrac{z}{3}=k$, suppose;

then $x=2k, \quad y=k, \quad z=3k.$

Substituting these values in the third equation, we get

$$4k^2-2k+6k^2-3k=5,$$

that is, $2k^2-k-1=0$, or $(2k+1)(k-1)=0$

From $k=-\frac{1}{2}$, we obtain $x=-1$, $y=-\frac{1}{2}$, $z=-\frac{3}{2}$.

" $k=1$, " $x=2$, $y=1$, $z=3$.

EXAMPLE 3 *Solve the equations*

$$(y+z)(z+x)=a^2, \quad (z+x)(x+y)=b^2, \quad (x+y)(y+z)=c^2.$$

By multiplying the three equations together, and taking the square root of the result, we obtain

$$(x+y)(y+z)(z+x)=\pm abc$$

Combining this with each of the given equations, we have

$$x+y=\pm\frac{bc}{a}, \quad y+z=\pm\frac{ca}{b}, \quad z+x=\pm\frac{ab}{c}.$$

From these equations we easily obtain

$$x=\pm\frac{a^2b^2+b^2c^2-c^2a^2}{2abc}, \quad y=\pm\frac{b^2c^2+c^2a^2-a^2b^2}{2abc}, \quad z=\pm\frac{c^2a^2+a^2b^2-b^2c^2}{2abc}.$$

EXAMPLE 4 *Solve the equations*

$$xy+2x+y=7, \quad yz+3y+2z=12, \quad zx+z+3x=15$$

By suitable additions to each side of these equations, they may be written

$$xy+(2x+y)+2=9, \quad yz+(3y+2z)+6=18, \quad zx+(z+3x)+3=18,$$

or $$(x+1)(y+2)=9, \quad (y+2)(z+3)=18, \quad (z+3)(x+1)=18.$$

Solving these equations, as in the last example, we obtain

$$x=2, \quad y=1, \quad z=3,$$

and $$x=-4, \quad y=-5, \quad z=-9$$

EXAMPLE 5 *Solve* (1) $x^2+y^2+z^2=84$, (2) $x+y+z=14$, (3) $xy=z^2$.

From (1) and (2), $(x+y+z)^2-(x^2+y^2+z^2)=112$,

$$xy+yz+zx=56$$

Hence from (3), $z^2+yz+zx=56$, or $z(x+y+z)=56$;

whence from (2), $z=4$

Equations (2) and (3) now become $x+y=10$, $xy=16$

From these equations we obtain $x=2$, $y=8$, or $x=8$, $y=2$.

Hence finally $x=2, \quad y=8, \quad z=4,$

and $x=8, \quad y=2, \quad z=4$

EXAMPLE 6 *Solve* $x^2-yz=1$, $y^2-zx=-5$, $z^2-xy=7$.

Multiply the equations by y, z, x respectively and add; then

$$7x+y-5z=0, \tag{1}$$

Multiply the equations by z, x, y respectively and add, then

$$-5x+7y+1=0 \tag{2}$$

From (1) and (2), $\dfrac{x}{1+35}=\dfrac{y}{25-7}=\dfrac{z}{49+5}$, by cross multiplication,

whence $\dfrac{x}{2}=\dfrac{y}{1}=\dfrac{z}{3}=k$, suppose

Substituting in one of the given equations we get $k=\pm1$.

$$\therefore x=\pm2, \quad y=\pm1, \quad z=\pm3$$

EXAMPLES XLVII. b.

Solve the equations:

1. $x^4+y^4=337,$
$x+y=7$

2. $x^4+y^4=272,$
$x-y=2$

3. $x^5+y^5=1023$
$x+y=3.$

4. $\dfrac{2x+y}{x-3y}-\dfrac{x-3y}{2x+y}=2\frac{2}{3},$
$5x+7y=19$

5. $\dfrac{2x-y}{x+y}+\dfrac{2y-x}{x+4y}=7,$
$x^2+y^2=29$

6. $3x^2+xy=2x+6,$
$y^2+3xy=2y-3.$

7. $x^2+xy+2x+y=11,$
$y^2+xy+2y+x=7$

8. $x^2-xy+x=35,$
$xy-y^2+y=15.$

9. $(x-y)^2=3-2x-2y,$
$y(x-y+1)=x(y-x+1).$

10. Find the rational roots of

(i) $(x+y)(x^3+y^3)=19,$
$x^2+y^2=13,$

(ii) $x-y=2,$
$(x^2+y^2)(x^3-y^3)=260.$

11. $x+6y+\dfrac{x}{y}=16,$
$3(x+y)+\dfrac{x}{y}=23.$

12. $xy+\dfrac{1}{xy}+\dfrac{x}{y}+\dfrac{y}{x}=13,$
$xy-\dfrac{1}{xy}-\dfrac{x}{y}+\dfrac{y}{x}=12.$

13. $(x+1)^2+(x+1)(y+2)+(y+2)^2=133,$
$(x+1)+\sqrt{(x+1)(y+2)}+(y+2)=19$

14. $2x(2x+y+3z)=34,$
$y(2x+y+3z)=102,$
$3z(2x+y+3z)=153$

15. $x(y+z-x)=39-2x^2,$
$y(x+z-y)=52-2y^2,$
$z(x+y-z)=78-2z^2.$

16. $3x-2y-3z=0,$
$x-10y+6z=0,$
$x^2+y^2-z^2=116.$

17. $4x-2y=7z,$
$y+z=x,$
$y^2+3z^2=4(2x+1)$

18. $(x-1)(y+5)=14,\quad (y+5)(z+8)=63,\quad (z+8)(x-1)=18$

19. $xy+x+y=29,\quad yz+y+z=23,\quad zx+z+x=19.$

20. $x^3y^2z=24,\quad xy^3z^2=18,\quad x^2yz^3=108$

21. $x^3y=2z,\quad y^3z=9x,\quad xyz=6$

22. $x+y+z=21,\quad x^2+y^2+z^2=189,\quad y^2=zx$

23. $x^2+y^2+z^2=133,\quad y+z-x=7,\quad yz=x^2$

24. $y+z-x=9,\quad x^2-y^2-z^2=15,\quad yz=3$

25. $x^2-(y-z)^2=a^2,\quad y^2-(z-x)^2=b^2,\quad z^2-(x-y)^2=c^2.$

26. $x^2-yz=64,\quad y^2-zx=88,\quad z^2-xy=4$

27. $x^2-y^2+z^2=6,\quad 2yz-zx+2xy=13,\quad x-y+z=2.$

Indeterminate Equations.

594 An equation such as $3x+17y=130$, in which two unknowns are connected by a single relation, is said to be **indeterminate.** For it is obvious that by giving any value we choose to x, we can obtain a corresponding value of y from the equation Thus in general the number of solutions of an indeterminate equation is unlimited. If, however, we are restricted to positive integral values of x and y, we may sometimes have a definite number of solutions

EXAMPLE 1 *A man is to spend £130 in buying sheep at £3 each, and cows at £17 each how many of each can he buy?*

Suppose he buys x sheep and y cows, then

$$3x+17y=130$$

Divide throughout by 3, the smaller of the two coefficients; then

$$x+5y+\frac{2y}{3}=43+\frac{1}{3};$$

$$\frac{2y-1}{3}=43-x-5y$$

Now since x and y must be integral, we have also

$$\frac{2y-1}{3}=\text{an integer}$$

Multiply by a number which will make the coefficient of y differ by unity from a multiple of the denominator, thus, multiplying by 2, we have

$$\frac{4y-2}{3}=\text{an integer};$$

that is, $$y+\frac{y-2}{3}=\text{an integer};$$

hence also $$\frac{y-2}{3}=\text{an integer}$$

$$=p, \text{ suppose};$$

$$y=3p+2 \qquad (1)$$

Substituting in the original equation we get $x=32-17p$ (2)

The required values of x and y are now found from (1) and (2) by giving integral values to p

From (1) it is evident that p cannot be negative, and from (2) we see that p cannot be >1

Thus we have $$\left.\begin{array}{l} p=0,\ 1, \\ x=32,\ 15, \\ y=2,\ 5 \end{array}\right\}$$

Thus the man may buy 32 sheep and 2 cows, or 15 sheep and 5 cows.

The solution may be verified graphically as follows:

The equation $3x+17y=130$ represents a straight line. The positive values of x and y which satisfy the equation are the coordinates of points on that portion of the line which lies in the first quadrant. If the graph is carefully drawn on a sufficiently large scale, it will be found that the only points in the first quadrant which have *integral* coordinates are (32, 2) and (15, 5).

This graphical illustration is left as an exercise for the student.

EXAMPLE 2. *Find the positive integral solutions of the equation*

$$13x-9y=151.$$

Divide by 9, the smaller coefficient; then

$$x+\frac{4x}{9}-y=16+\frac{7}{9};$$

$$\therefore \frac{4x-7}{9}=16-x+y$$

$$=\text{an integer.}$$

Multiply by 2; then

$$\frac{8x-14}{9}=\text{an integer};$$

that is,

$$x-1-\frac{x+5}{9}=\text{an integer};$$

$$\therefore \frac{x+5}{9}=\text{an integer}$$

$$=p, \text{ suppose};$$

$$x=9p-5.$$

By substituting in the original equation,

$$y=13p-24.$$

These two results furnish the *general solution* of the equation.

By giving to p any positive integral value greater than 1 we obtain positive integral values of x and y.

Thus we have

$$\left.\begin{array}{l} p=\ 2,\ \ 3,\ \ 4,\ \ 5, \\ x=13, 22, 31, 40, \\ y=\ 2, 15, 28, 41, \end{array}\right\}$$

the number of solutions being infinite.

As before the solution may be illustrated graphically. It is obvious that the graph of $13x-9y=151$ has a positive intercept on the axis of x and a negative intercept on the axis of y. Hence the graph will lie in the first, fourth, and third quadrants. Since that portion which lies in the first quadrant can be produced to an infinite distance, it follows that there is no limit to the number of points which may have positive integral coordinates.

EXAMPLE 3 *In how many ways may £6 be paid in half-crowns and shillings, using both kinds of coins or only one?*

Let x be the number of half-crowns, y the number of shillings; then

$$2\tfrac{1}{2}x+y=120$$

or

$$2x+\frac{x}{2}+y=120;$$

$$\frac{x}{2}=\text{some integer } p,$$

that is,

$$x=2p \qquad (1)$$

Also

$$y=120-\frac{5x}{2}$$

$$=120-5p \qquad (2)$$

Since the sum may be paid in half-crowns alone, or in shillings alone, a zero value for x or y is not excluded Hence from (1) and (2) we see that p may have the values 0, 1, 2, 3, 24

Thus there are 25 ways of paying £6, using no coins except half-crowns and shillings

EXAMPLE 4 *A man spent £4 1s in buying fowls at 2s 6d, ducks at 3s 6d, and pheasants at 4s The number of birds bought was 25, how many were there of each?*

Suppose there were x fowls, y ducks, and z pheasants,

then

$$2\tfrac{1}{2}x+3\tfrac{1}{2}y+4z=81;$$

or

$$5x+7y+8z=162 \qquad (1)$$

Also

$$x+y+z=25 \qquad (2)$$

Eliminating x, we have $2y+3z=37$

This equation may be solved as before, but we shall here give a different method of solution.

The equation is obviously satisfied by $y=8$, $z=7$

$$2\times8+3\times7=37.$$

By subtraction,

$$2(y-8)+3(z-7)=0;$$

$$\frac{y-8}{3}=\frac{7-z}{2};$$

that is, $y-8$ is the same multiple of 3 that $7-z$ is of 2.

we may put $y-8=3p$, $7-z=2p$, where p is an integer;

that is,

$$y=3p+8, \qquad z=7-2p$$

Here p may have the values 0, 1, 2, 3

$$\therefore\ y=\ 8,\ 11,\ 14,\ 17;$$

$$z=\ 7,\ 5,\ 3,\ 1;$$

and from (2),

$$x=10,\ 9,\ 8,\ 7.$$

EXAMPLES XLVII. c.

Solve in positive integers

1. $5x+3y=41$ 2. $7x+4y=85$ 3. $2x+5y=36$

4. $7x+11y=58.$ 5. $12x+65y=640$ 6. $11x+13y=390.$

Solve the following equations in positive integers, and verify the solutions graphically:

7. $9x+4y=35$ 8. $3x+5y=56$ 9. $4x+11y=70$

Find the general solution in positive integers, and the least values of x and y which satisfy the equations:

10. $6x-13y=8$ 11. $5y-7x=29.$ 12. $8x-21y=33$

13. $8x-7y=31$ 14. $7x-11y=34$ 15. $10x-13y=46$

16. A man spends £5 10s in buying two kinds of books at 3s 6d and 6s each respectively: how many of each kind does he buy?

17. In how many ways can £3 2s 6d be paid in shillings and half-crowns, including zero solutions?

18. Divide 152 into two parts so that one may be a multiple of 7 and the other of 12

19. Find two fractions, having 7 and 11 for their denominators, such that their sum is $1\frac{34}{77}$.

20. What is the simplest way for a person who has only florins to pay 13s 6d to another who has only half-crowns? In how many ways can the payment be made?

21. A dealer in furniture has £183 to spend in buying tables and sofas costing £4 12s and £5 each respectively. How many of each can he buy?

22. Divide 112 into two parts one of which when divided by 3 leaves remainder 2, and the other divided by 8 leaves remainder 7

Solve the following pairs of simultaneous equations in positive integers:

23. $2x+5y+z=21,$
$x-y+2z=11$

24. $11x-3(y+z)=17,$
$4x-6y+3z=25$

25. A farmer buys 36 animals consisting of rams at £4, pigs at £2, and oxen at £17: if he spends £214, how many of each does he buy?

26. Given that $x=h$, $y=k$ is one solution of the equation $ax+by=c$, shew that the general solution is of the form

$$x=h+bp, \qquad y=k-ap,$$

where p is an integer.

MISCELLANEOUS EXAMPLES X.

[*The following Examples are arranged in three sets I may be taken after* Chap XLI , II *after* Chap XLIII , III *after* Chap XLVII]

I. (*After* Chap XLI)

1. Find the divisor when $(4a^2+7ab+5b^2)^2$ is the dividend, $8(a+2b)^2$ the quotient, and $b^2(9a+11b)^2$ the remainder

2 Resolve $4a^2(x^3+18ab^3)-(32a^5+9b^3x^3)$ into four factors

3 Prove that $(y-z)^3+(x-y)^3+3(x-y)(x-z)(y-z)=(x-z)^3$

4. A man has a stable containing 10 stalls, in how many ways could he stable 5 horses?

5 Solve (i) $(x^2-5x+2)^2=x^2-5x+22$, (ii) $x-15\frac{3}{4}+\dfrac{5}{x-15\frac{3}{4}}=6$

6 If α, β are the roots of $x^2+px+q=0$, shew that p, q are the roots of the equation

$$x^2+(\alpha+\beta-\alpha\beta)x-\alpha\beta(\alpha+\beta)=0$$

7. Simplify $\log\frac{133}{65}+2\log\frac{13}{7}-\log\frac{143}{90}+\log\frac{77}{171}$

8 Find the coefficient of x^{14} in the expansion of $(2x^2-3x)^{10}$

---

9 Find the factors of (i) x^4+2x^2+9, (ii) $9(a+b)^3-4(a+b)$

10 If $x-\dfrac{1}{x}=y$, prove that $x^5-\dfrac{1}{x^5}=5y+5y^3+y^5$, and find a corresponding formula for $x^3-\dfrac{1}{x^3}$

11 Solve the equations

$$\text{(i)}\ \frac{x+a}{x+b}=\frac{2x-a+b}{2x+a-b}, \qquad \text{(ii)}\ x-cy=cx-y=c$$

12 Write down the product of $(1+a)(1+b)$, and thence that of 1 01 and 1 02 If the last term is neglected, what is the resulting error per cent?

13 If x is the harmonic mean between a and b, shew that it is also the harmonic between $\dfrac{1}{x-a}$ and $\dfrac{1}{x-b}$

14. How many numbers greater than a million can be formed with the digits 2, 5, 0, 5, 1, 2, 5?

15 In an action between two battleships A and B, A fired 3 times as many shells as B The total number of misses was 7 times the total number of hits The number of B's misses was 357, but B's hits exceeded A's hits by 66 What was the number of shells fired and the number of hits made by each?

16 Find the coefficient of x^r in the expansion of $\dfrac{3-x-2x^2}{(1-x)^2}$

---

17. Find A and B so that the equation

$$x^4+x^3+x^2+x+1=(x^2+Ax+1)(x^2+Bx+1)$$

may be an identity

18. If $\dfrac{y+z-x}{a}=\dfrac{z+x-y}{b}=\dfrac{x+y-z}{c}$, prove that

$$\frac{a+b+c}{x+y+z}=\frac{ay+bz+cx}{x^2+y^2+z^2}.$$

19. Prove that x^2-3x+2 is a common factor of

$$x^3-7x+6,\quad 3x^3-7x^2+4,\quad\text{and}\quad x^4-3x^3+6x-4.$$

For what values of x do these three expressions simultaneously vanish?

20. Draw the three graphs of

$$y=x,\quad y=3x,\quad\text{and}\quad y=(2x-5)(x+5)$$

between the values $x=-1$ and $x=+1$ Find by trial with the graphs, or otherwise, the two values of x at which the slopes of the third graph are equal to those of the other two respectively

21. Find $\sqrt{14}$ to three decimal places by the Binomial Theorem, and check the result by logarithms

22. Solve the simultaneous equations

$$x+y+z=1,\quad ax+by+cz=0,\quad a^2x+b^2y+c^2z=0.$$

23. Prove by Mathematical Induction that

$$1\cdot 4+2\cdot 5+3\cdot 6+\ldots\text{ to } n \text{ terms}=\tfrac{1}{3}n(n+1)(n+5)$$

24. A man receives a pension starting with £100 the first year, but each year he receives 90% of what he received the previous year. Find, to the nearest penny, the total amount he receives in the first 6 years; find also the greatest amount he could possibly receive, even if he were to live for ever

---

25. If $(a+b+c)x=(-a+b+c)y=(a-b+c)z=(a+b-c)w$, shew that

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}$$

26. Find x and y from the equations $\dfrac{1}{2}(by-ax)=\dfrac{a^2x+b^2y}{b-a}=ab$

27. If a, b are the roots of the equation $x^2-10x+17=0$, calculate the values of

(i) $(1-a)(1-b)$; (ii) $(1+a-a^2)(1+b-b^2)$.

28 In a certain town eggs are being sold at $2x$ pence a dozen, and in another town they are sold at x eggs for a shilling By buying six dozen eggs in the latter and selling them in the former town a profit of 1s. is made, find the buying and selling prices of the six dozen eggs.

29. A number of squares are described whose sides are in G P Prove that the areas of the squares are also in G P The side of the $2m^{th}$ square is a feet and the side of the $2n^{th}$ square is b feet, find the area of the $(m+n)^{th}$ square

30 At an election there are 4 candidates and 3 members to be elected, and an elector may vote for any number of candidates not greater than the number to be elected In how many ways may an elector vote?

31 If $a+b+c=0$, prove that

$$\frac{a^2+b^2+c^2}{a^3+b^3+c^3}+\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0$$

32 Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$, and $\log 7 = \cdot 845098$, find the logarithms of 0 005, 6 3, and $\left(\frac{49}{216}\right)^{\frac{1}{3}}$

II (*After* Chap XLIII)

33 The manufacturer's list price of an article exceeds the cost of making it by p per cent, and it is sold to a retailer at a discount of q per cent What is the manufacturer's percentage profit?

34. If $xx_1=b^2$, $y+y_1=2a$, and $xy_1=x_1y$, prove that

$$\frac{1}{x^2}=\frac{a}{b^2}\left(\frac{2}{y}-\frac{1}{a}\right)$$

35. If $x^2+3x+4=y$, find what value of y will give equal roots for x. Illustrate graphically

36. If $t+\frac{1}{t}=x$, prove that

$$t^8-\frac{1}{t^8}=\left(t-\frac{1}{t}\right)(x^7-6x^5+10x^3-4x)$$

37 A man rows down a river from a place A to a place B and back again from B to A without stopping in 2 hrs 36 min If the speed of the current is $1\frac{3}{4}$ miles per hour, and the distance from A to B is 3 miles, find the speed of the man in still water, and the times of his two journeys

38 Find by logarithms the number of integral digits in $(7\cdot 2)^{16}$, and the number of digits in 3^{45}

39 How many different arrangements, beginning with r and ending with n, can be made from the letters of the word *rotation*?

40. Find the general term of $\frac{1+5x}{1-2x-3x^2}$ when expanded in ascending powers of x

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41. If $\frac{x}{a+p}+\frac{y}{b+p}=1=\frac{x}{a+q}+\frac{y}{b+q}$, shew that $x=\frac{(a+p)(a+q)}{a-b}$, and find y

42. If $a^3+b^3+c^3=3abc$, prove that either

$$a+b+c=0, \quad \text{or} \quad a=b=c.$$

43. If α, β are the roots of the equation $x^2+mx=2-m^2$, shew that

$$\alpha^3-\beta^3=2(\alpha-\beta)$$

44. Find the square root of

(i) $27-7\sqrt{5}$; (ii) $a^2+x^2+\sqrt{a^4+a^2x^2+x^4}$

45. Justify the following graphical construction for finding approximately 1 414 of any number up to 10 Join the origin to a point P whose coordinates are 10 and 14 14 (or 5 and 7 07), taking 1 inch as unit; then the ordinate of any point on OP is 1 414 times the corresponding abscissa. Read off from the diagram as correctly as possible to two places of decimals, $1\,414\times 2$, $1{\cdot}414\times 3{\cdot}5$, $1\,414\times 8\,6$, $\frac{1}{1\,414}\times 7\,8$

46. A and B, starting from the same place, make a journey of 56 miles; A starts 3 hours and 20 minutes after B, but travels 5 miles an hour faster If they arrive at the same time, find the pace of each

47. Shew that the number of men required to form a hollow square containing a men in the front rows and b men deep is $4b(a-b)$ Hence find the values of a and b for all the possible ways of arranging 1000 men in a hollow square.

48 Find correct to three decimal places the tenth root of the sum of the infinite series

$$1+1+\frac{1}{1.2}+\frac{1}{1\,2\,3}+\frac{1}{1\,2\,3\,4}+\ldots$$

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49. Resolve $2(a^6+b^6)-ab(a^2+b^2)(2ab-3a^2+3b^2)$ into five simple factors

50. Divide $1-x+x^2-x^3+\ldots-x^{2p+1}$ by $1+x^2+x^4+\ldots+x^{2p}$

51 The price of coffee being raised a pence per pound, b pounds fewer can be purchased for $2ab$ pence How much per cent. is the increase of price?

52. Solve the equations

$$\text{(i)} \quad 4\sqrt{\frac{x}{x+2}}-3\sqrt{1+\frac{2}{x}}=11,$$

$$\text{(ii)} \quad \sqrt{5-2x}+\sqrt{15+3x}=\sqrt{26-5x}$$

53. A man can row at the rate of a miles an hour in still water He rows a distance of x miles down a river, which flows at the rate of b miles an hour, and back again Find how long he will take, and shew that the time taken is longer than that which he would require to row $2x$ miles in still water

54. If $2n+1$ quantities are in A P shew that the $(n-r+1)^{\text{th}}$, the $(n+1)^{\text{th}}$, and the $(n+r+1)^{\text{th}}$ terms are also in A P

55 Solve the following problem graphically

X and Y are two towns 30 miles apart A cyclist A leaves X at 2 p m and rides towards Y at the rate of $12\frac{1}{2}$ miles an hour, a second cyclist B leaves Y at 2 5 p m and rides towards X at the rate of 13 miles an hour; a third cyclist C leaves X at 2 15 p m and rides towards Y at the rate of 17 miles an hour Shew that C will overtake A before A meets B, and find to the nearest half-mile how far B will be from Y when he meets C

56. Shew that $\log_e\sqrt{\dfrac{x}{x-1}}$ may be expanded in the form

$$\frac{1}{2x-1}+\frac{1}{3(2x-1)^3}+\frac{1}{5(2x-1)^5}+\dots$$

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57. If $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, prove that $(x+y+z)^3=27xyz$

58 If $x=(b-c)(a-d)$, $y=(c-a)(b-d)$, $z=(a-b)(c-d)$, find the value of $x^3+y^3+z^3-3xyz$

59 If $x\sqrt{a^2-y^2}+y\sqrt{a^2-x^2}=a^2$, then $x^2+y^2=a^2$

60 Divide 1 by $(1-x)^2$ so as to obtain a quotient in ascending powers of x What is the remainder after n steps of the division have been performed? Deduce the sum of $1+2x+3x^2+4x^3+\dots$ to n terms

61 If x may have any real value, find the least value of $\dfrac{x^2-4}{x^2+4x+4}$

62 Find the coefficient of x^3 in the expansion of $(1-x)^n(1-x^2)^{2n}$

63 A sells his motor car, which cost him £378, to B, who in turn sells it to C for £512 Given that A and B each make the same profit per cent on their outlay find to the nearest shilling the price for which A sells the car

64 Draw a graph of $y=\dfrac{6x}{1+x^2}$ from $x=0$ to $x=9$, paying special attention to the shape of the curve between $x=0$ and $x=2$

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65 If a, b, c are any three numbers whose sum is zero, prove that the square of the sum of their products two at a time is equal to the sum of the squares of these products

66 Express $(x+2)(x+3)(x+4)(x+5)-15$ as the product of two quadratic factors

67 Prove that $(x+y+z)^3-(x^3+y^3+z^3)\equiv 3(y+z)(z+x)(x+y)$

Thence, or otherwise, prove that

$$a(x^3+y^3+z^3)+b(x+y)(y+z)(z+x)+cxyz$$

is divisible by $x+y+z$ if $3a-b+c=0$

68. If $a : b = c : d$, prove that

$$\text{(i)}\ \left(\frac{1}{a}+\frac{1}{d}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{(a-b)(a-c)}{abc};$$

$$\text{(ii)}\ 4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2.$$

69. Solve the following pairs of equations:

$$\text{(i)}\ xy+x+y=11, \qquad \text{(ii)}\ x^4-x^2y^2+y^4=117,$$
$$x^2y+xy^2=30, \qquad x^2+xy\sqrt{3}+y^2=39$$

70. Find the 13^{th} term in the expansion of $(2^3+2^6x)^{\frac{11}{2}}$.

71. Express $\dfrac{5-x}{(1-x)(1+x^2)}$ in partial fractions.

72. Find the sum of the infinite series $x+\dfrac{x^3}{\lfloor 3}+\dfrac{x^5}{\lfloor 5}+ \ldots$

III (*After* Chap XLVII)

73. To complete a piece of work A takes m times as long as B and C together, B takes n times as long as C and A together, and C takes p times as long as A and B together Find the relation between m, n, and p

74. Solve the equation $\dfrac{a}{x-b}+\dfrac{b}{x-a}=\dfrac{2p}{x-p}$ where $2p=a+b$

75. The perimeter of a rectangular table is 200 inches; its area is halved when a strip 6 inches wide is cut off all round, find the dimensions of the original table to an accuracy of one-tenth of an inch

76. Form the quadratic equation whose roots are the squares of those of $ax^2+2bx+c=0$, and shew that the equation can be put in the form $(ax+c)^2=4b^2x$. Can you give a reason for this result?

77 Plot on as large a scale as you can the graph of $y=(1\cdot5)^x$ for values of x between 0 and 6, using the Tables when necessary

From the curve verify that $(1\cdot5)^{\frac{3}{2}}(1\cdot5)^{\frac{10}{3}}=(1\cdot5)^{\frac{29}{6}}$

78. Express $\dfrac{3x^2-11}{(x-2)^3}$ in partial fractions.

79. Find the value of $\dfrac{2\pi l(t_2-t_1)}{\log r_2-\log r_1}$, given that

$\pi=3\cdot142$, $l=0\cdot74$, $t_1=69\cdot4$, $t_2=82\cdot3$, $r_1=1\cdot25$, $r_2=1\cdot55$.

80. If
$$\frac{p}{bc-a^2}=\frac{q}{ca-b^2}=\frac{r}{ab-c^2},$$
prove that
$$\frac{a}{qr-p^2}=\frac{b}{rp-q^2}=\frac{c}{pq-r^2}.$$

81 Express $\sqrt{2x^2+xy-6y^2}\ \sqrt{3x^2+5xy-2y^2}\ \sqrt{6x^2-11xy+3y^2}$ in its simplest form

82 Simplify $\dfrac{(p^2+q^2)(x+y)^2+2(px-qy)(qx-py)}{(p^2-q^2)(x^2+y^2)}$.

83 If $9x^4-12x^3y+Px^2y^2+4xy^3+y^4$ is a perfect square, find P

84. Find x and y from the equations

$$x^2-2xy+y^2+2x+2y-3=0=y(x-y+1)+x(x-y-1)$$

85 A, B, C, D are four stations on a railway, the distances AB, BC, CD being 10 miles, 10 miles, and 8 miles respectively The following is an extract from a time-table

Up Train	*Down Train*
A, dep , 7 57 a m	D, dep , 8 28 a m
B, dep , 8 18 a m	C, —
C, dep , 8 40 a m	B, —
D, arr , 8 55 a m	A, arr , 9 10 a m

Draw graphs to shew the positions of the trains at any intermediate time, assuming that each runs at a uniform speed between the stations, and that the up train stops 3 minutes at each of the stations B, C When and where do the trains pass each other? Shew that the down train passes B just as the up train reaches D

86 A man borrows £20 from a money lender and he has to repay £24 in monthly instalments of £2, the first to be paid at the end of the first month Reckoning simple interest at the rate of r per cent per annum, find the sum to which £20 amounts in a year, and shew that the sums repaid, together with interest on repayments, amount to £$\left(24+\dfrac{11r}{100}\right)$ He imagines that he is paying 20 % interest, determine the actual rate

87. Shew that 1030301 is a complete cube in any scale whose radix is greater than 3

88 If $\quad y^2+yz+z^2=a,\quad z^2+zx+x^2=b,\quad x^2+xy+y^2=c,$

shew that $\qquad 3x=t+\dfrac{b+c-2a}{t},\qquad a+b+c=t^2+\dfrac{p}{t^2},$

where $\qquad t=x+y+z,$ and $p=a^2+b^2+c^2-bc-ca-ab$

Solve the equations for x, y, z when $a=3$, $b=13$, $c=7$

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89 If $x+\dfrac{1}{y}=1$, and $y+\dfrac{1}{z}=1$, shew that $z+\dfrac{1}{x}=1$

90. If $a+b+c=0$, prove that

$$\frac{a^4}{b^3+c^3-3abc}+\frac{b^4}{c^3+a^3-3abc}+\frac{c^4}{a^3+b^3-3abc}=0.$$

91. Find what values, if any, of x and y satisfy all three of the equations

$$2x+3y=5, \quad y=3x+1, \quad \frac{x}{2}+\frac{y}{8}=1$$

Illustrate by drawing graphs of the equations

92. Simplify the following expressions

(i) $2(2+\sqrt{3})(\sqrt{6}-\sqrt{2})\sqrt{2-\sqrt{3}}$,

(ii) $x^3+y^3+z^3+3(x+y+z)(yz+zx+xy)-(x+y+z)^3$.

93. Solve the equation

$$\frac{2x-11}{x-2}+\frac{x+4}{x-3}=\frac{x-5}{x+2}+\frac{2x+9}{x+1}.$$

94. A party of four people is to be chosen from nine, among whom are A and his wife and B and his wife A, if invited at all, must be invited with his wife, similarly B and his wife must be invited together, if at all In how many ways can the party be chosen?

95 If a farthing is put out at compound interest for 1000 years at 5 %, how many digits will be required to express the amount in pounds?

96 In a certain machine P kilograms is the effort required to move a load of W kilograms The following values were obtained experimentally:

P=	10 6,	12 2,	15 2,	18 4,	21 6,	24 6,	27 8,
W=	5,	10,	20,	30,	40,	50,	60

Plot these values, and assuming the relation between P and W to be of the form P$=a$W$+b$, find the values of a and b

---

97. If $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$, prove that

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=\frac{3xyz}{abc}$$

98. A man receives $\frac{x}{y}$ of 10s and afterwards $\frac{y}{x}$ of 10s He then gives away a sovereign, shew that he cannot lose by the transaction

99. Solve the equation

$$\sqrt{x+a-c}+\sqrt{x+b-c}=\sqrt{c-a}+\sqrt{c-b}$$

100. Eliminate x and y from the equations:

$$x+y=a, \quad x^2+y^2=b^2, \quad x^3+y^3=c^3$$

101. An express leaving P at 3 p m reaches Q at 6 p m, a slow train leaving Q at 1 30 p m arrives at P at 6 p m, if both trains are supposed to travel at a uniform speed, find graphically the time when they will meet Shew also that the time does not depend upon the distance between P and Q

102 Prove that the sum of n terms of the series

$$1,\quad 1+r,\quad 1+r+r^2,\quad 1+r+r^2+r^3,$$

is
$$\frac{n-(n+1)r+r^{n+1}}{(1-r)^2}.$$

103 On a bookstall there are 2 copies of one work, 3 of another, and 4 of a third, in how many ways can a purchaser make a selection by taking one or more from the 9 volumes?

104. The value of P has to be found from the formula $P=k\frac{l}{t^2r^3}$, where k is a constant, and l, t, r are found by experiment If there is an error of 0 4% too much in the value of l, 1 5% too little in the value of t, and 0 2% too much in the value of r, find the percentage error in the value of P

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105 Using Detached Coefficients, find the first four terms of

$$(1+2x-4x^2+x^3)(2-x^2+3x^3+x^5)$$

106 If $xy^{p-1}=a$, $xy^{q-1}=b$, $xy^{r-1}=c$, prove that

$$(q-r)\log a+(r-p)\log b+(p-q)\log c=0$$

107 If $(a+b+c+d)(bc+ca+ab)=abc+abd+acd+bcd$,

shew that
$$(b+c)(c+a)(a+b)=0$$

108 Find the square root of 223141 in the scale of five

109 Rationalize the equation

$$(y+z-x)^{\frac{1}{3}}+(z+x-y)^{\frac{1}{3}}+(x+y-z)^{\frac{1}{3}}=0$$

From the resulting equation, shew that

$$(x+y+z)^4-27(x^2+y^2+z^2)^2+54(x^4+y^4+z^4)=0$$

110 In how many ways can 5 men take their places in an empty railway carriage with 8 seats, if one of them must always have a corner seat, and another must travel facing the engine?

111 If P and Q vary respectively as $y^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ when z is constant, and as $z^{\frac{1}{2}}$ and $z^{\frac{1}{3}}$ when y is constant, and if $x=P+Q$, find the equation between x, y, z, it being known that when $y=z=64$, $x=12$; and that when $y=4z=16$, $x=2$

112 The sum of n terms of an A P is s; shew that the sum of their squares is

$$\frac{s^2}{n}+\frac{1}{12}n(n^2-1)d^2,$$

where d is the common difference

113. Given that $x^4+4x^3+px^2+qx+9$ is the square of x^2+ax+b, find all the possible values of a, b, p, and q

114. If $a+b+c=0$, prove that $\Sigma a\left(\frac{b^3-c^3}{b-c}\right)=0$

115. Sum to n terms the series whose n^{th} term is $2^n+n(n-1)$

116. If ${}^nC_r$ is the number of combinations of n things r at a time, prove by general reasoning that

$$^{n+2}C_{r+1}={}^nC_{r+1}+{}^nC_{r-1}+2\times{}^nC_r.$$

117. Solve the equations

$$\text{(i)}\ \frac{1}{x+a}-\frac{1}{x+b}-\frac{1}{x+a+b}+\frac{1}{x+2b}=0;$$

$$\text{(ii)}\ \frac{x-2a}{b+c-a}+\frac{x-2b}{c+a-b}+\frac{x}{a+b+c}=3$$

118. Shew that the sum of all the products in a multiplication table going up to n times n is $\frac{n^2(n+1)^2}{4}$

119. Shew that the coefficient of x^{2n+1} in the expansion of

$$(1+x+x^2+\quad+x^{n-1})^3$$

is $\frac{(n-2)(n-3)}{2}$ Verify this result when $n=3$

120 To provide for his two infant sons, a man left by his will two sums of money as separate investments at different rates of interest, on the condition that the principal sums with simple interest were to be paid over to his sons when the amounts were the same After 5 years the first sum amounted to £451, and after 15 years to £533 After 10 years the second sum amounted to £432, and after 20 years to £544 Draw graphs from which the amounts may be read off for any year, and find after how many years the sons were entitled to receive their legacies

Also determine from the graphs what the original sums were at the father's death.

ANSWERS.

I a. Page 3. 1 23 2 14. 3 34 4. 17
5 16 6 43 7 7, 6

I. b Page 6. 2 900, 400 3 The first by 205 5 $5p$, 125
8 $\frac{x}{6}$, 7 9 $\frac{54}{c}$, 9 10 18, 210
12 (i) £7 8*s*, (ii) $(25p+8q)$ shillings 13 $mp+nq$, 125
14 9 15 10 16 7 17 12 18 50 19 2
20 2 21 30 22 80 23 6 24. 1000 25 100
26 3 27 6 28 21 29 24 30 12 31 12
32 60 33 7 34 1 35 1 36 $\frac{1}{4}$ 37 1
38 2 39 $\frac{19}{20}$ 40 84

I c Page 8 2 6, 8 3 12, 81 4. 21 5 343
6 16 7 256 sq ft 8 14 9 49 10 15
11 125 12 4 13 1 14. 7 15 40
16 243 17 64 18 4802 19 196 20 75
21 250 22 5 23 1 24. $\frac{1}{2}$ 25 $\frac{1}{25}$
26 1 27 3 28 $\frac{9}{32}$ 29 $\frac{1}{5}$ 30 $\frac{3}{2}$
31 $\frac{1}{8}$ 32 $\frac{1}{5}$ 33 8 34. 3 35 3
36 1 37 $\frac{1}{12}$ 38 $\frac{1}{81}$ 39 $-\frac{5}{12}$ 40 3
41. $\frac{16}{9}$ 42 $\frac{1}{125}$

I d Page 10. 1 75 2 45 3 5000 4. 1250
5 1000 6 100 7 0 8 0 9 0
10 0 11. 600 12 3000 13 21600 14. 0
15 36000 16 40 17 12 18 0 19 0
20 48 21 192 22 288 23 0 24 4
25 $\frac{1}{2}$ 26 0 27 $\frac{1}{4}$ 28 78 29 0 30 $\frac{1}{4}$

I e Page 12 1 12 2 4 3 3 4. 27 5 0
6 31 7 1 8 5 9 84 10 0 11. 25 12 0
13 21 14 81 15 18 16 0 17 0 18 46
19 13 20 8 21 $3\frac{1}{2}$ 22 49 23 $\frac{1}{4}$ 24 $\frac{7}{16}$
25 2, 10, 32, 62, 100 26 8, 4, 2, 2, 4. 27 20 28 8, 8, 10, 14.
29 16, 28, 58, 106 30 The first by 2 33 The first

II. a. Page 15. 5 100 8 +4, −4.

15 9 3−7.2=13, 5 3−11 2=−7

16 A, C, B, with +16, +2, −5 points

17 A, +240 yds E, −160 yds N, B, −252 yds E, +168 yds N

18 A, $£p-£a+£b$; B, $£q-£b+£a$

II. b. Page 18 3 $20a$ 4 $27x$ 5 $35p$ 6 $1111d$

7 $-24y$ 8 $-40m$ 9 $-47z$ 10 $-54c$

11 $-6b$ 12 $-10x$ 13 $-ab$ 14 0

15 $-11cd$ 16 $6pq$ 17 $-10c^2$ 18 $-b^3$

19 a^2b^2 20 0 21 $-y^3z^2$ 22 $4c^4d$

23. $-13abcd$ 24. $-26xyz$ 26 10 27 1. 28 2

II. c Page 20. 1 12, 84 2 14, 56 3 80, 5 4. 60, 3

5 4, 48 6 11, 99 7 8, 16, 32 8 12, 12, 60 9 3

10 A, £10, B, £12; C, £2 11 A, 13*s*, B, £2 12*s*

12 A and C, 21*s*, B, 3*s* 13 £10 14. £16

15 Man, 24*s*, woman, 18*s*; boy, 6*s* 16 Man, 24*s*, woman, 18*s*

17 Man, 48 yrs, daughter, 12 yrs

18 Man, 60 yrs, son, 30 yrs; grandson, 6 yrs

19 Man, 45 yrs, daughter, 15 yrs, son, 5 yrs

II d. Page 23 1 32 2 11 3 10 4 x

5 $6x$. 6 $-7a$ 7 $5p$ 8 $3y$ 9 $8xy$

10 0 11 $8y^2$ 12 $19y$ 13 $7p^2$ 14. $-11x^2$

15 $-9c^3$ 16 0 17 $7x^4$

II e. Page 25. 1. $4a+3b+2c$ 2 $7x+5y$

3 $9p+3q-5r$ 4. $2c$. 5 $3y-2z$ 6 $l+4m+n$

7 $13x-11y-8z$ 8 $2c+3e$ 9 $7a-4b+6c$

10 $5l-4m+n$ 11 $-3a+5c$ 12 $b+8c$

13 $7ab-bc$ 14 $-3xy+3zx$ 15 $3a-2b+2c+3d$

16 $5x+3z-l$ 17 $2pq+rp+4qr$ 18 $12ab-5bl+5xy$

19 $8+2a-2c$. 20 $4p+q$ 21 $7+5x-12y$

22 $11xy-10yz+9zx$ 23 $7a-2b+6c+5$

II. f. Page 27. 1 $4a^2-ab+2b^2$ 2 $6x^2-5$

3 $-c-3$ 4. $4p^2-3pq$ 5 $10a^2+a+3$

6 $2m^2+m$ 7 $2x^2+7x-2$ 8 $2x^3+3x^2-5$

9 $3-a-7a^3$ 10 $1+3b+10b^2+6b^3$ 11 $2y^3+10y^2-3y-7$

12 $6x^4+x^3$ 13 $7a^4+2a^2-2a$ 14 $8b^4-5b^3+2b^2+2b$

15 $-2a^3-5a^2b+4ab^2$ 16 $3x^2y+xy^2$ 17. $m^5+m^4+2m^2-2m-5$

18. $3a^3+a^5$ 19 c^4+2c^3 20 $p^4+4p^3-p^2+p-3$

21 $2p^2+2p$ 22 $2x^2y$ 23 7th and 9th respectively

24. Second and last are like; all homogeneous except the first

III a. Page 29 1 -2 2 6 3 -6 4 2
5 $4a$ 6 $10a$ 7 $-10a$ 8 $-4a$ 9 $4y$
10 $22y$ 11 $-4y$ 12 $-22y$ 13 $4x$ 14 $2x$
15 $23z$ 16 $-54y$ 17 $-2ab$ 18 0 19 $5cd$
20 $19cd$ 21 $-9x^2$ 22 $5y^2$ 23 $-3x^2y$ 24 $3yz^4$
25 $4a-b$ 26 $-4x^2z$ 27 $2c^2-5$ 28 $5-2c^2$ 29 $2c^2+5$
30 a^2-b^2 31 a^2+b^2 32 $z^3+x^2z^2$ 33 $16xy$

III b Page 30 1 $a-b$ 2 $3a-b$ 3 $2b$
4 $x-7y$ 5 $10c+7d$ 6 $3m+8n$ 7 $-b+c-2d$
8 $-2x+5y-3z$ 9 $p-7q-3r$ 10 $2y$
11 $3a+5b-3c$ 12 $8a-14b+4c$ 13 $2x-2z$
14 $-2m+2n+2p$ 15 $3ab-4cd+5ac$ 16 $4xy+2yz+zx$
17 $mn-14np+5pm$ 18 a^3-a^2-a+2 19 $2ax-3$
20 $-x^4+x^3+x$ 21 $-x^2y+4xy^2-4xyz$ 22 $a^4+2a^3+3a^2-2a$
23 $m^4-m^3+3m^2-m+1$ 24 $1-3a^3+3a^2b-2ab^2+b^3$
25 $3x^2y+6xy^2+y^3$ 26 $-3x^2y+3xy^2+2y^3$
27 $2+2x-2x^2-x^3$ 28 $-m^3-8m^2n+8mn^2+4n^3$
29 $2-2d+d^2-d^3-2d^4+d^5$ 30 $x^4-3x^3y+3x^2y^2-3xy^3+y^4$
31 d 32 $-x$ 33 $b+2c$ 34 $x^2-3y^2+5z^2$.
35 $-x^4+x^3y-x^2y^2+xy^3+1$ 36 $a^3-a^2b+ab^2-b^3$
37 c^3+2c^2-2 38 $3a-2c$
39 $m^3n-5m^2n^2+mn^3-n^4$, -1 40 $x^3-3x^2y+xy^2-y^3$

Miscellaneous Examples I Page 32

1 (i) $2x+x^2$, (ii) $-3a+b$ 2 $2a+2c$ 3 (i) 21, (ii) 108
4 (i) 11, (ii) 18 5 $7x^3-10x^2$ 6 175 9 $2x^3-2x^2$
10 $16x$, $8y$, $\frac{z}{112}$; 48, 40, 7 11 $2a-(3b+5c)$
12 47, 12 13 $-y^2+y$ 15 36 16 0 17 a^2b
18 $x+2z$ 20 $7xy$ 21 -4 miles 23 $4a$, 20
24 118 25 $240a+12b+c$, 860 26 27, 9, 1
27 30 B C 28 $2x^3-2x$ 30 $x-ab$ miles, 12
31 $a+3b$ miles south of O 32 $2y-z$, 14, 67, 193
33 $m^2-2mn-2n^2$ 34 7 yrs, 14 yrs, 42 yrs

IV. a Page 36. 1 -20 2 -21 3 54 4 $56a$
5 $6b$ 6 $-36x$ 7 $-60y$ 8 $108m$ 9 $35cd$
10 $30mn$ 11 $-104xy$ 12 $-105ab$ 13 $2abx$
14 $-3mnp$ 15 $7abc$ 16 $-9xyz$ 17 $20abcd$
18 $-3dxyz$ 19 $121lmn$ 20 $156abxy$

IV b Page 36 1 -10 2 12 3 -1 4 1
5 9 6 -20 7 4 8 -4 9 0 10 -1

11 27   12. 27.   13 4   14 10   15 -18 16 -1.
17 -16 18 32   19 -54 20 0   21 4   22 -14.
23 13   24 -10 25 -2 26 6   27 -13 28 -1
29 -32 20 3, 2, 6   31 18, 40, 70, 108
32 $-18, -2, -2, 2, 18$

IV. c Page 38.   1 b^5.   2 x^5 3 $-30z^3$
4 $45y^7$ 5 $-56c^6$ 6 $-6m^6$ 7 $4d^5$
8 $-p^{10}$ 9 $12a^3x$ 10 $-8c^2d^3$ 11 $9a^2b^2$
12 $-28a^3cd$ 13 c^4d^6 14 $-20x^4y^5z^2$ 15 m^6n^7
16 $-a^4b^7c^3$ 17 $-60ab$ 18 $-xyz$ 19 $60xyz$
20 $14xyz$ 21 $-abcdx$ 22 $84axy$ 23 $3a^4b^3$
24 $-72a^2b^3c^2$ 25 $x^3y^3z^3$ 26 $12x^5y^7$ 27 $a^3b^5c^6d^5$
28 $60a^3b^3$ 29 $b^8, x^{15}, y^{12}, -a^6b^6$ 30 $-a^3b^{12}, x^3y^{15}, -p^6q^3r^{12}$.

IV. d. Page 39   1 $4a+12b-20c$ 2 $a^3x+a^4x^2-a^5x^3$
3 $4a^3b^2-4a^2b^3$ 4. $x^5y^3-x^4y^4$ 5 $-6x^4y+14x^2y^4$
6 $-c^5d^3+c^3d^3-cd^3$ 7 $x^2y^2-xy^2z-x^2yz$ 8 $x^5y^2+x^4y^3-x^2y^4$
9 $-6c^4d^2+9c^3d^3-15c^2d^4$ 10 $-20a^4b^3c^2+12ab^5c^3+32ab^3c^2$.
11 $-a^6b^5+a^5b^6-a^4b^7$ 12 $3a^5bx^4-12a^6bx+6a^4bx^3$
13 $-a^3b^4c^4+a^3b^2c^5-a^4b^2c^4$ 14 $-abc+a^3bc-2ab^3c+3abc^3$
15 $-6xy^3z+9x^2y^2z^2-3x^3yz^3$ 16 $-6ax^3y^2+12abx^2y+4a^2x^2$

IV. e Page 41.   1 $x^2+7x+12$ 2 $x^2+6x-27$
3 $c^2-12c+35$ 4 x^2-36 5 $d^2-5d-84$
6 m^2+m-12 7 x^2-25 8 $f^2-18f+77$
9 $a^2+4a-77$ 10 $s^2-4s-77$ 11 $a^2-10a+9$
12 z^2-1 13 a^2+2a+1 14 c^2-2c+1
15 $y^2+18y+81$ 16 $-d^2+8d-16$ 17 $m^2-14m+49$
18 x^2-16 19 x^2-100 20 $a^2+10a+25$ 21 $-36+e^2$
22 $6c^2-5c-21$ 23 $5d^2-41d-36$ 24 $9m^2-4$
25 $-12+23x-10x^2$ 26 $10x^2-54x+56$ 27 $2x^2+ax-a^2$
28 $2a^2-ab-6b^2$ 29 $20c^2-9cd-20d^2$ 30 $3x^3+4x^2y-4xy^2$
31 x^4-9y^2 32 $25p^6-5p^4q-2q^2$ 33 $z^4-4z^2a^2+4a^4$
34. $12a^2+5ab^3-2b^6$ 35 $3x^4+2x^2y^2-y^4$ 36 $3b^4c-2b^2c^2-c^3$
38 $2x^2+22$, 94   39 $21x$, -63 40 243
41 69 6   42 200

IV f Page 43.   1 a^2-a-2 2 $a^2-11a+30$ 3 c^2+c-42
4 x^2-x-42 5 d^2-2d-3 6 x^2-1
7 y^2-y-20 8 $p^2+13p+42$ 9 $y^2+y-110$
10 x^2-81 11 $c^2-18c+81$ 12 $a^2+18a+81$
13 $a^2-2ax+x^2$ 14. $c^2+2cz+z^2$ 15 x^2-y^2
16 $4x^2+4x-3$ 17 $12+7c-12c^2$ 18 $25x^2+20x+4$
19 $1-49y^2$ 20 $a^2-ax-6x^2$ 21 $m^2-6mn+9n^2$.

22 $6x^2+11xy+3y^2$ 23 $25c^2-9d^2$ 24 $49a^2-14ab-3b^2$
25 $6x^2-5ax-6a^2$ 26 $-a^2+b^2$ 27 a^4-3a^2-18
28 $4a^4-7a^2b^2-2b^4$ 29 $a^2c^2+2ac-3$ 30 $1-12a+20a^2$
31. $1+b-42b^2$ 32 $2-5x-12x^2$ 33 x^4-y^4 34 $9a^4-b^4$
35 $4m^2+11mn-3n^2$ 36 $27-6ab-a^2b^2$ 37 £$(12x^2+5xy-2y^2)$
38 $6p^2-5pq-4q^2$ 39 $(25a^2-4b^2)$ miles 40 $3p^2+2pq-q^2$

V a Page 46 4 $2y^5$ 5 y^2 6 $3xy$
7 $7x^2y$ 8 $3a^2b^2$ 9 $2p^3q$ 10 $-2xy^2$ 11 $-3m^2n^3$
12 $-2b^2c^5$ 13 $-8qr$ 14. $9l m^3$ 15 $x^3y^2z^2$ 16 $-3l^7$
17 $-q$ 18 $-4l^2m$ 19 $7z^7$ 20 $-9bc^2$ 21 $9a^2b^3$
22 $-9b^2cx^2$ 23 $2ax-x^2$ 24. $5x^3+y^3$
25 $3a+4$ 26 $2m^2n^4-3n$ 27 x^3-5x^2+3
28 x^3-2x-1 29 $-2a+b+3c$ 30 $a-b^2+a^2b$
31 $-m^2+3mn-4n^3$ 32 $-p+9q^3+4p^2$

V. b Page 48 1 $a-2$ 2 $b+3$ 3 $x+2$
4. $x+2$ 5 $b+6$ 6 $z+4$ 7 $x-9$
8 $y-9$ 9 $p-5$ 10 $b+13$ 11 $x+4$
12 $3y-2$. 13 $x+2$ 14 $x+3$ 15 $4x+3$
16 $5a+1$ 17 $2m+1$ 18 $2m-3$ 19 $2c-1$
20 $3L-2$ 21 $4c+3$ 22 $3b+2$ 23 $p-3$
24. $2a-3$ 25 $-3x+4$ 26 $-7y+3$ 27 $3x+2$
28 $4x-3y$ 29 $7y-2z$ 30 $-x-9$ 31 $-6c-d$
32 $3c-5d$ 33 x^2+6x+5 34 $6+5x+x^2$ 35 $a^2+4a-21$
36 $2a^2+a-1$ 37 $3-11b+6b^2$ 38 $9m^2+9m-5$
39 $4a^2-12ax+9x^2$ 40 $a^2x-2abx^2-3b^2x^3$ 41 $7a+3$
42 $2x+7$ 43 $3p-4$ 44 $a-3b$

VI a Page 49. 1 $3x+y$ 2 $-x+5y$ 3 $3m-7$
4. $-m+1$ 5 $-a-b$ 6 $2c$ 7 $-3x+2y$
8 $-4x$ 9 $2a$ 10 $2b$ 11 $3a^2-8b^2$
12 $-5n+p$ 13 $2c^2+d^2$ 14 $2a^2$ 15 $-6x-y$
16 $m-4n$ 17 $2a-2b-2c$ 18 $4x+2y-11z$
19 $x^2+y^2+z^2$ 20 $2mnyz+2nxyz$ 21 39
22 -64 23 9 24. 8

VI. b. Page 51 1 $x+2y-3z$ 2 0 3 $12a$
4 $10c^2-5c$ 5 $2l-4m+4n$ 6 $5a-b$ 7 $2x$
8 $y-9z$ 9 1 10 $5a+9$ 11 0
12 $3-3x^2$ 13 x 14 $6c^2-3c-37$
15 $2x^2-2y^2-2z^2$ 16 $5a+c$ 17 $3d+3e$ 18 $a-3c-2d+2e$
19 $-4p+6q$ 20 $2m$ 21 a^2-b^2 22 $5x-10y-10$
23 $2a-6b$ 24. $a^2x-8ax+12x^3$ 25 a^4-a^2+2a-2
26 $x-6$, 16 27 10 28 1

VI. c. Page 53. 1 $3(x+2y)$ 2 $7(a-3b)$ 3 $5(a^2+2b^2)$.
4 $2(x^2-2xy+y^2)$. 5. $7(c^2-3d^2+4e^2)$ 6 $2(a^2c^2-3b^2y^2)$
7 $x(a-b)$ 8 $d(a-d)$ 9 $ax(a+x)$.
10 $5cd(cd-2)$ 11 $3ab(a-b)$ 12 $3a(a^2-2ab+b^2)$.
13 $x^2+(a+b)x$ 14 $y^2-(a+b)y$ 15 $z^2+(a-b)z$.
16 $ax-(a+b)x^2$ 17 $y^2-(2a+5b)y^3$ 18 $z^3-3(a^2-b)z^2$.
19 $(p+q)x^2-2(a+b)x+(a^2+b^2)y$
20 $(c^2-d^2)x^2+(2c-d)x-(a^2-b^2)z^2$
21 $(a-b-c)x-(a+b-c)y-(a-b+c)z$
22 $(3-c)x^3+(5-c^2)x^2$, $-(c-3)x^3-(c^2-5)x^2$
23 $(a^2-b^2)x^4+(b^2-c^2)x^2$, $-(b^2-a^2)x^4-(c^2-b^2)x^2$
24 $(1-2b^2)x^3+(1-2a^2)x$; $-(2b^2-1)x^3-(2a^2-1)x$
25 $(2-q)x^4+(p+r-3)x^3$; $-(q-2)x^4-(3-p-r)x^3$
26 $(a-b+c)x^3+(a+b-c)x^2$, $-(b-c-a)x^3-(c-a-b)x^2$
27 $(p-2m)x^5+(n-2p)x^3+(m-2n)x$,
$-(2m-p)x^5-(2p-n)x^3-(2n-m)x$
28 $(a-1)x^3+(5-c)x^2+(2-b)x$, $-(1-a)x^3-(c-5)x^2-(b-2)x$

Miscellaneous Examples II. Page 53. 1 $3a-6b+6c+6d$
2 (i) $12(x-y)$, (ii) $\frac{1}{20}(x-y)$ 3 $9a^2b^2+3ab+1$
4. $-3x^2+xy+2z-1$ 5 $ab(a+b)$ 6 $x+3a$
7. (i) -125, (ii) 125, (iii) 125, (iv) 15 8 $18b^2$
9 $n+1, n-1$, $p-1, p, p+1$ 10 $\frac{31}{36}$ 11 c
12 £$(b-a)$ 13 350 14. $x-7y$ 15 $p+52$, 28
17 $7pq+7q^2$ 18 $15x-15y$, 2250 19 $a^2+2ab+b^2$, $a^2-2ab+b^2$
21 $2a+5c$ 22 3, 0, 0, 3, 8 23 52 24. $2-3x+x^2$
25 x^2-ax+a^2, x^2+x+1 26 $-n$ 27 $3p-8$
28 (i) $x^2-9x+14$, (ii) x^2+5x+6, (iii) $2x^2-13x+20$ $9x$ 29 5
30 $12xy$ $144x^2+24xy+y^2$ 31 $5a^3b^4$, $5a^4b^3$ 32 $13x-5y$
33 a^2-ab+b^2, a^2+ab+b^2 34 $8n$
35 (i) $a^2+b^2+c^2$, (ii) $-4a+5b$ 37 420 38 $400m^2+40mn+n^2$
39. $5p-2q$ 40 $8a-5x-21$
41. $-x^4-x^3+2x^2y+2xy$ 42 50, 180, 280, 140 43 $y-y^2$.
44. $4x^2+5x$ 45. a^2+3b^2 46 $2x-y$
47 10, 6, 4, 4, 6 48 $(a-b+1)x^3-(b+c+3)x^2+(c-a-1)x$.

VII. a. Page 57. 1 $x^2+10x+25$ 2 d^2-4d+4
3 $16-8y+y^2$ 4. c^2+2c+1 5 $1+4a+4a^2$
6 $x^2-18x+81$ 7 $1-14b+49b^2$ 8 x^4-2x^2+1
9 $9x^2+6xy+y^2$ 10 $x^2-4xy+4y^2$ 11 $81+72z+16z^2$
12 $a^2+2ab^2+b^4$ 13 $x^2-2xyz+y^2z^2$ 14. $a^2b^2+2abc+c^2$
15 $y^4-4y^2z+4z^2$ 16 $a^6+2a^4+a^2$ 17 $25a^2+40ab+16b^2$
18 $1-2y^2+y^4$ 19 $a^2d^2-2ad^3+d^4$ 20 $x^2y^2-2x^2yz+x^2z^2$
21. $a^2-4acd+4c^2d^2$ 22 $9a^4-24a^3b+16a^2b^2$

23 $25x^6+10x^5y+x^4y^2$ 24. $9c^4-12c^6+4c^8$
25 12544. 26 39601 27 92·16 28 2490·01
29 a^2-c^2 30 a^2-1 31 $1-4x^2$ 32 $4a^2-b^2$
33 $9x^2-4y^2$ 34. $16x^4-1$ 35 a^2b^2-9 36 $4x^4y^2-1$
37 m^4-n^6 38 $25x^6-16x^2$ 39 $49a^2x^4-16a^4x^2$ 40 9991
41 9775 42 249375 43 39999·75
44. $241\times 1=241$ 45 $658\times 20=13160$ 46 $500\times 74=37000$
47 $3000\times 2462=7386000$ 48 $20\times 2·6=52$
49 $100\times 74·4=7440$ 50 x^2-x+1 51 c^2-c+1
52 $x^2-2xy+4y^2$ 53 $9+3x+x^2$ 54. $a^2-2ac+4c^2$
55 $x^4+x^2y^2+y^4$ 56 a^4-4a^2+16 57 $4a^4-2a^2+1$
58 p^3+q^3 59 $1-m^6$ 60 $27-b^3$ 61 x^3+8y^3
62 c^6-1 63 $64+27d^3$ 64. x^2+70 65 $x-5$
66 $2ab$ 67 $4a^2-5ab-6b^2$ 68 x^3+45

VII b Page 60 1 $-\frac{1}{12}x+\frac{5}{6}y$ 2 $\frac{5}{4}m$ 3 $\frac{3}{2}a-b$
4. $a-\frac{1}{3}b+\frac{2}{3}c$ 5 $\frac{5}{3}x^2+xy-\frac{9}{4}y^2$ 6 b
7 $2m^2-\frac{4}{3}mn-\frac{1}{2}n^2$ 8 $-\frac{1}{4}c^2-cd+\frac{2}{5}d^2$ 9 $\frac{1}{4}a$
10 $-\frac{2}{3}a^2+2ab-4ac$ 11 $\frac{4}{15}xy-\frac{2}{3}y^2+\frac{2}{5}y$
12 $18x^3y-x^2y^2$ 13 $\frac{1}{2}m^5n^2-3m^3n^4$
14. $\frac{1}{3}x^2-\frac{5}{12}xy+\frac{1}{8}y^2$ 15 $\frac{2}{3}a^2-\frac{4}{45}ab-\frac{1}{30}b^2$
16 $4x^4-\dfrac{x^2}{9}$ 17 $\frac{1}{36}x^4-\frac{1}{64}y^4$ 18 $x^8+x^4+\frac{1}{4}$
19 $\dfrac{m^6}{16}-\dfrac{m^3}{2}+1$ 20 $4a^2c^2-acd+\dfrac{d^2}{16}$
21 $\dfrac{m^4n^6}{4}-\dfrac{m^5n^5}{3}+\dfrac{m^6n^4}{9}$ 22 $2a-3b+4c$
23 $3x-2y-4$ 24. $-\frac{1}{3}x^2+2y^2$ 25 $3x^2-\frac{2}{7}xy$
26 $\frac{5}{6}x-5y$ 27 $a-\dfrac{2b}{3}$ 28 $\dfrac{m^2}{3}+\dfrac{mn}{4}+\dfrac{n^2}{6}$
29 $\dfrac{a^2}{2}+\dfrac{a}{3}+\dfrac{1}{4}$ 30 $\frac{1}{12}x-\frac{11}{6}y$ 31 $-3m+n$
32 $\frac{4}{3}x^2-\frac{4}{3}xy-\frac{1}{6}y^2$

VII c Page 62 1. $7a+15x$ 2 $x+3y$ 3 $\frac{1}{2}(m+n)$
4. $4b$ 5 $\frac{4}{3}y$ 6 $\frac{11}{3}a-2b$ 7 $2(a^2-b^2)-8(a+b)-4$, 5
8 $8(a+b)x+(a-b)y$, x 9 $3x^2(a+b)+4y^2(a-b)$
10 $\dfrac{11x+2}{18}$ 11 $\dfrac{10x-45}{21}$ 12 $\dfrac{2x+33}{24}$
13 $\dfrac{5x+21}{12}$ 14 $\dfrac{5x+6}{10}$ 15 $\dfrac{7x+13}{15}$
16 $\dfrac{17a}{36}$ 17 $\dfrac{5-y}{12}$ 18 $\dfrac{81-35c}{60}$
19 $25x-33$ 20 $7x-36$ 21 $15x+108$

VII. d. Page 64.

1 9 2 16 3 8 4 1
5 2 6 6 7 6 8 12 9 12 10 18
11 36 12 6 13 20 14 −10 15 −192 16 72

VII e. Page 65.

1 4 2 18 3 10 4 16
5 36 6 −16 7 −6 8 −9 9 72 10 −216
11 −2 12 −4 13 2 14. $3\frac{5}{9}$ 15 0 16 $3\frac{7}{9}$

VII f. Page 66.

1 2 2 10 3 22 4 1.
5 −2 6 1 7 6 8 9 9 6 10. 4
11 −5 12 10 13 −7 14 6 15 $\frac{1}{2}$ 16 1
17 0 18 29 19 −13 20 $-13\frac{3}{4}$ 21 $-\frac{3}{4}$ 22. $-1\frac{2}{3}$.
23 6 24 $1\frac{2}{3}$

VIII b Page 71

1 3 2 5 3 4
4 6 5 2 6 5 7 2 8 3
9 5 10 7 11 $3\frac{1}{2}$ 12 2 13 2
14 4 15 3 16 5 17 $\frac{1}{2}$ 18 $2\frac{1}{2}$
19 1 20 $1\frac{1}{2}$ 21 2 22 1 23 6
24 12 25 8 26 20 27 18 28 60
29 12 30 15 31 $4\frac{1}{2}$ 32 7 33 9
34 6 35 $2\frac{1}{2}$ 36 $6\frac{2}{3}$ 37 $\frac{5}{8}$ 38 $\frac{7}{12}$
39 $\frac{8}{21}$ 40 $1\frac{13}{27}$ 41 $\frac{4}{7}$ '42 $1\frac{1}{7}$ 43 $\frac{7}{40}$
44 $2\frac{1}{2}$ 45 $7\frac{1}{2}$ 46 $\frac{3}{4}$ 47 $2\frac{4}{5}$ 48 $8\frac{1}{6}$

VIII. c Page 73.

1 −10 2 6 3 2
4 5 5 7 6 1 7 $3\frac{1}{3}$ 8 $2\frac{2}{5}$
9 $1\frac{1}{3}$ 10 −3 11 3 12 9 13 $-16\frac{1}{2}$.
14 2 15 −5 16 $1\frac{1}{2}$ 17 2 18 15
19 $1\frac{1}{3}$ 20 $\frac{1}{3}$ 21 −1 22 4 23 11
24 −7 25 $\frac{1}{7}$ 26 6 27 41 28 1
29 9 30 7 31 $5\frac{1}{2}$ 32 60 33 3
34 5 35 $4\frac{1}{2}$ 36 $\frac{1}{2}$ 37 18 38 11
39 $-5\frac{1}{6}$ 40 $\frac{3}{26}$ 41 $3\frac{1}{2}$

VIII. d Page 75.

1 2 2 3 3 5 4 5
5 8 6 1 7 $\frac{1}{2}$ 8 $\frac{1}{3}$ 9 2 10 1
11 7 12 $\frac{1}{2}$ 13 2 14 5 15 1 16 6
17 4 18 −2 19 7 20 2 21 $\frac{3}{7}$ 22 4
23 $3\frac{1}{7}$ 24 −15 25 $7\frac{1}{13}$ 26 −6 27 18

IX. a. Page 78

1 $15-x$, $5x-5$ 2 $\frac{a}{b}$ 3 $p-q$
4 37 5 6 6 $7x$ 7 $\frac{y}{5}$ 8 $16-c$
9 $(a-5)$ years, $(a-p)$ years 10 $(15-b)$ years

11 $(q+p)$ years 12 $15-p$ 13 $\frac{c}{d}$ 14 $\frac{3x}{b}$

15 $\frac{12y}{x}$ 16 $\frac{9}{2}x$ 17 $\frac{xz}{20}$ 18 kx.

19 d^4 20 $14x-y$ 21 $2240a - 112b$

22 $c(a-b)$ miles 23 $20m+2n-x$ 24 $\frac{xy}{9}$. 25 $\frac{100}{c}$.

26 $\frac{n}{3}$ hours 27 xy miles 28 $\frac{y}{x}$ miles 29 $5p$

30 $\frac{44}{x}$. 31 $240x+12y-z$ 32 (i) $8m$, (ii) $\frac{n}{12}\times m$.

33 64, $\frac{4}{5}x$ 34 (i) $\frac{x}{24}$, (ii) $3y$, (iii) $2x$

35 $\frac{3}{4}x$, $\frac{4}{3}y$

IX b Page 80 1 $m(m+1)(m+2)(m+3)$ 2 $3n-3$

3 $k-2,\ k-1,\ k,\ k+1,\ k+2$ 4 $(2p-1)(2p+1)(2p+3)$

5 $(2n-2)(2n)(2n+2)=d$ 6 $(x+15)$ years

7 $(n+18)$ years 8 $(2y-10)$ years 9 $lm+n$ 10 $x-yz$.

11 $\frac{a-c}{b}$. 12 $3ab$ 13 $\frac{m^2}{9}$ 14 $\frac{pq}{2}$

15 $\frac{xyz}{60}$ 16 $\frac{50}{p}$. 17 $\frac{60a}{b}$ 18 $\frac{mn}{18}$

19 $24p$ miles 20 $\left(\frac{a}{7}+\frac{b}{35}\right)$ hours 21 $100m-10n+r$

23 $\frac{15mn}{22}$ 24 $20x-25+\frac{y}{12}$ 25 $\frac{5a^2}{b}$.

26 $a+b=x$ 27 $xy=5(c-d)$ 28 $\frac{m}{n}=p+q-12$.

29 $x-7=6(y-7)$ 30 $\frac{x}{4a}=6mn-9$

31 $a+x+5=2(a+5)$, 35, 24. 32 $c+5=\frac{1}{2}(a+2)$

33 $z=x+xy$ 34 $x+4=y-4$ 35 $a-c=\frac{b}{20}-c$.

36 $20mx=ny$ 37 $dy=cx-9$

IX c Page 84 1 (i) 204 sq ft, (ii) 16 ft, (iii) 4 8 chains; (iv) 39 sq cm

2 (i) 32 cu ft, (ii) 12 cu ft; (iii) 6 ft

3 (i) 11 in, 14 ft 8 in, (ii) $9\frac{5}{8}$ sq in, $17\frac{1}{9}$ sq ft

4 (i) 55 44 sq in; (ii) 1 ft 2 in

5 $A=\pi(R^2-r^2)$ (i) 13 86 sq cm, (ii) 4·2 cm

6 (i) 5 6 sq in, (ii) 107 9 sq cm 7 1 5 in

8 (i) 18, (ii) 1 hr 30 min, (iii) 45

9 (i) 144 9 ft, (ii) 5 secs 10 $A=bl$, $P=2(b+l)$, $S=2h(b+l)$

11 (i) A, 198 sq ft, P, 58 ft, S, 522 sq ft;
(ii) A, 297 sq ft, P, 69 ft 10 in, S, 838 sq ft
12 10 ft 6 in 13 27 sq ft 14 328. 15 37.
16 (i) and (iii) 19 (i) 45150, (ii) 500500, (iii) 455350 20 15.
21 (i) 17, (ii) 24, (iii) 40, (iv) 1 5
22 $I = \frac{P \times n \times r}{100}$ (i) £52 4*s*, (ii) $3\frac{1}{2}$, (iii) 5 yrs; (iv) £670
23 40 24. 12 25 (i) 9780, (ii) 1, (iii) 12, (iv) −40 5.
26 4, $5\frac{1}{5}$, $6\frac{2}{5}$, $7\frac{1}{5}$, $8\frac{4}{5}$, 10

X. a. Page 89. 1 17, 9 2 35°, 13°, 132° 3 9
4 7 5 15 6 13 7 5
8 14, 15, 16 9 7, 21 10 6, 7, 8 11 *A*, £24, *B*, £16.
12 36, 24 13 24, 26, 28 14 21, 22, 23 15 12, 8
16 £15, £5 17 60 18 45, 54 19 4
20 36, 56 21 45, 36 22 20, 30 23 13, 9
24 204 25 49, 50 26 96 27 72

X. b. Page 91 1 *A*, £35; *B*, £20, *C*, £12
2 *A*, £14, *B*, £28, *C*, £24 3 *A*, £15, *B*, £25, *C*, £45
4 *A*, £35, *B*, £72, *C*, £81 5 1st, £78, 2nd, £42, 3rd, £36.
6 36 tons, 31 tons, 33 tons 7 £25; £35
8 420 yds, 230 yds 9 £2 5*s*
10 *A*, £9 5*s*, *B*, £2 15*s* 11 *A*, £4 10*s*, *B*, £1 10*s*
12 17*s*, 13*s* 13 *A*, 16 yrs, *B*, 8 yrs
14 *A*, 24 yrs, *B*, 8 yrs 15 45 years
16 *A*, 9 yrs, *B*, 36 yrs 17 Man, 33 yrs, son, 3 yrs
18 *A*, 15 yrs, *B*, 5 yrs 19 180, 205
20 140 at 1*s* 6*d*, 60 at 2*s* 6*d* 21 *A*, £2 5*s*, *B*, 18*s*
22 84 lbs at 4*s*, 28 lbs at 2*s*. 23 *A*, 45 yrs, *B*, 48 yrs
24 *A*, 16 yrs, *B*, 25 yrs, *C*, 10 yrs
25 12 tea, 9 coffee 26 20 oxen, 40 sheep
27 9 h crs, 27 sh, 6 pence 28. 4 ft 10 in, 4 ft 5 in

X. c. Page 93 1 40 2 36 3 221
4. 108, 144 5 51 yrs 6 36 yrs; 18 yrs ago
7 36 8 360 9 36 yrs, 8 yrs; 4 yrs
10 $2\frac{1}{2}$ mi, 5 mi, per hr 11. 8 mi, 12 mi., per hr
12 10 mi 13. Horse, £75, carriage, £60, harness, £9.
15 60 mi, 5 p m, 10 p m
16 2 hrs, +18 must be changed to −18 17 At noon and 2 p m.

XI. a. Page 98 12 100 13 A square; 36.
15 36 units of area 18 32
19 (i) 13; (ii) 10; (iii) 13, (iv) 15, (v) 26, (vi) 20

21 15 mi 22 5 mi 23 10 mi. 24. 10 units
25 42 units 26 (3, 7) 27 10, 13, 5, 5, 3 units respectively.
28 (i) All lie on a line through the origin,
(ii) all lie on the axis of x,
(iii) all lie on a line parallel to the axis of x,
(iv) all lie on a line parallel to the axis of y,
30 At the point (0, 9)
32 A circle of radius 13 whose centre is the origin

XI b Page. 103 9 (1, 2) 10 (3, −2)
11 $y=5$ 12 21, 28 5 13 21
14. 0 9 sq in 15 1 25 sq in 16 1 3

XI c Page 107 3 9, 2 4 4. 20, 1 84
5 5 in each case 18 5 units 6 (1 3, 2 0) 7 15 5, 2 43

XI. d Page 109 2 2 56 cm, 1 56 in
3 (i) 11 4 litres, (ii) 4 6 gals 4. 18, 40, 51
5 5s 9d, 16s 5d, 42s, 62 days
6 22s 3d, 36s, 39s 5d 7 1s 5d, 2s, 3s 8d, 5 hrs
8 $7\frac{1}{2}d$, 1s 8d, 2s 11d, 1s, 2s 1d, 3s 7d
9 104, 72 10 £350, 4250
11 6 p m, 48 mi from London At 4 and 8 p m
(i) B 4 mi behind A, C 6 mi behind B (ii) 4 20 p m
12 6 p m, (i) 3 30 p m, (ii) 7 30 p m

XI. e. Page 112 1 (4 2, 0) 2 25 6, 0·7
3 4 or −3, 13 75. 4 15

XI. f Page 116 7 29 8 in, 6 in 8. (i) £180, (ii) 23
9 17, 34 3 millions 10 2 5 sq ft
11 3s 7d, 3s, 2s 1d 12 7 6, 5 13 £1 18s, £2 15s
14 8 6, 48 lbs 15 1880, 1896 16 9 4 ft, 13 5 lbs
17 14s 6d, £1 18 6 2 cu in

Miscellaneous Examples III Page 120

1 $8xy-4y^2$ 2 $\frac{5x}{4y}$ 3 $-\frac{3}{2}a^3-\frac{1}{2}a^2b+\frac{1}{4}ab^2-b^3$.
4 $(a+5b)x-8by$ 5 (i) 16, (ii) 3 6 0
7 £660, £340 8 $13y$ 9 3968600000 10 19
12 $ap+bq$, $\frac{4ap+5bq}{20}$ 13 (i) 9; (ii) 22 14 8 sovs, 16 h-cr.
15 x^2-34y^2, $2y^2$ 16 7 17 $\frac{45b}{22a}$, $\frac{88ab}{3}$
18 $-10a^2+24c^2$, $11a^2+b^2-23c^2$ 19 120 mi
20 233 21 −2. 22 $10x+y$, $x+\frac{y}{10}$, xy

23 $-(1+6b)-7ax+(36-7a-2b)x^2-(a-b)x^3$; 37

24 $2a^2+2ab-2b^2+a-b$ 26 (i) 2; (ii) $\frac{3}{5}$

27. Hens' 10*d*, ducks' 1*s* 2*d* per dozen

28 (i) £1 10*s*, (ii) £16, (iii) 15

XII. a. Page 124. 1 $x=9, y=3$ 2 $x=12, y=7$.

3 $x=8, y=8$ 4. $x=3, y=2$ 5 $x=2, y=1$

6 $x=7, y=9$ 7 $x=11, y=9$ 8. $x=5, y=-2$

9. $x=\frac{3}{7}, y=\frac{7}{3}$ 10 $x=-\frac{1}{2}, y=3$ 11 $x=-15, y=8$

12 $x=5, y=6$ 13 $x=13, y=5$ 14 $x=12, y=15$

15 $x=1, y=-4$ 16 $x=3, y=-4$ 17 $x=16, y=35$

18 $x=3, y=7$ 19 $x=-9, y=-2$ 20 $x=-\frac{1}{2}, y=3$

21 $x=\frac{1}{3}, y=-\frac{1}{6}$ 22 $x=-2, y=-3$ 23 $x=9, y=8$.

24 $x=1, y=2$ 25 $x=-2, y=3$ 26 $x=2, y=-1$

27 $x=2, y=4$ 28 $x=4, y=-7$ 29 $x=11, y=-2$

30 $x=2, y=1$ 31 5, 2 32 $a=1\frac{1}{2}, b=2$

XII b. Page 126 1 $x=9, y=4$ 2 $x=6, y=-4$

3 $x=4, y=9$ 4 $x=\pm\frac{2}{3}, y=\frac{3}{2}$ 5 $x=-\frac{2}{3}, y=-\frac{3}{4}$

6 $x=13, y=11$ 7 $x=7, y=11$ 8 $x=-\frac{1}{2}, y=3$

9 $x=3, y=-3$ 10 $x=2, y=3$ 11 $x=13, y=17$

12 $x=3, y=-4$ 13 $x=12, y=-4$ 14 $x=8, y=2$

15 $x=0\ 02, y=2\ 9$ 16 $x=\frac{1}{4}, y=\frac{1}{7}$ 17 $x=3, y=2$

18 $x=6, y=10$ 19 $x=\frac{1}{5}, y=\frac{1}{6}$ 20 $x=\frac{1}{3}, y=3$.

21 $x=4, y=\frac{1}{2}$ 22 $x=2, y=-3$ 23 $x=\frac{7}{2}, y=-\frac{5}{2}$

XII. c. Page 128 1 $x=2, y=3$ 2 $x=3, y=3$

3 $x=6, y=4$ 4 $x=3, y=-2$ 5 $x=2, y=-2$.

6 $x=-2, y=5$ 7 $(4, -2)$

8 (i) $x=2\ 5, y=3\ 6$, (ii) $x=3\ 2, y=2\ 4$. 9 $(2\ 5, 1\ 7)$

10 $(-3, 2), (4, 1), (3, 4)$ 12 $a=\frac{1}{5}, b=4\frac{3}{5}$. 13 $7y=6x+11$

15 $2x+10y=31$ 16 $a=2, b=6$ 17 $a=\frac{3}{4}, b=-5$

XII. d. Page 131. 1. $y=0\ 21x+1\ 37$. 2 $y=0\ 4x+1\ 6$ 9 2; 3

3 54 5° F, 86 9° F, $F=32+\frac{9}{5}C$ 4 $P=0\ 6\ G-14\ 4$; 24

5 7 $P=0\ 08\ W+1\ 4$ 26 2 lbs, 1 ton 6 $a=\frac{1}{4}, b=3$ 2, 12

XII e. Page 133. 1 $x=1, y=1, z=5$ 2 $x=6, y=4, z=2$

3 $x=2, y=3, z=1$ 4 $x=3, y=-2, z=4$

5 $x=5, y=4, z=-6$ 6 $x=2, y=1, z=0$

7 $x=1, y=2, z=3$ 8 $x=1, y=3, z=5$

9 $x=8, y=10, z=14$ 10 $x=12, y=18, z=6$

11. $x=y=z=12$ 12 $x=8, y=4, z=5$

13 $x=6,\ y=11,\ z=6$ 14 $x=8,\ y=-2,\ z=12$

15 $x=5\cdot 4,\ y=4\cdot 5,\ z=7\cdot 6$ 16 $x=5,\ y=-1,\ z=-4,\ w=5$

17 $a=4,\ b=3$ 18 $4x-3$ 21 $a=5,\ b=2$

22 The equations are inconsistent The equations become consistent, but are not independent

XIII a. Page 135 1 16, 9 2 38, 23 3 35°, 17°

4 29, 13 5 50, 30 6 30, 18 7 33, 3 8 £3, £4

9 $\frac{3}{5}$ 10 $\frac{7}{12}$ 11 $\frac{9}{13}$ 12 $\frac{7}{9}$

13 Horse, £27, cow, £15 14 Table, £7 10*s*, chair, £1 10*s*

15 10 sheep, 5 horses 16 Tea, 2*s* 3*d*, coffee, 1*s* 9*d*

17 Man, 3*s* 6*d*, boy, 2*s* 18 13 yds, 17 yds

19 Tea, 2*s* 8*d*, coffee, 1*s* 6*d*

XIII. b. Page 137 1 60 eggs, 30 apples

2 50 penholders, 60 lead pencils 3 80 of first, 40 of second

4 34 5 72 6 75 7 18 8 59

9 72 boot-laces, 108 buttons 10 Larger, 4*d*, smaller, 2*d*

11 *A*, 15*s*, *B*, 18*s*, *C*, 20*s* 12 100

13 63, 36 14 , 49 15 57 16 275 17 50

18 Horse, £31, cow, £14 10*s* 19 Half-way at 5 p m

20 $2\frac{1}{2}$ hrs 21 Boat, 8 mi per hr, stream, 3 mi per hr.

22 210 mi 23 540 mi

24 Rabbit, 1*s* 3*d*, pheasant, 3*s* 9*d*, chicken, 3*s*

25 432 26 200 mi, $33\frac{1}{3}$ mi per hr

XIV a Page 140 1 $a(a+b)$ 2 $a^2(a-x)$ 3 $2a(a-1)$

4 $b^2(1-b)$ 5 $c(d-c)$ 6 $c^2(c-d)$

7 $5a(a-2)$ 8 $3a(1-3a)$ 9 $3x(x-2y)$

10 $p^2(2q+1)$ 11 $y^2(1-x)$ 12 $y^4(y-1)$

13 $4a^2(1-4b)$ 14 $15d(1+3d)$ 15 $9c(2c^2-d^2)$

16 $16m(1-4mn)$ 17 $13y^2(x^2+3y^2)$ 18 $3x^2(3y^2-z^2)$

19 $27(3x-2)$ 20 $5p^3(2+5pq)$ 21 $17(3x^2y^2-1)$

22 $x(4x^2+x-1)$ 23 $2a(a^2-2a-1)$ 24 $3x(x^2-2x+3)$

25 $x(x^2-xy+y^2)$ 26 $3x(4y^2+3xy+x^2)$

27 $2c^2d(d^2-3d+c)$ 28 $2a^3(a^2-3ab-b^2)$

29 $3x^2y(x^2-2xy+3y^2)$ 30 $7a(a^2-ab+2b^2)$

XIV b Page 141 1 $(m+n)(y+z)$ 2 $(c+d)(x-y)$

3 $(2a-b)(y^2+z^2)$ 4 $(c^2-2)(x-2y)$ 5 $(x-y)(5-n)$

6 $(ab+y)(l+m)$ 7 $(a+b)(a+c)$ 8 $(a-c)(a+b)$

9 $(ac+d)(ac+b)$ 10 $(a+3)(a+c)$ 11 $(2+c)(x-c)$

12 $(x-a)(x+5)$ 13 $(5+b)(a+b)$ 14 $(a-y)(b-y)$

15 $(a-b)(a-z)$ 16 $(p+q)(r-s)$ 17 $(x-y)(m-n)$

18 $(x-a)(m+n)$ 19 $(2x+y)(a+b)$ 20 $(3a-y)(2c-1)$

21	$(2x+y)(3x-a)$	22	$(x-2y)(m-n)$.	23	$(a+b)(x^2+2)$
24	$(x-3)(x-y)$	25.	$(2x-1)(x^3+2)$	26	$(x+1)(x^3+2)$
27	$(y-1)(y^2+1)$	28	$(a+bc)(xy-z)$	29	$(f^2+g^2)(x^2-a)$
30	$(2x+3y)(ax-by)$	31.	$(a-b-c)(x-y)$.	32	$(a+b)(ax+by+c)$.

XIV c. Page 143.

		1	$(x+1)(x+2)$	2	$(x+2)(x+3)$
3	$(x+1)(x+3)$	4	$(x-1)(x-2)$	5	$(x-2)(x-3)$
6	$(x-1)(x-3)$	7	$(y+1)(y+4)$	8	$(y+2)(y+4)$
9	$(y+3)(y+4)$	10	$(y-4)(y-5)$	11	$(y-1)(y-7)$
12	$(y-5)(y-2)$	13	$(z+3)(z+5)$	14	$(z-2)(z-5)$
15	$(z+3)(z+6)$	16	$(z-1)(z-15)$	17.	$(z+6)(z+7)$
18	$(z+4)(z+4)$	19	$(a-8)(a-1)$	20	$(a+7)(a+3)$
21	$(a+6)(a+4)$	22	$(a+7b)(a+2b)$	23	$(a-6)(a-2)$
24	$(a+8b)(a+3b)$	25	$(b-3)(b-3)$	26	$(b-1)(b-13)$
27	$(b+7)(b+4)$	28	$(b-9c)(b-c)$	29	$(b+8c)(b+c)$.
30	$(b+11)(b+1)$	31	$(x+7y)(x+9y)$	32	$(x+5y)(x+5y)$
33	$(x-2y)(x-12y)$	34.	$(ab-2)(ab-2)$	35	$(ab+2)(ab+8)$
36	$(ab+7)(ab+5)$	37	$(n^2+5)(n^2+13)$	38	$(n^2-8)(n^2-17)$.
39	$(n^3-5)(n^3-5)$	40	$(p-17q)(p-q)$	41.	$(p^2+23)(p^2+3)$.
42	$(pq-11)(pq-4)$				

XIV. d. Page 144.

		1	$(a-2)(a+1)$	2	$(a-3)(a+1)$
3	$(a-3)(a+2)$	4.	$(a+2)(a-1)$	5	$(a+3)(a-1)$
6	$(a+3)(a-2)$.	7	$(b-5)(b+1)$	8	$(b+5)(b-3)$.
9	$(b-6)(b+2)$	10	$(b+4)(b-1)$	11	$(b-5)(b+2)$
12	$(b-4)(b+3)$	13	$(c-5d)(c+4d)$	14	$(c-6)(c+2)$
15	$(c+5)(c-4)$	16	$(c+8)(c-7)$	17	$(c-7d)(c+3d)$
18	$(c+8)(c-5)$	19	$(x+12)(x-3)$	20	$(x-8y)(x+3y)$
21	$(x-9)(x+5)$	22	$(x-9y)(x+4y)$	23	$(x-6)(x+4)$
24	$(x-y)(x+5y)$	25	$(y+11)(y-10)$	26	$(y+9)(y-7)$
27	$(y-15)(y+4)$	28	$(y+13z)(y-12z)$	29	$(y^2-7)(y^2+5)$
30	$(y^2+20)(y^2-3)$	31	$(z-17)(z+5)$	32	$(z-15)(z+6)$
33	$(z^2+13)(z^2-6)$	34	$(z-8)(z+9)$	35	$(z^2+9)(z^2-6)$
36	$(z+25)(z-3)$	37	$(x-4y)(x+2y)$	38	$(x+8y)(x-3y)$.
39	$(x-11y)(x+7y)$	40	$(x-13y)(x+2y)$	41	$(x+17y)(x-6y)$.
42	$(x+13y)(x-7y)$	43	$(ab+5)(ab-3)$	44	$(ab-8)(ab+7)$
45	$(ab+9)(ab-6)$	46	$(2-m)(1+m)$	47	$(7+x)(2-x)$
48	$(14+y)(7-y)$				

XIV. e. Page 144.

		1	$(x-1)(x-2)$	2	$(a+2b)(a+5b)$
3	$(b+4)(b-3)$	4	$(y-7)(y+3)$	5	$(c+1)(c+11)$
6	$(x-5)(x+1)$	7	$(n+2)(n+10)$	8	$(y+10)(y-1)$
9	$(p-6q)(p+4q)$	10	$(y+11)(y-10)$	11	$(z-15)(z+6)$
12	$(k-6)(k-8)$.	13	$(a+9b)(a+9b)$	14.	$(b-27c)(b+3c)$.

15 $(c+27)(c+3)$ 16 $(x-7)(x-7)$ 17 $(y+7z)(y+3z)$
18 $(z+9)(z-7)$ 19 $(n+8)(n+3)$ 20 $(p-8q)(p-3q)$
21 $(l+12)(l-3)$ 22 $(ab-2)(ab-2)$ 23 $(ab+8)(ab+2)$
24 $(b-9c)(b+5c)$ 25 $(m+11)(m-8)$ 26 $(n-15)(n-3)$
27 $(p+13)(p-3)$ 28 $(xy-9)(xy+8)$ 29 $(z-5)(z+4)$
30 $(x+8y)(x-7y)$ 31 $(a-13b)(a+2b)$ 32 $(ab-8)(ab+7)$
33 $(y^2+13)(y^2-12)$ 34 $(z^2-13)(z^2+6)$ 35 $(y^2+5)(y^2-7)$
36 $(x+13y)(x-7y)$ 37 $(9-y)(7+y)$ 38 $(13+x)(4-x)$
39 $(12+a^2)(11+a^2)$

XIV f Page 145 1 $(x+1)(x-1)$ 2 $(x+2)(x-2)$
3 $(x+3)(x-3)$ 4 $(x+5)(x-5)$ 5 $(x+4)(x-4)$
6 $(3a+b)(3a-b)$ 7 $(6+c)(6-c)$ 8 $(d+7)(d-7)$
9 $(y+8)(y-8)$ 10 $(10+z)(10-z)$ 11 $(pq+1)(pq-1)$
12 $(cd+2)(cd-2)$ 13 $(3+xy)(3-xy)$ 14 $(4+x^3)(4-x^3)$
15 $(5+2y)(5-2y)$ 16 $(9+5p)(9-5p)$ 17 $(10m+7)(10m-7)$.
18 $(z^2+11)(z^2-11)$ 19 $(3a^2+5b^2)(3a^2-5b^2)$
20 $(x^3y^3+4)(x^3y^3-4)$ 21 $(2xy+ab)(2xy-ab)$
22 $(12+ax^2)(12-ax^2)$ 23 $(4ax+7)(4ax-7)$.
24 $(k+13)(k-13)$ 25 $(abc^2+8)(abc^2-8)$
26 $(l+9mn)(l-9mn)$ 27 $(5m+8n)(5m-8n)$
28 $(a^4+2b^2)(a^4-2b^2)$ 29 $(x^2a+7)(x^2a-7)$
30 $(3x^2+5y^2)(3x^2-5y^2)$ 31 $(4x^4+yz)(4x^4-yz)$
32 $(1+5b^3)(1-5b^3)$ 33 $(p^2q+11)(p^2q-11)$
34 $(7z+9)(7z-9)$ 35 $(5b^2+9c)(5b^2-9c)$.
36 $(x^2y^2+8)(x^2y^2-8)$ 37 400 38 200
39 400 40 6200 41 1,002,000
42 3200. 43 5000 44 800 45 750,000.

XIV g Page 146 1 $(a-1)(a^2+a+1)$ 2 $(x+1)(x^2-x+1)$
3 $(1+m)(1-m+m^2)$ 4 $(1-n)(1+n+n^2)$
5 $(2-b)(4+2b+b^2)$ 6 $(c+3)(c^2-3c+9)$
7 $(d+4)(d^2-4d+16)$ 8 $(1+2p)(1-2p+4p^2)$
9 $(3y-1)(9y^2+3y+1)$ 10 $(xy+z)(x^2y^2-xyz+z^2)$
11 $(ab-2)(a^2b^2+2ab+4)$ 12 $(m+3n)(m^2-3mn+9n^2)$
13 $(4-pq)(16+4pq+p^2q^2)$ 14 $(5p-2)(25p^2+10p+4)$
15 $(x+10y)(x^2-10xy+100y^2)$ 16 $(7-y)(49+7y+y^2)$
17 $(b+9)(b^2-9b+81)$ 18 $(x+5y)(x^2-5xy+25y^2)$
19 $(6-ab)(36+6ab+a^2b^2)$ 20 $(n-4m)(n^2+4mn+16m^2)$
21 $(5-z)(25+5z+z^2)$ 22 $(8a+b)(64a^2-8ab+b^2)$
23 $(2c-7)(4c^2+14c+49)$ 24 $(xyz-3)(x^2y^2z^2+3xyz+9)$
25 $(x^2+4y)(x^4-4x^2y+16y^2)$ 26 $(5a^2+1)(25a^4-5a^2+1)$
27 $(9p-2q)(81p^2+18pq+4q^2)$ 28 $(2-10a^2)(4+20a^2+100a^4)$
29 $(4x^2-5y)(16x^4+20x^2y+25y^2)$ 30 $(cd^2e^3-1)(c^2d^4e^6+cd^2e^3+1)$.

31. $(p^2+2q)(p^4-2p^2q+4q^2)$ 32 $(1-3ab)(1+3ab+9a^2b^2)$.
33 $(z+6)(z^2-6z+36)$ 34. $(7a-5b)(49a^2+35ab+25b^2)$
35 $(4p^2q^2+1)(16p^4q^4-4p^2q^2+1)$ 36 $(9xy-8z)(81x^2y^2+72xyz+64z^2)$.

XIV. h. Page 147. 1 $m^2n^2(m-3n)$ 2 $5x^3(2+5xy)$
3 $(y-5)(y+3)$ 4. $(a+b)(p+q)$ 5 $(x+y)(x-z)$
6 $(a-b)(4-c)$ 7 $(a^3+2)(a+1)$ 8 $x(x^2+5)(x^2-5)$
9 $(b^2c^2+1)(bc+1)(bc-1)$ 10 $(z^2+9)(z+3)(z-3)$
11 $(m^2-20)(m^2+5)$ 12 $(ab-11)(ab+10)$
13 $(p-7)(p-7)$ 14. $(pq+4)(pq+4)$ 15 $z(z-3)(z+2)$
16 $a(a+7)(a-6)$ 17 $(5+9a)(5-9a)$ 18 $(a^2b^2+3)(a^2b^2-3)$.
19 $(3+l)(9-3l+l^2)$ 20 $(1-4m)(1+4m+16m^2)$
21 $(l^2+5l)(l^2-5l)$ 22 $(pq-1)(p^2q^2+pq+1)$.
23 $(2z+1)(4z^2-2z+1)$ 24. $(1+8a)(1-8a)$
25 $(m^3+2)(2m-1)$ 26 $a^2(a-b)(a-3)$
27 $(p-5q)(p+4q)$ 28 $l(l-7)(l+6)$ 29 $(abc+9d)(abc-9d)$
30 $(x+9)(x+12)$ 31 $(a+13)(a-7)$ 32 $(x-12y)(x-8y)$
33 $(ab+17)(ab-3)$ 34. $c(c+13)(c-12)$ 35 $n(m-3n)(m-3n)$
36 $(a+b)(a-3b+1)$ 37 $(x+y)(x-y+1)$ 38 $9a^4-25$.
39 a^4-16 40 $(1-49x^6)$ 41 $a^2b^2-3ab+9$
42 $4x^2+20x+100$ 43 $1-4p$ 44. $81-9y^2+y^4$
45 $x+11$. 46 $a+12b$ 47 $cd+12$

XV. a. Page 149 1 $2x^3-3x^2+3x-1$ 2 $3a^3-4a^2-a+2$
3 $6x^3+5x^2-21x+10$ 4. $12z^3-11z^2-25$ 5 $c^3-6c^2+3c+18$
6 $-9b^3+15b^2+8b-16$ 7 $15x^4-29x^3+12x^2+24x-32$
8 $-12d^3+8d^2+13d-7$ 9. $x^3-7x^2+13x-7$
10 $a^3b-2ab^3-b^4$ 11 $12y^3+11y^2-19y+3$
12 $a^2x^3+abx^2-acx^2-acx-bcx+c^2$ 13 a^3+b^3 14. a^3-b^3
15 $x^3-9x^2+27x-27$ 16 $-c^3-4c^2d-5cd^2-2d^3$
17 $1-3x-9x^2+47x^3-60x^4$ 18 a^6-b^6
19 m^4-n^2+4n-4 20 $x^4-6x^3+11x^2-6x+1$
21 $a^2-b^2+2bc-c^2$ 22 $4x^2-y^2+6yz-9z^2$
23 $1-9d^2+6d^3-d^4$ 24. $-2x^4+9ax^3-14a^2x^2+9a^3x-2a^4$
25 $3y^5+y^4-17y^3+13y^2+16y-12$ 26 $a^2-2ab+b^2-c^2+2cd-d^2$
27 $x^2-3xy+y^2+1$ 28 $a^3-3abc+b^3+c^3$
29 $x^6-3x^4y^2+3x^2y^4-y^6$ 30 $2a^6b^5-5a^5b^4c+3a^4b^3c^2-a^2bc^4+ac^5$.

XV. b Page 150 1 $4a^6-16a^4-8a^2-1$.
2 $3x^5-x^4-15x^3+7x^2-10$ 3 $1+a-2a^2-2a^3+a^4+a^5$
4. $3p^4+5p^3q-4p^2q^2+3pq^3-q^4$ 5 $-2x^4-11x^3+2x^2+17x-6$
6 x^6-y^6 7 $x^8+x^7+x^6+2x^3-3x-2$
8 $1-6y+15y^2-20y^3+15y^4-6y^5+y^6$ 9 $x^6-6xy^5+5y^6$
10 $1+x^2$ 11. $6-4x+3x^2-x^3$ 12 $y^2-3y^3+3y^4$
13. $2+x-8x^2$. 14. (i) $1-4x+10x^2-10x^3$, (ii) $1+3a+6a^2+10a^3$.

XV. c. Page 152 1 $2a-1$ 2 $2a+3$ 3 $3b-1$
4 $2x-7$ 5 $2y-5$ 6 $3c-2$ 7 $3d-4$
8 $3x-5$ 9 $2y-1$ 10 $5b-2c$ 11 $2x-7$
12 m^2+3m+2 13 $2x-y$ 14 $4x^2+3x-2$
15 a^2+a+1 16 $4a^3-ax^2+2x^3$ 17 b^2+5b-3
18 $6k^3-6k+7$ 19 $5c^2-2c+3$ 20 $3p^2-2p+5$
21 $2a^2+a-1$, rem $3a+4$ 22 c^2-2c+3, rem $31c-15$
23 $4x^4-2x^2y^2+y^4$ 24 $27x^3+9x^2+3x+1$ 25 $3x^3+6x^2+4x+2$
26 c^4+4c^2+8 27 $x^6+x^3y^3+y^6$
28 $7y^2+5y-3$, rem $-39y+27$ 29 $3m^2-2m-4$
30 $4x^2+14x+9$ 31 $a+2b+3c$ 32 $5a^3+2a^2-4a+3$

XV. d Page 153 1 $1+a+a^2-a^3+a^4+a^6$
2 $x^2-xy+y^2+x+y+1$ 3 $a^2+ab+ac+b^2-bc+c^2$
4. $a^2-ab-2ac+b^2-2bc+4c^2$ 5 $x^2+3xy-2xz+9y^2+6yz+4z^2$
6 $a^6x^6-2a^4x^4-5a^3x^3+4a^2x^2-10ax+25$
7 $-x^2-3xy-2x-9y^2+6y-4$ 8 $-4x^2-2xy-2xz-y^2-yz-z^2$

XV e Page 155 1 p^2-pq+q^2 2 x^2-2x+4
3 $a^3+a^2b+ab^2+b^3$ 4 $x^3-x^2y+xy^2-y^3$ 5 $9+3x+x^2$
6 $8-4d+2d^2-d^3$ 7 $x^4-x^3y+x^2y^2-xy^3+y^4$
8 $a^4+a^3+a^2+a+1$ 9 $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$
10 $c^5-c^4d+c^3d^2-c^2d^3+cd^4-d^5$ 11 $a^6-a^5+a^4-a^3+a^2-a+1$
12 $16+8z+4z^2+2z^3+z^4$ 13 c^2+d^2 14. $x^5+x^4+x^3+x^2+x+1$
15 a^4-4a^2+16 16 $4a^4-2a^2+1$ 17 a^3+b^3 18 c^3-d^3
19 $1-x^4$ 20 a^4-1 21 x^6-y^6 22 $8x^3+27y^3$
23 x^5+1 24. x^6-1 25 x^8-1 26 x^6-125

XV. f. Page 157 1 4, 112, 0
2 Zero in each case; hence $x-2$, $x+2$, $x-3$ are factors of $f(x)$
3 $2(n+1)$ 4. 48 5 0 6 0 7 2
13 11 14. $p=4$, $q=9$ 16 11
17 (i) $(x-1)(x-3)(x+2)$, (ii) $(x-2)(x-3)(x+5)$,
(iii) $(x-1)(x-2)(x+4)$, (iv) $(x-3)(x+1)(x^2+3)$,
(v) $(x+2)(x+6)(2x-3)$, (vi) $(x-2)^2(2x+5)$
18 $a=21$, $b=12$

XVI. a Page 159 1. $4a^6$ 2 $9x^2y^6$ 3 b^4c^6
4 $16a^2b^4$ 5 $49c^6d^{10}$ 6 $36a^6b^{12}$ 7 $25a^4b^{10}c^2$ 8 $9a^2b^6c^{10}$
9 $81p^8q^{12}$ 10 $16a^2b^{16}$ 11 $a^4b^6c^{10}d^8$ 12 $64m^6n^{12}$
13 $\dfrac{9x^2y^2}{16}$ 14. $\dfrac{4m^4n^6}{9p^{10}q^4}$ 15 $\dfrac{1}{9x^6}$ 16 $\dfrac{1}{16y^{16}}$
17 $\dfrac{49k^6l^{10}}{64p^2q^8}$ 18 $\dfrac{1}{81a^{10}b^6c^8}$ 19 $\dfrac{16x^8y^6z^{10}}{49}$ 20 $\dfrac{121}{100b^{10}c^{16}}$.
21. x^9y^6 22 $8x^6$ 23 $27y^9$ 24 $216x^6z^9$

25 $-27a^{27}$. 26 $-8x^6y^9z^3$. 27. $-64p^{12}q^6$ 28. $-125c^{15}d^{18}$

29 $\frac{1}{27a^9b^3}$ 30 $-\frac{8k^6l^9}{p^{15}q^6}$ 31 $-\frac{27x^{12}y^6}{343}$ 32 $-\frac{216}{125a^{16}b^6}$.

33 x^6y^3 34 x^4y^{12} 35 $-32m^{15}n^{10}$ 36 $-27p^6q^9$

37 $\frac{1}{64a^{15}}$. 38 $-\frac{32a^{15}x^5}{p^{10}q^{25}}$ 39 $\frac{1}{a^{10}b^{14}c^{10}}$ 40 $\frac{a^{30}b^{45}c^9}{x^{27}y^{54}z^{18}}$

XVI. b. Page 160

1 $a^2+4ab+4b^2$ 2 $4a^2-4ab+b^2$

3 $x^2+6xy+9y^2$ 4 $4x^2-12xy+9y^2$ 5 $p^2-10pq+25q^2$

6 $16-8x+x^2$ 7 $a^2+14a+49$ 8 $c^2d^2+2cd+1$

9 $4a^2b^2-12ab+9$ 10 $1+2x^2+x^4$. 11 $1+6xy+9x^2y^2$.

12 $x^4-4x^3+4x^2$ 13 $a^2+b^2+c^2+2ab-2ac-2bc$

14 $a^2+b^2+c^2-2ab-2ac+2bc$ 15 $x^2+y^2+4z^2+2xy+4xz+4yz$

16 $x^2+4y^2+z^2-4xy+2xz-4yz$ 17. $4p^2+q^2+r^2\ 4pq-4pr+2qr$

18 $x^4-2x^3-x^2+2x+1$ 19 $4x^4-4x^3+5x^2-2x+1$

20 $k^4+l^4+m^4-2k^2l^2+2k^2m^2-2l^2m^2$

21 $9k^4-30k^3+37k^2-20k+4$

22 $a^2+b^2+c^2+d^2-2ab+2ac+2ad-2bc-2bd+2cd$

23 $4x^2+y^2+9a^2+b^2+4xy-12ax+4bx-6ay+2by-6ab$

24 $m^2+n^2+p^2+4q^2+2mn-2mp+4mq-2np+4nq-4pq$

25 $a^2+\frac{b^2}{4}+\frac{c^2}{16}-ab+\frac{ac}{2}-\frac{bc}{4}$ 26 $\frac{x^2}{9}+9y^2+\frac{9}{4}-2xy-x+9y$.

27 $\frac{9}{4}-3m+3m^2-\frac{4}{3}m^3+\frac{4}{9}m^4$

XVI c Page 161.

1 $p^3+3p^2q+3pq^2+q^3$

2 $m^3-3m^2n+3mn^2-n^3$ 3 $a^3-6a^2b+12ab^2-8b^3$

4 $8c^3+12c^2d+6cd^2+d^3$ 5 $27x^3+54x^2y+36xy^2+8y^3$.

6 $64x^3-48x^2+12x-1$ 7 $1-15y+75y^2-125y^3$

8 $8a^3b^3-36a^2b^2+54ab-27$ 9 $a^6+9a^4b^2+27a^2b^4+27b^6$.

10 $p^6-9p^4q^2+27p^2q^4-27q^6$ 11 $27c^6-54c^4d^3+36c^2d^6-8d^9$.

12 $64x^9-144x^7+108x^5-27x^3$ 13 $a^3-\frac{3}{2}a^2b+\frac{3}{4}ab^2-\frac{b^3}{8}$

14 $\frac{c^3}{27}+\frac{c^2}{3}+c+1$ 15 $\frac{l^6}{27}-l^5+9l^4-27l^3$

16 $\frac{x^3}{216}+\frac{x^2y}{6}+2xy^2+8y^3$

XVI. d. Page 162

1 xy^3 2 $2c^2d^4$ 3 $4a^2b^8$

4 $3x^3y^6$ 5 $8pq^8$ 6 $9x^6$ 7 $12x^{12}y^3$ 8 $6m^{18}$

9 $\frac{1}{9a^9}$ 10 $\frac{8x^9}{5}$ 11 $\frac{4}{m^8n^4}$ 12 $\frac{13y^{13}}{7}$ 13 $\frac{17}{18p^6}$.

14 $\frac{10}{9x^5y^9}$ 15 $\frac{6c^6}{5a^5}$ 16 $\frac{9a^{12}}{10b^{20}c^{10}}$ 17. $2ab^2$

18 $3c^2d^3$ 19 $-4xy^3$ 20 $7a^6$ 21 $-\frac{5}{b^7c^4}$

22 $-\frac{3a^9}{2b^3}$ 23 $\frac{m^5}{2p^3q^4}$ 24 $-\frac{1}{9a^9}$ 25 a^2b^3

26 x^3y^4 27 $2a^3b^2$ 28 $-2d^2$ 29 $2a^9$

30 $-m^2n^3p^4$ 31 $-\frac{3}{x^3}$ 32 $-\frac{ab^2}{c^4}$ 33 $-\frac{2k^4l^3}{3n^{10}}$

XVI e Page 163 1 $x+5$ 2 $y-9$ 3 $11-m^2$ 4 $3-2x^3$

5 $3c+7$ 6 $4-5y^6$ 7 a^2-3b^2 8 x^3-12yz

9 $m^2n^2+13p^4$ 10 $7ab^3-8d^5$ 11 $\frac{a}{2}-3b$ 12. $\frac{m}{3}+\frac{n^2}{2}$.

13 $\frac{3a}{b}-\frac{5c}{d}$ 14 $\frac{3a}{4x^4}-\frac{2x^4}{3a}$ 15 $x+2a-1$ 16 $10c^6-a-b$

17 $x+y-2z$. 18 $a-2b-c$ 19 $a-b-3c$ 20 $p-3q+2r$

XVI f Page 165. 1 a^2-2a+1 2 a^2+a+2 3 x^2-x-1

4. $2x^2-x+1$ 5 x^2-3x+2 6 $2x^2-3x+4$ 7 $2p^2+2p-1$

8 a^2-3a+2 9 $a-x+1$ 10 $x^2-ax+2a^2$ 11 $2c^2+3cd+d^2$

12 $x^3-xy^2+y^3$ 13 $a^3-2a^2b-b^2$ 14 a^3-2a+7

15 $4m^3+2m^4-m^5$. 16 $1-2b+3b^2-4b^3$ 17 x^3-2x^2+3x-4

18 $3-2x+4x^2-x^3$ 19 $2a^3-3a^2+a-4$ 20 $a^3-4a^2x-3ax^2+2x^3$.

XVI. g Page 167 1 $2a^2+a-2$ 2 $1-5x+x^2$

3 $a^5-2a^4-4a^3$ 4. $3c^2-5c+7$ 5 $1+a-\frac{a^2}{2}$

6 $\frac{x^2}{8}+\frac{x}{2}-1$ 7 $2-3b-\frac{9b^2}{4}$. 8 $x^2-\frac{3x}{2}-\frac{2}{3}$

9 $2a^3-a^2+\frac{a}{3}$. 10 $x^2-1+\frac{1}{x^2}$ 11 $2x^2-\frac{3}{x}+\frac{4}{x^4}$.

12 $\frac{x}{2a}+1-\frac{2a}{x}$. 13 $\frac{3x}{a}-1+\frac{a}{3x}$

XVI h Page 168 1 $x-2y$ 2 $2a+b$ 3 $3x-5y$

4. x^2+4y^2 5 $a-\frac{2b}{3}$ 6 $\frac{a}{6}+2x$ 7 $1+a+a^2$

8 $2x^2+x-3$ 9 $a+b-c$ 10 $\frac{a}{b}-1+\frac{b}{a}$ 11 $x^2+10x+25$

12. $a^2-2ab+b^2$ 13 $4c^2-4cd+d^2$ 14 $a^2+b^2+c^2+2ab+2bc-2ac$

15 $2p^2+2q^2+4pr$ 16 a^3 17 $(x+1)(x+3)(x+7)$

18 $(a-2)(a+2)(a+3)$ 19 $2a$ 20 $8c^3$ 21 a^2+3a+1.

24. $2x^2+2x+1$, 221 25 (i) 0, (ii) y^3

XVII a Page 171 1 $(2a+1)(a+1)$ 2 $(2a+1)(a+2)$

3 $(3a+2)(a+1)$ 4. $(3a+1)(a+1)$ 5 $(2a+1)(a+4)$

6 $(2a+3b)(a+2b)$. 7 $(2b+1)(b+3)$ 8 $(2b+5)(b+2)$.
9 $(2b+1)(b+5)$ 10 $(5b-2)(b-1)$ 11 $(3b-2)(b-3)$
12 $(3b-1)(b-3)$ 13 $(2c-1)(c+2)$ 14 $(3c-2)(2c+1)$
15 $(2c-3d)(c-d)$ 16 $(2c+1)(c-1)$ 17. $(2c-7)(c+4)$
18 $(2c-1)(c-8)$ 19 $(2x-1)(2x+3)$ 20 $(3x-2)(2x+3)$
21 $(3x-5)(x+6)$ 22 $(2x-5y)(x-3y)$ 23 $(4x-7)(x+2)$
24 $(5x+1)(x+2)$ 25 $(2y-1)(2y-5)$ 26 $(3y-2)(4y+3)$
27 $(3y-1)(2y+3)$ 28 $(3x-2y)(4x-5y)$ 29 $(12a-7)(2a+3)$.
30 $(15y-2)(y-5)$ 31 $(3+4p)(1-3p)$ 32 $(2+5p)(3+p)$
33 $(4+5p)(1+2p)$ 34 $(5-3p)(3+5p)$
35 $(8-7p)(1+p)$ 36 $(7-p)(4+5p)$

XVII. b. Page 172

1 $(x+y+z)(x+y-z)$
2 $(x-y+z)(x-y-z)$ 3 $(a-b+2c)(a-b-2c)$
4 $(a+b+3c)(a+b-3c)$ 5 $(a-2b+c)(a-2b-c)$
6 $(a+2b+c)(a+2b-c)$ 7 $(c+d+2a)(c+d-2a)$
8 $(c-d+1)(c-d-1)$ 9 $(c+2d+3)(c+2d-3)$
10. $(m+n+p)(m-n-p)$ 11 $(m+n-p)(m-n+p)$
12 $(2m+n+p)(2m-n-p)$ 13 $(1+a+b)(1-a-b)$
14. $(2+a-b)(2-a+b)$ 15 $(3+a+b)(3-a-b)$
16 $a(a+2b)$ 17 $-d(2c+d)$ 18 $y(2x-y)$
19 $m(m-6n)$ 20 $m(m+4n)$ 21 $m(m-8n)$
22 $3a(a+2x)$ 23 $8y(3x-2y)$ 24 $(3a-c)(3a-4b+c)$.
25 $(3x-2y)(x+8y)$ 26 $3m(m-6n)$ 27 $(a+1)(a-10b-1)$
28 $(7a-13b)(a+b)$ 29 $(7a-13b)(b-a)$

XVII. c. Page 173

1 $(a+b+c)(a+b-c)$
2 $(x+c+d)(x-c-d)$ 3 $(2x-y+1)(2x-y-1)$
4 $(1+m-3n)(1-m+3n)$ 5 $(c-d+3)(c-d-3)$
6 $(c+d+4)(c-d-4)$ 7 $(5+y-z)(5-y+z)$
8 $(2p-3q+9)(2p-3q-9)$ 9 $(3y+4c+2d)(3y-4c-2d)$
10 $(11+5a+b)(11-5a-b)$ 11 $(x^2+x-1)(x^2-x+1)$
12 $(x^3+x^2+1)(x^3-x^2+1)$ 13 $(5a-b+x)(5a-b-x)$
14. $(x-2y+3xy)(x-2y-3xy)$ 15 $(x^2-3y+c)(x^2-3y-c)$
16 $(a^2-3b^2+2c^2)(a^2-3b^2-2c^2)$ 17 $(a+b+c+d)(a+b-c-d)$
18 $(a-b+c-d)(a-b-c+d)$ 19. $(x-7+y-z)(x-7-y+z)$
20 $(a^2+a+10)(a^2+a-10)$ 21 $(3a-2+b-4c)(3a-2-b+4c)$.
22 $(7y^3-2+6y)(7y^3-2-6y)$ 23 $(1+x+y-z)(1+x-y+z)$
24 $(x-y+c-d)(x-y-c+d)$
25 $(m^2+n^2+a^2+b^2)(m^2+n^2-a^2-b^2)$
26 $(a^2-3+a+b)(a^2-3-a-b)$
27 $(3a-2x+p+q)(3a-2x-p-q)$
28 $(c+3d+3a)(c+3d-3a)$

XVII d. Page 174

1 $(a^2+ab+b^2)(a^2-ab+b^2)$
2 $(m^2+2mn+4n^2)(m^2-2mn+4n^2)$ 3 $(a^2+ab+2b^2)(a^2-ab+2b^2)$
4 $(p^2+3pq+9q^2)(p^2-3pq+9q^2)$ 5 $(25c^2+5cd+d^2)(25c^2-5cd+d^2)$
6 $(x^2+3xy-y^2)(x^2-3xy-y^2)$ 7 $(2m^2+5mn+n^2)(2m^2-5mn+n^2)$
8 $(x^2+3xy-5y^2)(x^2-3xy-5y^2)$ 9 $(x+13)(x+19)$
10 $(a+17)(a-13)$ 11 $(y-11)(y-15)$ 12 $(a+23b)(a+9b)$
13 $(2c-9)(c-16)$ 14 $(3m-7n)(9m+16n)$ 15 $(6x-13)(2x-7)$
16 $(8p-9)(3p+4)$ 17 $(6a-7b)(5a+12b)$ 18 $6(x-8)(2x-1)$

XVII e Page 175

1 $(y+8)(y-9)$ 2 $(c+12)(c+9)$
3 $(m+17)(m-5)$ 4 $(2z-5)(z+3)$ 5 $(2a-5b)(2a+b)$
6 $(6p-q)(p-2q)$ 7 $(2x^2+3)(4x^2-5)$ 8 $(2m+3)(3m-1)$
9 $(a-3c)(a-19c)$ 10 $(z+17)(z+17)$ 11 $(x-12y)(x+6y)$
12 $(4-x)(1-x)$ 13 $(3+x)(2-x)$ 14 $(4a+3b)(3a-4b)$.
15 $(7x+5)(4x-3)$ 16 $(3x-4y)(2x+3y)$ 17 $x^3(2-x)(3-x)$
18 $x^2(x-9)(x+7)$ 19 $(a+17)(a-15)$ 20 $2x(x-9)(3x+8)$
21 $(18+x)(4-x)$ 22 $(2b-3)(3b+1)$
23 $(a^2+3)(a+1)(a-1)$ 24 $(c^2+7d^2)(c+3d)(c-3d)$
25 $2(5p+1)(25p^2-5p+1)$ 26 $4(5ab^2+1)(5ab^2-1)$
27 $(9+cd)(81-9cd+c^2d^2)$ 28 $x(3x+2y)(3x-2y)$
29 $(a+x+1)(a+x-1)$ 30 $(4+b-c)(4-b+c)$
31 $(l+17)(l-16)$ 32 $(p^2+3pq+q^2)(p^2-3pq+q^2)$
33 $(a^2+a+2)(a^2-a+2)$ 34 $(4x^2+2xy+y^2)(4x^2-2xy+y^2)$
35 $a^2(ax+2y)(a^2x^2-2axy+4y^2)(ax-2y)(a^2x^2+2axy+4y^2)$
36 $ab(3a+b)(9a^2-3ab+b^2)(3a-b)(9a^2+3ab+b^2)$
37 $20y(5x+y)(5x-y)$ 38 $\{(a+b)^2+1\}(a+b+1)(a+b-1)$
39 $(c+d-1)\{(c+d)^2+c+d+1\}$
40 $(1-x+y)\{1+x-y+(x-y)^2\}$ 41 $(x-19)(x+13)$
42 $(a+9)(a-31)$ 43 $2\{5(a-b)+1\}\{25(a-b)^2-5(a-b)+1\}$
44 $2c(c^2+3d^2)$ 45 $y(12x^2-6xy+y^2)$
46 $(x-2y)(x+2y+1)$ 47 $(a-b)(a+b+1)$
48 $(a+b)(a+b+1)$ 49 $(a+b)(a^2-ab+b^2+1)$
50 $(a+3b)(a-3b+1)$ 51 $(x-y)\{2(x-y)+1\}\{2(x-y)-1\}$.
52 $xy(x+y)(x-y)(x-y)$ 53 $(x+1)(x+7)(2x-3)$
54 (i) $3(x+2a)(x-a)^2$, (ii) $(x+3)(x+4)(x-7)$,
(iii) $(a+2)(3a-1)(2a-3)$, (iv) $(a+b)(a-b)(a+2b)(a-3b)$

XVII f Page 177

1 $x^2-y^2+2yz-z^2$
2 $x^2-y^2-2yz-z^2$ 3 $4a^2+4ab+b^2-c^2$
4 $a^2-6ab+9b^2-1$ 5 $1-3a^2+a^4$
6 $a^4-4a^2-12a-9$ 7 $x^2+2xy+y^2-c^2+2cd-d^2$
8 $x^2-2xy+y^2-a^2+2ab-b^2$ 9 c^4-d^4 10 $c^8-2c^4d^4+d^8$
11 a^6-b^6 12 x^4+4 13 a^6-729 14 $8x^2-5x+21$

XVII. g. Page 179. 1 1, 2. 2 $-3, -4$ 3 0, 3 4 0, 5
5 $0, -8$ 6 $0, \frac{6}{5}$ 7 $\frac{1}{2}, -4$ 8 $\frac{2}{3}, -\frac{3}{2}$
9 $\frac{6}{5}, \frac{6}{5}$ 10 1, 6 11 $7, -4$ 12 $11, -9$
13 $12, -11$ 14 $-4, -4$ 15 15, 8 16 $3, \frac{1}{3}$.
17 $\frac{3}{2}, \frac{2}{3}$ 18 $\frac{1}{2}, -\frac{2}{3}$ 19 $4, -\frac{3}{2}$ 20 $1, -\frac{5}{3}$.
21 $3, -\frac{5}{4}$ 22 $\frac{13}{2}, -3$ 23 $\frac{5}{2}, -3$ 24 $\frac{5}{8}, -3$
25 $6, \frac{3}{5}$ 26 $\frac{9}{10}, -\frac{5}{6}$ 27 $\frac{8}{3}, -\frac{3}{4}$ 28 $\frac{1}{2}, \frac{1}{4}$
29 $5, -\frac{5}{2}$ 30 $4, \frac{7}{5}$ 31 1 32 0
33 -4 34 $-\frac{5}{3}$ 35 $\frac{1}{3}$ 36 6, 9
37 13, 14 38 3 39 6 40 15, 17
41 39 42 13 43 24 yds, 35 yds. 44 55 ft, 30 ft.

Miscellaneous Examples IV. Page 180.

1 (i) $(x+12)(x-11)$, (ii) $(a+2b)(2a-b)$, (iii) $(b+c+3a)(b+c-3a)$
2 31 3 $2a^2-3a(x+y)-1$ 4 (i) 15, $x=2$, $y=3$
5 91 6 80 shillings, 16 half crowns 7 $x=\frac{1}{2}$, $y=1$
8 $x^5-11x-10$, $3p^3-5p^2+2p$ 9 1
10 $(ap+bq)$ miles, $\dfrac{ap+bq}{c}$ hours 55 mi, 5 hrs
11 (i) $(2a+3b)(2a+3b)$, (ii) $(a^2+a+1)(a^2+a-1)$,
(iii) $(a-b)(a-c)$, (iv) $(a+b-1)(a-b-1)$
12 (i) $7\frac{1}{13}$, (ii) $x=7$, $y=9$ 13 225 14 Sheep £4 10s, cow £16.
15 $9a^2b^2-16b^2c^2+40abc^2-25a^2c^2$ 16 $(2x+5)(2x+1)(x-3)$
17 $4(a^2+b^2+c^2)$ 18 £1 10s
19 (i) $(a^2+ab+b^2)(a^2-ab+b^2)$, (ii) $(2x-3)(3x-2)$,
(iii) $10x(x-2y)(x+2y)$
20 (i) $x=0{\cdot}25$, $y=-0{\cdot}4$, (ii) $x=\frac{1}{2}$, $y=3$ 21 6s 2d, 20 3
22 (i) 5; (ii) 1, or 3, (iii) 2, or 7, (iv) $-\frac{5}{2}$ 23 99,980,001
24 $2x^6-16x^5+27x^4+20x^3-20x^2-20x-25$
26 (i) $\frac{1}{5}, -1$, (ii) $x=\frac{1}{2}$, $y=2$, (iii) $x=6$, $y=7$, $z=8$
27 240 28 18 29 52, 2s 4d, 5s 30 $6(m^4-16)$.
31 (i) 4, (ii) $x=2$, $y=3$, $z=5$ 32 1
33 (i) $1+x+\frac{11}{12}x^2+\frac{1}{3}x^3+\frac{1}{9}x^4$, (ii) $\frac{1}{8}a^6-\frac{1}{2}a^4b^3+2a^2b^6-\frac{8}{27}b^9$
34 4 36 (i) 29s, (ii) 14 4 dollars
37 112, 168, 78 $y=\frac{10}{11}x-70$

XVIII a. Page 184. 1 xy 2 $2x^2$. 3 ab 4 cd 5 $2pq^2$.
6 5 7 $3ab$ 8 a^2bc. 9 $7x^2yz^3$ 10 ab
11. x^2y 12 $3x^2$ 13 p^3 14 $2c$. 15. $4ab$
16 $5a^3$ 17 $3xy^3z$ 18 $17p^2r^3$ 19 $5c^3$ 20 $7xyz^2$

XVIII b Page 185

				1	$a+b$	2	$c-d$
3	$x(x+y)$	4	$2x+1$	5	$n(m-2n)$	6	$2p+3q$
7	$c^2(c+d)$	8	$a-3b$	9	$ab(b+c)$	10	$n+2m$
11	$(a+2)$	12	$x-3$	13	$ax(a-x)^2$	14	$d^2(c-d)$
15	$(m-n)^2$	16	$x^2(x+y)$	17	$x-3a$	18	$2a+b$
19	$c-1$	20	$d+2$	21	$p+9$	22	$m+3$
23	$x+11$	24	$x(x-3)$	25	$3a+1$	26	$2c+1$
27	$a+b$	28	$c-d$	29	$x+2y$	30	$a-4$
31	$2ab-3$	32	$x(x-3)$	33	$a(b-a)$	34	$(4p-q)^2$
35	$x-2$	36	$x(x+4)$	37	$x+2$	38	$2+c$
39	$a+3$	40	$x+1$	41	$a-5$	42	$x-2$
43	$a-3$	44	$a+7$				

XVIII c Page 188

		1	x^2-3x+2	2	$2a^2-a+3$
3	c^2-2c+1	4	a^2-3a+7	5	$b^2-13b+5$
6	x^2+2x-1	7	$a+5$	8	$b-8$
9	x^2-3x+5	10	$m(2m^2+3m-4)$	11	$x(x^2+4x+4)$
12	$x^2-3xy+y^2$	13	c^2-2c+5	14	$a(a^2-3a+3)$
15	$y^2+4y-20$	16	$3a-5b$		

XVIII. d Page 189

		1	x^2-x-1	2	$2x^2-3$
3	$y+1$	4	$c^2+7cd+21d^2$	5	$2x^3-4x^2+x-1$
6	$x^3(x^4-5x^2+8x-4)$	7	$2x-3y$	8	$ab(2a^2+ab-3b^2)$
9	$2m^2(2m+7)$	10	$2x^2-7$	11	$2x^2-9x+9$
12	$y^2-10y+21$	13	$1+a$	14	$x(3+4x)$
15	x^2-2x+1	16	$1-x^3-x^4$	17	$x-2$
18	$a-3$	19	y^2-3y+2		

XIX a Page 192

		1	$\frac{2x^2}{3y}$	2	$\frac{2d}{3c}$	3	$\frac{b^2}{3a}$
4	$\frac{3m^2}{n}$	5	$\frac{m^3}{5}$	6	$\frac{2p^4}{3}$	7	$\frac{3c^2}{4e^2}$
8	$\frac{x^2}{ay}$	9	$\frac{3abc}{2}$	10	$\frac{3x}{13a^2b}$	11	$\frac{5dx^3}{8c}$
12	$\frac{a^2y^4}{19x}$	13	$\frac{3}{4k^2m}$	14	$\frac{3x^2}{4ay^4}$	15	$\frac{2bd^2}{3c^2}$
16	$\frac{ln^2}{5k^3}$						

XIX. b. Page 193

1	$\frac{1}{ab-1}$	2	$\frac{x}{y}$	3	$\frac{y}{x}$	4	$\frac{1}{4c}$
5	$\frac{3a}{4b}$	6	$\frac{a}{3b}$	7	$\frac{1}{2ab}$	8	$\frac{3}{2y}$
9	$\frac{q}{m}$	10	$\frac{a+2b}{a}$	11	$\frac{3x}{2x-y}$	12	$\frac{2x+y}{x(2x-y)}$
13	$\frac{2(xy-2)}{3x}$	14	$\frac{5p}{p-5}$	15	$\frac{m+2n}{m(m-2n)}$	16	$\frac{x}{x+1}$

17 $\frac{a-2}{a-3}$ 18 $\frac{x-7a}{x^2(x+3a)}$ 19 $\frac{x^2-xy+y^2}{x-2y}$ 20 $(x+1)(x+2)$.

21 $\frac{x+9}{3x+1}$ 22 $\frac{2a-7}{2a+7}$ 23 $\frac{3(x^2+2x+4)}{2(x+5)}$ 24 $\frac{d^2}{3c+d}$

XIX. c. Page 194.

In each of the following examples the H C F is given in [ ]

1 $\frac{2x+1}{x^2+x-2}[x+1]$ 2 $\frac{2a^2+3a-5}{7a-5}[a-1]$.

3 $\frac{c-1}{3c^2+3c+10}[c-1]$ 4 $\frac{2x+7}{x(x+2)}[x^2-2x+7]$

5 $\frac{x^2+3}{x(x^2+x+1)}[x-2]$ 6 $\frac{x+2}{3x-1}[x^3-2x^2+3x+1]$

7 $\frac{x-3}{x(2x+1)}[x^3+3x^2-2x+1]$ 8 $\frac{2a^2-ab-3b^2}{4a^2-3ab+b^2}[a^2+ab]$

9 $\frac{4x+3}{x(5x+2)}[x^2-5]$ 10 $\frac{7a-4}{5a+2}[a^2-3]$

11 $\frac{x-3}{x+1}[x^2-3x+2]$ 12 $\frac{3y+1}{y+3}[y^2-3y+1]$

13 $\frac{a^2-a+5}{a(5a^2+4a+16)}[a^2+a-4]$ 14 $\frac{3x+4}{4x-1}[2x^2-3x+4]$

15 $\frac{3c^3+6c^2+4c+2}{2c^3+4c^2+6c+3}[c^2-2c+1]$ 16 $\frac{3-7x-9x^2}{4-11x-8x^2}[1+x]$

XIX. d. Page 195.

1 $\frac{xy}{b}$ 2 $\frac{b^2d^2}{2c}$ 3 $\frac{c}{a}$

4 $\frac{5}{2x^2}$ 5 cd^2 6 $\frac{4abc}{d}$ 7 $\frac{x}{2y^3}$ 8 $\frac{z^3}{6x^3}$.

9 $\frac{acd}{5b^2}$ 10 $\frac{1}{32yz}$ 11 $\frac{1}{n}$ 12 1

XIX. e Page 196

1 $\frac{3a}{4x^2}$ 2 $\frac{3b^2}{5a^3}$

3 $\frac{x(a-2)}{a^2}$ 4 $\frac{b-10}{a+b}$ 5 $\frac{1}{2(x+1)}$ 6 $\frac{c(3c+4d)}{c+5}$.

7 $\frac{m^2n^2}{2(m+2n)}$ 8 $\frac{ab-3}{2a-1}$ 9 $\frac{a^2+ab+b^2}{a}$ 10 $\frac{3(c-3)}{2(c-4)}$

11 $\frac{x^2-1}{x^2-4}$ 12 $\frac{1}{x-3}$ 13 $\frac{x+1}{x+5}$ 14 x

15 $\frac{y-6}{y+3}$ 16 $\frac{a+2}{a^2-3a+9}$ 17 $\frac{1}{m-2}$

18 1 19 b^2+3b+9. 20 $\frac{(x-2)(x+3)}{(2x+1)(3x-1)}$.

21 1 22 x 23 $x-y$ 24 1

25 $a+b$ 26 x^2y 27. $\frac{c^2+2c+4}{2c^2}$ 28 1.

29 $\frac{a}{a+b}$ 30 $\frac{a+b+c}{a}$ 31 $\frac{x+y-a}{x+y-c}$ 32 a^2-ab+b^2

33 $\frac{m}{n}$ 34 $x(2+x)$ 35 $\frac{x+4}{x(x-4)}$

XX a Page 198. 1 a^3bc 2 $2a^3b$ 3 $6x^2y^3$ 4 $12x^2yz$

5 $15a^3bc^3$ 6 $15c^4d^3$ 7 xyz 8 $x^2y^2z^2$

9 ab^2xy 10 $12abc$ 11 $6x^2y^2z^2$ 12 $42a^2b^3$

13 $p^2q^2r^2$ 14 $24a^3b^2c^3$ 15 $340a^3b^5$ 16 $39c^5d^4$

17 $162m^3n^2p^2x^2$ 18 $96a^4b^3c^5$ 19 $84a^2bc^3$ 20 $72a^4b^2c^3$

XX. b Page 199 1 $a^3(a-2)$ 2 $y(y^2-1)$

3 $ab(a+b)$ 4 $28c^3(c+1)$ 5 $4p^2q(2p+1)$

6 $p(p^2-4)$ 7 $(x+2)(x^3-8)$ 8 $6c^2d(c+2d)$

9 $(x-1)(x^3-1)$ 10 $(a+1)^2(a+2)$ 11 $(x+9)(x-9)(x-12)$

12 $x(x+2)(x-2)$ 13 $x(x+1)(x-1)^2$ 14 $(x-1)(x+2)(x-3)$

15 $(x+3)(x-2)(x-7)$ 16 $(a+b)(a-2b)(a-3b)$

17 $(m+2)(m-11)(m+3)$ 18 $(c+3d)(c-5d)(c-13d)$

19 $(x-3)(x-15)(x-11)$ 20 $(x-6)(x-13)(x-8)$

21 $(x+y)(x-2y)(2x-y)$ 22 $(x+2)(x-2)(3x-7)$

23 $(2x+1)(x-3)(x+4)$ 24 $(3m+2)(m+1)(m-2)$

25 $2x(2x+1)(x-3)(3x-1)$ 26 $9(a^2-4x^2)(2a-x)^2$

27 $180a^3x^3(a^2-x^2)^3$ 28 $x^3y(x^6-y^6)(x-y)^2$

29 $12a^2(a-3x)^3(a+3x)^3$ 30 $8c^2(2c-3d)^2(8c^3-27d^3)$

XX. c. Page 200 1 $(a-2)(a+3)(a-3)(a-4)$

2 $(x+1)(x+2)^2(x-2)(x-3)$ 3 $(d-1)^2(d^2+d+1)(d+3)$

4 $(x-2y)(x+3y)(x+4y)(x-5y)$ 5 $y^2(x-2y)(x-3y)(x+3y)^2$

6 $(c-5)(c+7)(c^2-c+1)$ 7 $(x+y)(x^3-y^3)(x-2y)$

8 $(2x-3)^2(3x-2)(x+4)$ 9 $x^2(2x+3)^2(2x-3)^2(3x-1)$

10 H C F (x^2-2x+3), L C M $(x+3)(x-3)(x^2-2x+3)$

11 H.C F x^2+1; L C M $x^3(x^2+1)(x+1)(x-1)(x-2)$

12 H.C F $ab(3a+2b)$, L C M $a^2b^2(a-4b)(3a-2b)(3a+2b)^2$

XXI a Page 201 1 $\frac{5a^2,\ 4a^2}{10}$ 2 $\frac{4ab,\ 3cd}{36}$

3 $\frac{a^2b,\ 2bc}{2a}$ 4 $\frac{2a,\ 3ax}{2x^2}$ 5 $\frac{ayz,\ 2xz,\ c^2xy}{xyz}$

6 $\frac{2x^3,\ 2a^2x,\ a}{2ax^2}$ 7 $\frac{3a^2cx,\ 6ab^2,\ 2bc^2x}{6abcx^2}$ 8 $\frac{3x-9,\ 4x+16}{12}$

9 $\frac{6ab+3b^2,\ 2a^2-4ab}{6a^2b}$ 10 $\frac{a(x-a),\ x(x+a)}{(x+a)(x-a)}$

11 $\frac{c^2d(c-d),\ cd^2(c+d),\ c^2-d^2}{cd(c+d)(c-d)}$ 12 $\frac{4a^2(x+a),\ 9x^3}{6ax(x-a)(x+a)}$

13 $\frac{5x^2(a+b),\ 5xy(a-b),\ ay^2}{5a(a+b)(a-b)}$ 14 $\frac{(y+2)^2,\ (y+1)^2}{(y-3)(y+1)(y+2)}$.

15 $\frac{4d^3(c-7),\ 3c^4(2c-1)}{cd(2c-1)(c+7)(c-7)}$ 16 $\frac{3x(x+y)^2,\ 2(x^3-y^3)}{12(x-y)(x-y)(x+y)}$

XXI. b. Page 203

1 $\frac{2a}{3}$ 2 $\frac{b}{3}$ 3 $\frac{6x-5}{20}$

4 $\frac{cx+ay+bz}{abc}$. 5 $\frac{a^2bc+b^2-c^2}{abc}$. 6 $\frac{x^2-3xyz+z^2}{xyz}$.

7 $\frac{2x^2z+3a}{az}$ 8 $\frac{x(6y-x)}{2y}$ 9 $\frac{2a^3-3b^2}{3a^2}$

10 $\frac{c^2-3a^2+6c}{3c}$ 11 $\frac{x^3-2y^2+2x}{2x^2}$ 12 0

13 $\frac{3y+2}{5}$ 14 $\frac{5-b}{12}$ 15 $\frac{10x+13}{35}$. 16 $\frac{35c-81}{60}$

17 $\frac{5(a-1)}{2a}$ 18 $\frac{a+2x}{3a}$. 19 0 20 $\frac{7a-b}{2b}$

21 $\frac{5a+31}{102a}$. 22 -4 23 $\frac{6yz-6zx-xy}{6yz}$ 24 $\frac{3(2y+}{2y}$

XXI. c Page 204.

1 $\frac{2a}{a^2-b^2}$ 2 $\frac{x+10}{x^2-4}$ 3 $\frac{a(x-y)}{(a-x)(a-y)}$.

4 $\frac{2(a^2+b^2)}{a^2-b^2}$ 5 $\frac{8y}{y^2-4}$ 6 $\frac{x}{x^2-9}$ 7 $\frac{ab}{a^2-b^2}$

8 $\frac{4(a+1)}{a^2-16}$ 9 $\frac{3d(2c-3d)}{c^2-9d^2}$ 10 $\frac{4(x-1)}{(x-2)^2(x+2)}$

11 $\frac{x^2+y^2}{(x-y)^2(x+y)}$ 12 $\frac{b-2}{(b+1)^2}$ 13 $\frac{4y^2}{x^2-4y^2}$ 14 $\frac{2y^3}{1-y^4}$

15 $\frac{2ab}{a^4-b^2}$ 16 $\frac{5p^2}{25p^2-q^2}$ 17 $\frac{4d^2}{c(c+2d)}$ 18 $\frac{4(x-1)}{(x-2)^2(x+2)}$

19 $\frac{x^2+a^2}{ax(x+a)^2(x-a)}$ 20 $\frac{1}{1-x^4}$ 21 $\frac{x-9}{(x^2-9)(x-3)}$

22 $\frac{8m}{(m+1)(m+2)}$ 23 $\frac{2y}{y^2-1}$ 24 $\frac{7}{(x-4)(x+3)}$

25 $\frac{1}{(y-3)(y-2)}$ 26 $\frac{8}{(x+2)(x+10)}$ 27 $\frac{13}{(y+9)(y-4)}$

28 $\frac{2}{(x+1)(x-2)(x-4)}$ 29 $\frac{a^2+5ab-3b^2}{(a+3b)(a+6b)}$ 30 1 31 $\frac{a-x}{a+x}$

32 $\frac{17(1-2x)}{12(1-x^2)}$ 33 $\frac{23(1+2x)}{12(1-x^2)}$ 34 1 35 1

36 $\frac{(5a-1)(2a-1)}{a(a-1)(3a-1)}$ 37 $\frac{x^2-13xy+6y^2}{2x(x-y)(2x-y)}$ 38 $\frac{1}{x^3-1}$.

39 $\frac{1}{m^3-1}$ 40 2 41 0 42 0 43 $\frac{2}{1-y^4}$

44. $\dfrac{4x^3}{16-x^4}$. 45 $\dfrac{48a}{81a^4-16}$. 46 $\dfrac{8a^2}{(a+1)(a-1)^2}$

47 $\dfrac{200m}{81m^4-625}$ 48 $\dfrac{16(a^2+5)}{3(16-a^4)}$ 49 $\dfrac{1}{y+1}$

50 $\dfrac{1}{c+d}$. 51. $\dfrac{1}{2b+1}$ 52 $\dfrac{a^3-2b^3}{2(a^4-b^4)}$

XXI. d. Page 208 1 $\dfrac{a}{1-a^2}$ 2 $\dfrac{1}{1-x^2}$ 3 $\dfrac{4y}{4-y^2}$

4. $\dfrac{1}{x+3}$. 5 $\dfrac{2}{ab}$. 6 $\dfrac{1}{p}$ 7 $\dfrac{1}{1-4x^2}$

8 $\dfrac{1}{1-9y^2}$ 9 0 10 $\dfrac{1-6m^2}{1-4m^2}$ 11 $\dfrac{2}{1-a^4}$

12 $\dfrac{6(x-2)}{x^4-81}$. 13 $\dfrac{4}{x+1}$ 14. $\dfrac{4}{a-1}$ 15 3

16 $\dfrac{12(2m+1)}{4m^2-9}$. 17 $\dfrac{48}{(x^2-9)(x^2-1)}$. 18 $\dfrac{120}{(y^2-16)(y^2-1)}$

19 0 20 0 21 $\dfrac{x^3}{(x^3-1)(x+1)}$ 22 $\dfrac{y^3}{(1+y^3)(1-y)}$.

23 $\dfrac{2}{a}$ 24. $\dfrac{1}{a^3-1}$. 25 $\dfrac{1}{1-b^3}$ 26 $\dfrac{x}{(a-x)^2}$

XXI e Page 210 1 $\dfrac{2(a+2)}{(a-1)(a-2)(a+5)}$. 2 $\dfrac{4x^2}{(x-2)^2(x^2+4)}$.

3 $\dfrac{1}{(1-x)^2}$. 4. $\dfrac{2}{(2y-1)(y+1)(2y-3)}$ 5 $\dfrac{c}{(c-2)^2(c-4)}$.

6 $\dfrac{x+52}{(x^2-16)(x-4)}$. 7 0 8 $\dfrac{2}{(a-2)(a-3)(a-4)}$ 9 $\dfrac{8x^2}{(x^2-4)^2}$

10 0 11 $\dfrac{2(bc+ca+ab-a^2-b^2-c^2)}{(a-b)(b-c)(c-a)}$ 12 0 13 0 14 0

15 $\dfrac{2(bc+ca+ab-a^2-b^2-c^2)}{(a-b)(b-c)(c-a)}$ 16 $\dfrac{p(y-z)+q(z-x)+r(x-y)}{(y-z)(z-x)(x-y)}$

XXII. a. Page 212 1 $\dfrac{c}{ac+b}$ 2 $\dfrac{y}{x-yz}$ 3 $\dfrac{x^2}{x-1}$

4. $c(1-d)$ 5 $\dfrac{1}{a-b}$. 6 $\dfrac{1}{x-a}$ 7 $3+a$

8 $\dfrac{2x}{3y}$ 9 $\dfrac{4b}{3a}$ 10 $\dfrac{ac-b}{b}$ 11. $\dfrac{bx+ay}{xy+ab}$.

12 $\dfrac{xy}{ab}$ 13 $\dfrac{a+2}{a+5}$ 14 $\dfrac{2x(x-2)}{(x+4)}$ 15 $\dfrac{a-3}{a^2(a-5)}$.

16 $\dfrac{x-8}{x-6}$. 17 $\dfrac{1}{2m^2-1}$ 18 1

XXII. b. Page 213. 1 $\frac{1}{a+1}$ 2 $\frac{x-2}{a-1}$ 3 $\frac{a(2a-1)}{a-1}$.

4 $\frac{b(2a-b)}{a^3}$ 5 $\frac{2b^2}{3b-a}$ 6 $\frac{2x^2-y^2}{2x+y}$ 7 $\frac{a^2c^3+a^2-1}{ac^4+a+c}$

8 $\frac{x^2+y^2}{2y^2}$ 9 $\frac{4}{3(x+1)}$ 10 $\frac{1}{bc}$ 11 yz

12 $\frac{6a^2+8a-1}{9a^2+7a+1}$ 13 $\frac{6(x-1)}{x-4}$ 14 $2xy$

XXII. c. Page 216. 1 $1+\frac{6}{x+2}$ 2 $1+\frac{10}{x-2}$ 3 $1-\frac{10}{a+3}$.

4 $2+\frac{3}{x+1}$ 5 $6+\frac{11}{x-3}$ 6 $3+\frac{3}{x+2}$ 7 $2+\frac{14}{2x-1}$

8 $3+\frac{12}{3x-2}$ 14 $x-x^2+x^3-x^4$, Rem x^5

15 $1+2x+2x^2+2x^3$, Rem $2x^4$ 16 $1+x-x^3-x^4$, Rem x^6

17 $1+2x+3x^2+4x^3$, Rem $5x^4-4x^5$

18 $1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}$, Rem $\frac{b^4}{a^3}$ 19 $x-3+\frac{9}{x}-\frac{27}{x^2}$, Rem $\frac{81}{x^2}$

20 $\frac{x-3}{x-4}$ 21 $(2x+5a)(x-a)$ 22 $\frac{(2x-3)(2x+7)}{6}$.

XXII. d Page 216. 1 $\frac{6x+1}{(2x+1)^2(2x-1)}$ 2 $\frac{x+4}{(x+1)(x+2)}$

3 $\frac{x^2+y^2}{x^2-y^2}$ 4 $\frac{2x-3}{(2x+3)(x^2-1)}$ 5 $\frac{x-3y}{x+3y}$ 6 $\frac{2}{x}$

7 $\frac{x^4+x^3+x}{x^4+x^2+1}$ 8 $\frac{3}{a(x+1)}$ 9 1 10 $\frac{2}{x-3}$

11 $\frac{x^2-7x+4}{2x^2-15x-9}$ 12 $\frac{z^2-a^2}{z^2+a^2}$ 13 0 14 $\frac{1}{1+x}$

15 $\frac{1}{x}$ 16 $\frac{x-1}{x^2}$ 17 $\frac{x(x+y+z)}{z(x-y+z)}$ 18 $\frac{5bc-2ca-3ab}{(a-b)(b-c)(c-a)}$

19 $\frac{x-5}{x+5}$ 20 1 21 $\frac{1}{x+y}$ 22 2 23 $x(1+x-x^2)$.

24. $\frac{2(x^2+2xy+2y^2)}{y(x+y)}$ 25 x^2y^2 26 0 27 $\frac{x}{9}$

28 $\frac{4}{(1-a^2)^2}$ 29 1 30 $\frac{x^2-4xy+y^2}{x^2+y^2}$ 31 $\frac{2(x^4+1)}{x^8-1}$.

32 $a-b$ 33 0 34. $\frac{3n-m}{2}$, 35 $\frac{a-b}{1+ab}$

36 1 37 $2(ac+bd)(ad+bc)$ 38 $\frac{2+x+3x^2}{2(1-x^4)}$.

39 $\frac{1}{(3x-y)(x-3y)}$ 40 $\frac{2x}{x+5y}$ 41 $\frac{2x^2}{8x^3-y^3}$

42. $\frac{16x^4}{x^8-256}$ 44 0

XXIII. a. Page 223.

1	6	2	8	3	-2	4	16		
5	3	6	7	7	11	8	2	9	2
10	3								
11	$-6\frac{5}{8}$.	12	$1\frac{1}{2}$	13	-3	14.	4	15	$3\frac{1}{3}$
16	$-2\frac{1}{2}$	17	-2	18	6	19	4	20	$4\frac{1}{2}$
21	-7	22	$3\frac{1}{2}$	23	4	24	-11	25	9
26	$\frac{3}{2}$	27	$\frac{1}{2}$	28	11	29	2	30	$2\frac{1}{2}$

XXIII. b Page 225

1	$a-b$	2	$\frac{5a}{4}$	3	$\frac{ac}{b}$				
4	$-a$	5	$a-2b$	6	$\frac{bc}{a}$	7	c	8	$2a+b$.
9	$2p-q$	10	$-\frac{a}{a+b}$	11	$p-q$	12	$a+b$	13	$\frac{6}{7}a$
14	$\frac{5(m-n)}{4}$.	15	abc	16	$\frac{4abc}{bc+ca+ab}$	17	a		
18	$\frac{a+b}{a-b}$.	19	$-\frac{ac}{2b}$.	20	q	21	$\frac{c(a-b)}{a}$		
22	$\frac{a^2-b^2}{4a-b}$.	23	$a+2b$	24	$bc+ca+ab$	25	$-a$		
26	$-c$	27	$\frac{a}{3}$	28	$-2a,\ \frac{a}{3}$	29	$2a,\ 3b$		
30	$-m,\ \frac{n}{2}$	31	$c+2d,\ c-2d$	32	$3c,\ \frac{3c}{2}$				
33	$13a,\ -a$	34	$3p,\ -14p$	35	$-2c-d,\ -2c+d$.				

XXIII c. Page 227.

1	$x=\frac{l+m}{2a},\ y=\frac{l-m}{2b}$				
2	$x=a,\ y=a$	3	$x=a,\ y=2b$	4	$x=a,\ y=b$
5	$x=b,\ y=a$	6	$x=a-b,\ y=0$	7	$x=c+d,\ y=0$
8	$x=c+d,\ y=d$	9	$x=p-q,\ y=q$	10	$x=a,\ y=b$
11	$x=\frac{a}{a+b},\ y=\frac{b}{a+b}$	12	$x=\frac{nm'-n'm}{lm'-l'm},\ y=\frac{ln'-l'n}{lm'-l'm}$		
13	$x=c(c-d),\ y=d(c+d)$	14.	$x=\frac{21a-10b}{5},\ y=\frac{20b-9a}{5}$.		
15	$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1},\ y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$	16	$x=1,\ y=2$		
17	$x=\frac{3n+m}{4},\ y=\frac{3n-m}{4}$	18	$x=\frac{m^2-n^2}{ma-nb},\ y=\frac{m^2-n^2}{mb-na}$.		
19	$x=\frac{pr}{p^2-q^2},\ y=-\frac{qr}{p^2-q^2}$	20	$x=\frac{al-bm}{cl},\ y=\frac{al-bm}{cm}$.		
21	$x=\frac{a+b}{2},\ y=\frac{b-a}{2}$	22	$x=p-2q,\ y=2p-q$.		
23.	$x=\frac{7a+8b}{9},\ y=\frac{8a+7b}{9}$	24.	$x=c,\ y=b$		

XXIII. d. Page 230. 1 7 miles 2 72
3 A goes 15 miles and takes $3\frac{1}{3}$ hrs 4 $2\frac{1}{3}$ miles
5 $1\frac{1}{4}$ miles 6 60 miles 7 72 miles.
8 £225 at 4 %, £330 at 5 % 9 £600 at 3 %, £120 at 5 %.
10 Cloth, 15*d*, canvas, 6*d* 11 15 horses, 20 cows
12 £3000 at 3 %, £2000 at $3\frac{1}{2}$ % 13 Currants, 5*d*, figs, 8*d*
14 288 15 240 16 12
17 £2 10*s* 18. £17 10*s* 19 £1800

XXIII e. Page 232. 1 720 miles 2 20 persons, 3*s* each.
3 100 boys, 6*d* each 4 Length, 150 yds, breadth, 50 yds.
5 30*s*, 18 boys 6 4′ past 10 7 45 min
8 660 yds 9 600 10 27 11 36,000
12 325 13 £140 at $2\frac{1}{2}$ %: £275 at 4 %.
14 2 miles per hour 15 At 12 o'clock, 125 miles from Bristol.
16 240 17 45 miles, at 5 miles per hour

Miscellaneous Examples V. Page 234.

1 (i) $(p^2+7q^2)(p^2-8q^2)$, (ii) $6(2y-1)(y-2)$, (iii) $(m+n+1)(m+n-1)$; (iv) $(x+3y)(1+x^2-3xy+9y^2)$
2 H C F $3x-5$, L C M $(3x-5)(x+2)(4x^2-9)$
3. (i) $-7, 3$, (ii) $0, -8$, (iii) ± 5, (iv) $0, \frac{3}{2}$
4 (i) $\dfrac{1}{2a^2-1}$, (ii) $\dfrac{1}{a^4-c^4}$ 5 (i) $-\frac{5}{2}$, (ii) $x=\frac{7}{2}, y=4$.
6 91 7 $8-8x^2+10x^3$ 8 $5y^4+4y^3+3y+2$
9 (i) $x^4-x^3-x^2-4x+6$, (ii) $2x^3+7x+3$ 11 $2(x^2-16)(x^2-9)$
13 $x=-1, y=2\frac{1}{2}, z=-\frac{1}{2}$ 15 -1
16 (i) $(5x+1)(2x-3)$, (ii) $(3b-c+4)(3b-c-4)$
17 $a-2b-c$ 18 $(x-2)(x+3)(x+6)$
19 (i) $\dfrac{1}{(1-x)^2}$, (ii) $\dfrac{x^2-x+1}{x(2x+1)}$, (iii) 2 20 (i) 5, (ii) $x=2\frac{1}{2}, y=1\frac{1}{2}$
21 $2\frac{2}{3}$ mi per hour 22 y^2-3y+1 23 2,999,080.
24 (i) $(x-4)(5x-8)$, (ii) $(2x-1)(x^2+4)$ 25 (i) $6p$; (ii) $\dfrac{x(3x+1)}{4x^2+2x-1}$.
26 (i) $4a^4-12a^3b+9a^2b^2-25b^4$, (ii) $1-64a^{12}$
27 £36 28 $(0, 2), (\frac{1}{2}, 1), (-1\frac{1}{2}, 1)$.
29 (i) $2a(1+3a)(1-3a)$, $(a+2b)(x-a)$, $(p+7)(p-6)$, (ii) $(3d+2)(9d^2-6d+4)$, $(2a+b)(3a-2b)$, $(m+n-r)(m-n+r)$; (iii) $(a+3)(a-3)(a+2)(a-2)$, $(x+17)(x-15)$, $(a+b)^2(a-b)^2$
30 $\dfrac{x}{n}=\dfrac{y}{m}-6$ 31 $x^4-x^3-x^2-2x+4$ 32 $(x-2a)(x+a)$.
33 $-\dfrac{3}{(x-1)(x-2)(x-3)}$ 34 $x=-2, y=7, z=6$

35 $(4, 20), (-2, 2)$ 36 $\dfrac{Cc}{C+c}$ 37 $36(x^2-9)(x^2-4)(x^2-1)$

38 (i) $x=3,\ y=5$; (ii) $x=-4,\ y=3$

39 (i) $(x+23)(x+17)$, (ii) $(p+29)(p-19)$ 40 $\dfrac{a+b}{a-b}$

41 45 persons; 2s each 42 19·2 inches 43 $\dfrac{200d}{c-d}$

44 $a^2+ax-2x^2$ 45 The expression$=0$ when $d=1$

47 (i) $-(a+b+c)$, (ii) $x=\frac{5}{2},\ y=-\frac{1}{2}$ 48 440 yds 49 14, 27 6

51 (i) $-\dfrac{2ab}{a+b}$, $x=\frac{1}{3},\ y=7$ 52 (i) $\dfrac{a+b}{b}$, (ii) $\dfrac{1}{1-9x^2}$ 53 9 ft

54 $3x-7$ 55 6 miles 56 2 11 Kg, 4 Kg per sq cm, 36 2

57 (i) $(4x+y)(3x-y)$, (ii) $(a+1)(a^2+a+1)$ 58 $a^3b^4(a^4-b^4)(a^3-b^3)$

59 (i) 9; (ii) $\dfrac{p+q}{2}$ 60 (i) $\dfrac{4a^2}{a^2-b^2}$, (ii) $\dfrac{a-x}{a+x}$

61 $\frac{3}{5}$ of a pint, $\frac{2}{5}$ of a pint 63 5s, 8?

XXIV a Page 241.

3 $y=x$

4 (i) 7 6, (ii) 13, (iii) 17 6, (iv) 3·2

XXIV b Page 248

1 (iii) 6·25, (v) −16

2 (i) $(4, 4)$; (ii) $(-1, -5)$

3 (i) 1 46, −5 46, (ii) 3·24, −1 24, (iii) 3 32, 0 68, (iv) 4, −8,
(v) 4, −2, (vi) 1 5, 2 5

4 2 38, 4 62, −1·25 5 −1 5 6 2 62, 0 38, 4, −1

7 6 46, −0 46 8 Max 7, min −5 (1, 7), (1, −5)

9 −0·25; 3 79, −0 79, 4 54, −1 54 12 2 55 13 0 6, −3

14 6·25, 2 56, −1 56 16 (2, 7), (−7, −2)

17 (i) $x=12,\ y=3$, or $x=3,\ y=12$,
(ii) $x=6,\ y=-3$, or $x=-3,\ y=-6$,
(iii) $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$, $\left.\begin{matrix}x=3\\y=2\end{matrix}\right\}$, $\left.\begin{matrix}x=-3\\y=-2\end{matrix}\right\}$, $\left.\begin{matrix}x=-2\\y=-3\end{matrix}\right\}$

XXV a Page 251

1 $-3, \frac{1}{2}$ 2 $a, -2a$ 3 $0, c$

4 $-\frac{7}{3}, \frac{5}{2}$ 5 $\frac{m}{2}, -2m$ 6 $\pm p$ 7 $3, \frac{1}{3}$ 8 $\frac{3}{2}, \frac{2}{3}$

9 $3a, -\dfrac{5a}{2}$ 10 $\frac{13}{2}, -3$ 11 $\frac{5}{2}, -3$ 12 $0, \dfrac{2a-b}{3}$

13 $\frac{1}{2}, \frac{1}{4}$ 14 $\dfrac{5b}{3}, -\dfrac{8b}{3}$ 15 $3a, -2a$ 16 $15, -4$

17 $2, -\frac{1}{2}$ 18 $1, \frac{3}{5}$ 19 $0, \frac{2}{3}$ 20 $4, \frac{3}{2}$

XXV. b. Page 254

1 17, −5 2 7, −15

3 10 −24 4 8, −7. 5 7, −14 6 13, −23

7 3, 19 8 11, −8 9 31, −11 10 21, 17.

11 $-19, 13$, 12 $17, -14$ 13 $\frac{1}{2}, -2$ 14 $\frac{2}{3}, -3$

15 $3, -\frac{5}{2}$ 16 $\frac{1}{4}, -3$ 17 $\frac{1}{3}, -5$ 18 $\frac{5}{3}, \frac{7}{2}$

19 $3, -\frac{1}{4}$ 20 $6, \frac{3}{5}$ 21. $\frac{4}{5}, -\frac{5}{4}$ 22 $\frac{2}{3}, \frac{3}{4}$

23 $\frac{1}{5}, -6$ 24. $\frac{4}{5}, -7$ 25. $\frac{9}{10}, -\frac{3}{5}$ 26 $\frac{13}{6}, -\frac{2}{3}$

27 $\frac{13}{3}, -\frac{11}{3}$ 28 $\frac{1}{3}, -\frac{3}{2}$ 29 $14, \frac{9}{4}$ 30 $8, \frac{5}{2}$

31 $\frac{3}{4}, -\frac{4}{3}$ 32 $\frac{25}{9}, \frac{25}{9}$ 33 $-4, -\frac{11}{2}$ 34 $\frac{2}{5}, -\frac{3}{2}$

35 $\frac{5}{2}, -\frac{2}{7}$ 36 $3, -\frac{5}{3}$ 37 $4, -\frac{1}{7}$ 38 $\frac{9}{8}, \frac{9}{8}$

39 $3, -\frac{4}{3}$ 40 $4, -\frac{2}{9}$ 41 $2, \frac{20}{3}$ 42 $3, \frac{13}{5}$

43 $3\pm\sqrt{2}$, 4 41, 1 59 44 $4\pm\sqrt{5}$, 6 24, 1 76

45 $2\pm\sqrt{3}$, 3 73, 0 27 46 $\frac{3\pm\sqrt{3}}{2}$, 2 37, 0 63

47. $-1\pm\sqrt{5}$, 1 24, -3 24 48 $\frac{5\pm\sqrt{13}}{2}$, 4 30, 0 70

49 $\frac{1\pm\sqrt{6}}{3}$, 1 15, -0 48 50 $\frac{6\pm\sqrt{15}}{7}$, 1 41, 0 30

51 $\frac{-8\pm\sqrt{29}}{7}$, -1 91, -0 37 52 $\frac{7\pm\sqrt{85}}{6}$, 2 70, -0 37

53 $\frac{9\pm\sqrt{181}}{10}$, 2 25, -0 45 54 $\frac{5\pm\sqrt{73}}{4}$, 3 39, -0 89

55 $\frac{1\pm\sqrt{33}}{4}$, 1 69, -1 19 56 $12\pm\sqrt{137}$, 23 71, 0 29

XXV. c. Page 258. 1 $4, -\frac{3}{2}$ 2 $15, 8$

3 $\frac{3\pm\sqrt{17}}{2}$, 3 56, -0 56 4. $5a, -4a$ 5 $\frac{2c}{3}, -\frac{4c}{5}$

6 $\frac{-3\pm\sqrt{5}}{2}$, -2 62, -0 38 7 $\frac{15\pm\sqrt{5}}{10}$, 1 72, 1·28

8 $\frac{7b}{6}, -\frac{5b}{6}$ 9 $\frac{7c}{6}, -\frac{4c}{7}$ 10 $\frac{3\pm\sqrt{21}}{2}$; 3 79, -0 79

11 $\frac{1\pm\sqrt{-15}}{8}$ 12 $\frac{2}{3}, -\frac{1}{4}$ 13 $\frac{9\pm\sqrt{-7}}{2}$

14 $-1\pm\sqrt{4\ 2}$, 1 05, -3 05 15 $\frac{3\pm\sqrt{23\ 04}}{2}$, 3 90, -0 90

16 $16, -17$ 17 $\frac{-1\pm\sqrt{5\ 3824}}{2}$, 0 66, -1 66 18 $29, -27$

19 $\frac{15a}{2}, -12a$ 20 $\frac{4}{3}, -2$ 21 $\frac{14\pm\sqrt{148}}{2}$, 13 08, 0 92

22 0 55, -0 22 23 $\frac{a}{2}(\sqrt{5}-1), -\frac{a}{2}(\sqrt{5}+1)$, 7 416 -19 416

24. $\frac{1}{2}(a\pm\sqrt{a^2-4c^2})$, 13 202, 2 708

XXV. d. Page 260 1 4, −2

2 (i) 1·46, −5·46, (ii) 3·24, −1·24, (iii) 3·32, 0·68

3 (i) 3, −2, (iii) 2, −3, the roots of (ii) and (iv) are imaginary

4 0·5, −1·5 6 1·81, 0·69

7 (i) 3·79, −0·79, (ii) 1·64, 0·61, (iii) 2·25, −0·45

XXV e Page 261 1 ±1, ±2 2 ±2, ±3

3 $\pm 2c, \pm\frac{c}{2}$ 4 1, −2 5 3, −2 6 $c, -3c$

7 $\pm 4, \pm\frac{1}{4}$ 8 $\pm a, \pm b$ 9 2, 3, −3, −4

10 ±3, ±4 11 −1, −2, 4, 5 12 $4a, -2a, a, a$

XXV f Page 262 1 1, −1, −1 2 1, −1, 2 3 1, 2, −2

4 1, −3, −5 5 2, −1, −1 6 0, 1, 1, −2 7 3, 2, −5

8 1, 3, −2 9 $2a, a, -3a$ 10 $\frac{3}{2}, -2, -6$ 11 5, 2, −7

12 7, −3, −4 13 $-2a, -2a, 4a$ 14 $0, 6a, 6a, -12a$

15 0, −2, 7·24, 2·76 16 $-\frac{1}{4}$, 3·73, 0·27

17 (i) 1·4, 0·6, (ii) 3·2, 2·3 18 (i) $4, -\frac{1}{4}, 3, -\frac{1}{3}$, (ii) $2p, 3p, \frac{p}{2}, \frac{p}{3}$

19 (i) and (iv) by formula, (ii) and (iii) by factors

20 4, −2, $x^2-2x-8=0$

21 (i) unreal, (ii) real but irrational, (iii) rational, (iv) rational.

22 −1·5, 3·5 23 3·30, −0·30 25 −5 and 1

XXVI a. Page 267 1 $x=5, y=3$, $x=-2, y=-4$

2 $x=4, y=1$, $x=5, y=2$ 3 $x=3, y=5$, $x=-\frac{5}{2}, y=-6$

4 $x=2, y=1$, $x=\frac{14}{3}, y=-\frac{13}{3}$ 5 $x=1, y=1, x=\frac{3}{2}, y=\frac{1}{4}$

6 $x=5, y=4$, $x=-1, y=1$ 7 $x=6, y=-4$, $x=\frac{2}{5}, y=\frac{36}{5}$

8 $x=5, y=-2$, $x=\frac{5}{4}, y=\frac{1}{2}$ 9 $x=3, y=2$, $x=\frac{13}{17}, y=-\frac{20}{17}$

10 $x=5, y=6$, $x=6, y=5$ 11 $x=9, y=5$, $x=5, y=9$

12 $x=7, y=5$, $x=-5, y=-7$ 13 $x=34, y=11$, $x=-11, y=-34$

14 $x=4, y=3$, $x=9, y=\frac{4}{3}$ 15 $x=7, y=3$, $x=-\frac{3}{2}, y=-14$

16 $x=5, y=-4$, $x=1, y=-20$ 17 $x=10, y=\frac{9}{5}$, $x=9, y=2$

18 $x=31, y=34$, $x=-34, y=-31$

19 $x=32, y=40$, $x=-40, y=-32$

20 $x=7, 5, -5, -7$,
$y=5, 7, -7, -5$

21 $x=9, 8, -8, -9$,
$y=8, 9, -9, -8$

22 $x=13, 14, -14, -13$,
$y=14, 13, -13, -14$

23 $x=6, 4, -4, -6$,
$y=2, 3, -3, -2$

24 $x=12, y=9$, $x=9, y=12$ 25 $x=15, y=12$, $x=12, y=15$.

26 $x=15, y=2$, $x=-2, y=-15$ 28 $x=5, y=6$, $x=6, y=5$

29 $x=7, y=8$, $x=-8, y=-7$ 30 $x=10, y=5$, $x=5, y=10$

31 $x=13, y=4$; $x=-4, y=-13$ 32 $x=4, y=3$; $x=3, y=4$

33 $x=5, y=7$; $x=7, y=5$ 34. $x=\frac{1}{13}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{13}$,
$y=\frac{1}{3}, \frac{1}{13}, -\frac{1}{13}, -\frac{1}{3}$

35 $x=2, y=3$, $x=3, y=2$ 36 $x=\frac{1}{2}, y=2$; $x=-2, y=-\frac{1}{2}$

37 (i) $x=6, y=3$, $x=-9, y=-2$; (ii) $x=9, y=2$, $x=-3, y=-6$;
(iii) $x=6, y=3$

XXVI. b. Page 269 1 $x=16, y=2$ 2 $x=5, y=3$

3 $x=3, y=2$, $x=2, y=3$ 4 $x=5, y=6$, $x=6, y=5$

5 $x=12, y=8$, $x=8, y=12$ 6 $x=3, y=2$; $x=-2, y=-3$

7 $x=2, y=5$; $x=-5, y=-2$ 8 $x=7, y=3$, $x=-6, y=-\frac{7}{2}$

9 $x=4, y=2$, $x=6$, $y=\frac{4}{3}$ 10 $x=\pm 1, y=\pm 2$; $x=\pm 2, y=\pm 1$.

11 $x=\pm 3, y=\pm 2$, $x=\pm 2, y=\pm 3$

12 $x=\pm 5, y=\pm 2$, $x=\pm 2, y=\pm 5$

13 $x=4, y=3$, $x=-3, y=-4$ 14. $x=7, y=2$, $x=2, y=7$

15 $x=3, y=1$, $x=\frac{1}{2}, y=6$ 16 $x=\pm 6, y=\pm 1$

17 $x=\pm 10, y=\pm 2$ 18 $x=\pm 2, y=\pm 1$, $x=\pm 3, y=\pm 2$.

19 $x=\pm 2, y=\pm 1$, $x=\pm 5, y=\pm 3$

20 $x=\pm 1, y=\pm 3$, $x=\pm\frac{7}{2}, y=\mp 12$

21 $x=\pm 5, y=\pm 3$, $x=\pm\frac{19}{3}, y=\pm\frac{8}{3}$

22 $x=\pm\frac{3}{2}, y=\pm\frac{1}{2}$; $x=\pm\frac{1}{2}, y=\pm\frac{3}{2}$

XXVI. c Page 271. 1 $x=4, y=1$, $x=1, y=4$

2 $x=5, y=2$, $x=-2, y=-5$ 3 $x=5, y=7$, $x=-7, y=-5$

4 $x=4, y=3$, $x=3, y=4$ 5 $x=4, y=1$, $x=-1, y=-4$

6 $x=\pm 3, y=\pm 5$, $x=\pm\frac{13}{2}, y=\mp\frac{1}{4}$ 7 $x=5, y=3$, $x=3, y=5$

8 $x=5, y=1$, $x=-1, y=-5$ 9 $x=6, y=-3$, $x=3, y=-6$

10 $x=8, y=3$, $x=\frac{15}{2}, y=\frac{16}{5}$ 11 $x=5, y=2$, $x=-\frac{7}{2}, y=-\frac{11}{3}$

12 $x=3, 2, 4, 1$;
$y=2, 3, 1, 4$ 13 $x=\pm 9, y=\pm 3$, $x=\pm 3, y=\pm 9$

14. $x=4, 3, 6, 2$,
$y=\frac{3}{2}, 2, 1, 3$. 15 $x=2, \quad 2, -3, -3$,
$y=3, -4, \quad 3, -4$

16 $x=7, 5, 10, \frac{7}{2}$, $y=5, 7, \frac{7}{2}, 10$ These values are the coordinates of the points of intersection of $x+y=12$, and $x+y=\frac{27}{2}$, with $xy=35$

17 $x=2, 1$, $y=1, 2$ These values are the coordinates of the points of intersection of $x^2+y^2=5$, and $x+y=3$ Two other roots are unreal

18 $x=7, y=5$; $x=5, y=7$

19 (i) $x=8, 6, -6, -8$;
$y=6, 8, -8, -6$ (ii) $x=3, y=5$, $x=5\,8, y=-0\,6$
(iii) $x=3, y=4$ [The line is a tangent]
(iv) $x=5\,2, y=-3\,0$, $x=-1\,4, y=5\,8$

20 $x=\pm 3, y=\pm 4, z=\pm 1$ 21 $x=4, y=1, z=3$

22. $x=\pm\frac{a^3}{bc}, y=\pm\frac{b^3}{ac}, z=\pm\frac{c^3}{ab}$ 23 $x=\pm 5, y=\mp 3, z=\pm 8$

24. $x=\pm 5, y=\pm 2, z=\pm 4$ 25 $x=\pm 4, y=\pm 1, z=\pm 7$

26 $x=6, y=9, z-1$, or $x=6, y=-1, z=-9$ 27 $x=7, y=2, z=4$

XXVII a Page 274 1 11, 4 2 8, 9 3 5 4 2, 7
5 2, 6 6 121 yds, 120 yds 7 £25 8 £80
9 6 mi. per hr 10 9 days 11 30 mi and 45 mi per hr
12 5 mi and 4 mi per hr 13 10 hrs, 15 hrs
14. 15 min and 25 min 15 4*s*, 2*s* 16 15 17 64

XXVII b Page 276 1 30 2 100 3 24. 4. £75
5 3*s* 4*d* 6 £750 7 £50 or £150 8 9*d* per doz 9 2*s*
10 40*s* per doz 11 *A*, 25, *B*, 20 12 18 13 7*s*, 8*s* 9*d*.
14. 7, 2 15 91 16 40 ft, 30 ft 17 8 ft, 15 ft, 17 ft
18 *A*, 18, *B*, 12 19 6, 5 20 15 at 2*d* each 21 £50
22 175 mi at 35 mi per hr 23 20 24. 75 25 4 mi per hr
26 108 min, 135 min 27 48 mi per hr 28 $6\frac{2}{3}$ mi per hr
29 3, 1 30 24 31 *A* 5, and *B* $3\frac{1}{2}$ mi per hr.
32 3 7 cm, 2 3 cm 33 AP=20 9 cm, BP=12 9 cm. 35 8 4 cm.
36 2 6 cm, 1 6 cm 37 9 cm, 4 cm.
39 (i) 3, 4, (ii) 5, 6 (iii) 5 2, 0 8, (iv) 5 7, 2 3

XXVIII a Page 281 1 2 30 p m, 1 30 p m and 3 30 p m
2 2 p m; 2 51 p m
3 6 p m, 36 mi from the start At 3, 4, and 5 p m
4 47 mi from *A*'s starting place at 12 42 p m 11 12 a m and 2.12 p m.
5 35 mi from London at 3 33 p m 3 9 p m; 3 57 p m, 36 mi
6 $49\frac{1}{2}$ mi, 6 15 p m
7 (i) 1 p m, 28 mi from P, (ii) 20 mi, (iii) 11 30 a m
8 5 hrs from the start 9 $4\frac{1}{2}$ mi per hr
10 1 30 p m., As 3 to 1 11 24 min

XXVIII b Page 284 1 2 4, 3 2, 11 6, 2 75, 2 30, 2 19
3 5 385, 5 745 4. 3 031, 3 240 5 2 40 6 3 53, 3 92, 40, 54.

XXVIII c Page 287.
1 9 mi from Y at 12 48 p m 12 18 p m and 1 18 p m.
2 12 12 p m (i) 11 a m, (ii) 57 mi 3 27 mi.
4. (i) 15 mi after *C*'s start, 1 mi from Bath;
(ii) 45 mi ,, ,, $3\frac{1}{2}$ mi ,,
(iii) half a mile behind *A* and *B*
5 *A* 16 yds ahead, *C* 16 yds behind
6 *B* 30 yds ahead of *A*; *A* 54 yds ahead of *C* 27 yds.
7 *B*, 10 yds, *C*, 20 yds

8 $1\frac{1}{2}$ hrs from the start, 18 mi, 20 mi Half an hour

9 5 mi 10 10 4 a m 11. 400 yds 12 £420. 20 for £480

13 After 10 secs 400 ft per sec; 600 ft per sec

14 At Northampton 48 4 mi, 84 8 mi The quickest run is from Willesden to Northampton The slowest is that from London to Willesden

Miscellaneous Examples VI Page 290. 1 $121\frac{3}{10}$

2 $14a-13b-16c$ 3 (i) $x^2+6x-91$, (ii) $4y^2-9$, (iii) $6a^2+13a+6$; (iv) $15p^2-52p+32$, (v) $16m^2-9n^2$, (vi) $40x^2-69x+27$

4 -13 5 The first by 6 6 $\dfrac{y}{m}=\dfrac{x}{n}+6$ 7 345

8. Father, 42, Son, 14 9 (i) x^2-3x+2, (ii) $7x-12$

10 $2x+3$, 7 12 $12(3x-y)=12y+c$

13 288 English, 104 French, 60 Latin 14 3 11, 3 48, 8 76, 23

15 (i) (-2), (ii) 1 16 A, £27, B, £36, C, £54

17 $5x^3+2ax^2-5a^3$ 18 $(7-12a)x^3-(3a+60)x^2+(4a-15)x-3$

19 0 20 $16m-4n$ 21 (i) $\dfrac{22p}{15}$, (ii) $\dfrac{36x}{5}$, (iii) $\dfrac{28y}{3}$

22 $\mathrm{N}=\dfrac{lbt}{\mathrm{V}}$ (i) 1296, 14 4 cu cm 23 8

24 80 lbs at 1s 6d, 60 lbs at 1s 8d 26 $4a+c$ 27 $\dfrac{y^2}{x}$ 28 $\frac{5}{2}$

29 (i) $(p-13)(p+5)$, (ii) $2y(x-y)(2x+3y)$, (iii) $3x^2(2x+3a)(2x-3a)$

30 $x=1, y=-1$ 31 A, 24, B, 12, C, 6 32 13 3, 17 6

33 240 34 $\dfrac{bcd}{ae}$ days 35 x^2+5x+6

36 0, 42, -48, $(x+2)(x-3)(x+3)$ 37 $1-2x+3x^2-3x^3$

38 (i) 4, (ii) $x=-5, y=-10$ 39 £39

40 $x=3\ 5, y=2\ 5$ 41 $a+b+c$

42 (i) $[2z(x+y)+4z^2]$ sq ft, (ii) $\dfrac{400}{x}$, $\dfrac{400}{x(x-1)}$ 43 1

44 (i) $2x^2-5x+2$, (ii) $6x^2-x-15$, (iii) $6x^2-4x-16$ $-\frac{1}{3}$

45 (i) $x(x-7)(x+13)$; (ii) $(3+2x)(9-6x+4x^2)$, (iii) $(x+a)(x-a)(y+a)$

46 $x=-1, y=3, z=-5$ 47 26 49 $4a^2+2ab-2ac+b^2+bc+c^2$.

50 (i) $(x-2y)(2x+y)$, (ii) $(x-1)(x-2)(x-3)$, (iii) $(a-b)(a+b+1)$

51 $\dfrac{3pq}{88x}$ 52 (i) $\dfrac{7x-5}{x^2-x+1}$; (ii) $\dfrac{6b}{b^2-a^2}$

54 (i) 10, (ii) $x=5, y=11$ 55 33 9, -7 56 93.

57. (i) $ab(ab-5)(ab-4)$, (ii) $(x+1)(x+2)(2x-1)$

58 (i) $\dfrac{x^2+y^2}{2xy}$, (ii) $\dfrac{1}{2+x}$ 59 $1+x^2-x^7$

60. $(x+3y)(2x-y)(3x-y)$ 61 10 62 (i) $-\frac{3}{2}$, (ii) $x=3, y=-2$.

63. 18 shillings, 9 half-crowns. 64 $x=\frac{1}{2}, y=\frac{7}{4}$.

65 18 5, 26 66 (i) -8, (ii) $0, -6$, (iii) $\frac{3}{2}, 7$, (iv) $6, \frac{5}{2}$

67 $a(a-3)(a+2)$, $a(a+3)(a-3)$, $a^3(a+2)$, L C M $=a^3(a-2)(a^2-9)$

68 (i) 0, (ii) 6, (iii) $2(x^3-10x-12)$, $(x-4)(x+1)(x+3)$

69 (i) 16, (ii) $x=1$, $y=-3$, $z=-2$ 70 $\dfrac{a^3}{(a-x)(a^3+x^3)}$

72 $136\frac{1}{2}$ miles 73 $\left(\dfrac{240}{y}-\dfrac{x}{5}\right)$ pence 15 74 $a=13$, $b=-6$

75 (i) $\dfrac{b^2+2bc}{a^2-b^2}$, (ii) x^2-x+1 76 (i) $6, -2, \dfrac{3\pm\sqrt{5}}{4}$, 1 31, 0 19

77 (i) $4+4x+x^2-4x^3-6x^4$, (ii) $x^{12}+4x^{10}-2x^9+4x^8$

79 45 mi per hr 80 4 3, 0 7

81 (i) $9ab-\dfrac{c}{2}(a+b)+\dfrac{c^2}{36}$, (ii) $\dfrac{c}{2}(a+b)-\dfrac{c^2}{36}$

83 (i) $\dfrac{x-2}{x^2-x-3}$, (ii) $\dfrac{a(1-a+a^2)}{1-a^4}$ 84 -13

85 3750 tons 136 ft 86 $\dfrac{3x}{y}+1+\dfrac{y}{2x}$ 87 325

88 (i) 4 6 Kg per sq cm, (ii) 91 lbs per sq in

89 (i) $(x-1)(x+1)(x-3)(x+3)$, (ii) $(x+y)(x+y-1)$

90 (i) -2, (ii) $x=2a+b$, $y=a+2b$

92 (i) $\dfrac{x-23}{(x-2)(x-3)(x-7)}$, (ii) $\dfrac{1}{b}$ 94 $3\frac{1}{2}$ mi per hr 95 3 05

98 (i) $x=10$, $y=9$, (ii) $x=\pm6$, $y=\pm7$, $x=\pm7$, $y=\pm6$

99 12 25 100 (i) $(x-y)(x+y+2)$, (ii) $(a+b)^3(a-b)$

102 20 days 103 4 16, -2 16, -10

104 Walks $10\frac{1}{2}$ mi, rides 15 mi 3 hrs from the start 105 1

107 (i) $\dfrac{7}{x^2-4}$, (ii) 1 108 2 109 $(c-d)(2c-3d)(3c+2d)$

110 (i) $x=b-\dfrac{a}{2}$, $y=\dfrac{b}{2}-a$, (ii) $x=-3$, $y=2$, $x=-\frac{18}{7}$, $y=\frac{13}{7}$

111 374 112 $x=0$, $y=-4$, or $x=1$ 5, $y=-2$ 5 113 -2

114 (i) $\dfrac{2}{(x-1)(x-2)(x-3)}$, (ii) $\dfrac{x}{x-z}$ 115 $6-2x-3x^2+14x^3$.

116 Earned income $=(4A-80T)$ pounds, Unearned income $=(80T-3A)$ pounds; £800, £350

118 148, 12, 13 119 $x=2$, $y=1$, or $x=1$ 41, $y=-0$ 34.

120 4 mi from Q at 5 p m 8 mi apart at 4 30 p m, 18 mi apart at 3 18 p m, and also at 2 42 p m approximately

122 5 123 $0, 3\frac{3}{5}$.

124 $3(2x-y)(5x+4y)$ 127 4 yds, 5 yds

128 £22 10s, £28 (i) 230, (ii) 350

1100E $=49G+8140$, 550R $=52G-2255$, 20P $=G-230$

PART II

XXIX. a. Page 302. 1. 39, 99 2 0, −36 3 −32, −104.
4. 21, 63 5 $22a$, $58a$ 6 $-20x$, $-56x$ 7 4·2, 18·6
8 $-23p$, $-71p$ 9 38·4, 100·8 10 $-2d$, $c-12d$ 11 y, $x+4y$
12 $a-b$, $8a-3b$. 13 $a+b$, $9a+5b$ 14. 22, 20, 18, . .
15 5, 9, 13, . . 16 35, 38, 41, 17 $62\frac{1}{2}$, 60, $57\frac{1}{2}$, .
18 45, 37, 29, . 19 $a=b-c$, $d=b+c$. 20 $3x$, $4x-2y$, $5x-4y$
21 (i) $2n+1$; (ii) $10-2m$, (iii) $(4p-3)x$, (iv) $(7r-8)y$.
22 $74-14n$ 23 8 24. 3, 39, 51, 63, 75

XXIX. b Page 306 1 (i) 21, (ii) 34, (iii) $\frac{9}{32}$, (iv) $-5a$, (v) 6. (vi) a
2 115, 100, 85, 70 3 22, 19, −8 4. 9·5, 11·6, 17·9
5 $21\frac{1}{2}$, 24, · 34 6 $8a$, $5a$, $-10a$ 7 39·8, 37, 20·2
8 37, 39, 41 9 −113, −117, −121 10 $55p$, $61p$, $67p$
11 $-51x$, $-55x$, $-59x$ 12 188, 2058 13 199, 10,000
14. −6; 0 15 −14·5, 45 16 810 17. 652.
18 −345 19 2431 20 2480 21 −714.
22 231 23 −21 24 $2x^3-x^3$. 25 $\frac{5p^2}{2}(p-3)$.
26 $-45b$ 27 $-280m-210n$ 28 3500
29 44, 13266; 107034 30 23, 111, 3036 31. $2x^2$.
32 22969 beams, 103 layers 33 6 or 8 34. 25
35 13 or 20 36 23 37 3 or 10 38 9
39 20 40 8 secs, 104 in, 104 in.

XXIX c. Page 308 1. 25 2 14 or 15 3 (i) 6440, (ii) 40
4. 90 5 33, 36, 39, 6 (i) $12a+42b-54c$; (ii) $10p-25q+35r$
7. 1, $\frac{1}{5}$ 9 $x=\frac{2a+b}{3}$, $y=\frac{a+2b}{3}$ 10 49, 56, 63, 70, 77.
11 7, 12, 17 12 25, 37, 49 13 4, 6, 8, 10
14. 3, 7, 11, 15 15 $x=5$, $y=15$ 16 3, 7, 11, 15
17 25 weeks 18 In 6 days 19 Between 43 and 44 years.

XXIX d. Page 311. 1 $3\frac{3}{4}$ 2. $-5\frac{1}{11}$ 3 $\frac{2}{x+y}$.
4. $\frac{2pq}{p^2+q^2}$ 5 $-2, \frac{1}{2}, \frac{2}{9}$ 6 $1\frac{1}{5}, 1\frac{1}{2}, 2, 3$ 7 $-12, \infty, 12$.
8 $\frac{1}{3}, \frac{4}{15}$ 9 $3, -\frac{3}{5}$ 10 $5\frac{3}{5}, -4$. 11. $\frac{3}{7}$.
12 6, 8 13. $x=\frac{3ab}{a+2b}$, $y=\frac{3ab}{2a+b}$

XXIX. e. Page 313 1 48, 384 2 $4, -\frac{1}{2}$ 3 $67\frac{1}{5}, 537\frac{3}{5}$
4 $-27, 729$ 5 16, 1024 6 $8, -1$ 7 16
8 $-45, 75, -125$ 9 (i) 12, (ii) 9; (iii) $4x^2$
10 $3x^2-8ax-3a^2$ 11 (i) $\frac{1}{3}, \frac{1}{9}, \frac{2}{27}$, (ii) $\frac{3}{5}, \frac{1}{5}, \frac{1}{15}$
12 6, 12, 24, 48 13 $-28, 14, -7, 3\frac{1}{2}, -1\frac{3}{4}, \frac{7}{8}$
14 32, 18 15 $133\frac{59}{97}$ 16 $\frac{2}{3}, -2, 6$

XXIX. f. Page 314. 1 $682\frac{1}{2}$ 2 $2\frac{20}{81}$ 3 $42\frac{5}{8}$.
4. $\frac{1281}{2560}$ 5 $\frac{1365}{2048}$ 6 $1\frac{31}{432}$ 7 $311\frac{29}{81}$
8 $\frac{1261}{2916}$ 9 $-58\frac{73}{144}$ 10 $13\frac{885}{1024}$ 11 $2(2^m-1)$
12 $\frac{3}{4}(1-3^{2p})$ 14 (i) $\frac{1}{8}, 127\frac{7}{8}$, (ii) $-1458, -1092$
15 $\left(1-\frac{1}{2^n}\right)-\frac{1}{2}\left(1-\frac{1}{3^n}\right)$

XXIX. g. Page 316. 1 $13\frac{1}{2}$ 2 $\frac{1}{7}$. 3 $7\frac{1}{7}$
4. $\frac{27}{70}$ 5 $\frac{27}{29}$ 6 $2\frac{2}{15}$. 7 $r>1$, impossible
8 $\frac{x}{1+x}$, if $x<1$ 9 $\frac{p^2}{p-1}$, if $p>1$ 10 $\frac{ar}{r-1}$, if $r>1$
11. 2 12 $\frac{a^4b}{a+b}$, if $b<a$ 13 $\frac{7}{9}$ 14 $\frac{31}{45}$
15 $\frac{4}{11}$ 16 $\frac{95}{108}$ 17 $3\frac{1}{8}$ 18 12, 6, 3,
19 $\frac{3}{4}, \frac{1}{4}, \frac{1}{12}$, 20 $12, -6, 3$, 21 $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$,
22 $n, \frac{n}{n+1}, \frac{n}{(n+1)^2}$, 23 $\frac{(x+1)^2}{x^2(2x+1)}$ 26 (i) $2p$, (ii) $\frac{4A}{3}$

XXIX. h. Page 319. 1 (i) H P, ∞, ∞, (ii) G P, $\frac{3125}{512}, \frac{11529}{512}$. (iii) A P, $4\frac{1}{2}, 19\frac{1}{2}$
2 (i) $36\frac{987}{1024}$, $17\frac{1}{4}$ 3 (i) $1\frac{46}{1215}$, (ii) -18 4 343
5 (i) $\frac{3^n-1}{2}+n$, (ii) $4(2^n-1)-3n$, (iii) $\frac{3}{2}(3^n-1)-n$
6 $\frac{3n(n+1)}{2}-2n=\frac{3n^2-n}{2}$ 7 $\frac{3(3^n-1)}{2}-2n$
8 $690a-390b$ 9 $500a-220b$ 10 $\frac{t(t+1)(t+3)}{4}$
11 $1, 1\frac{1}{7}, 1\frac{2}{7}$, 12 The 48th, viz 53 13 $\frac{n(n+1)}{2}+\frac{1}{9}\left(1-\frac{1}{10^n}\right)$
14. 9, 12 15 $2, \frac{2}{3}, \frac{2}{9}$, ; $4, -\frac{4}{3}, \frac{4}{9}$,
17 7, 13, 19, . 20 $2q$, $p+(2r-1)q$.
21 (i) $\frac{x^2(x^{2n}-1)}{x^2-1}+\frac{xy(x^ny^n-1)}{xy-1}$; (ii) $4ap^2+\frac{2}{9}\left(1-\frac{1}{2^{2p}}\right)$
23 17. 24 £204 15s 1d 27 16 years
28. (i) £1,503,160, (ii) £2,000,000.

XXX a Page 325. 11. $\frac{2}{x^{\frac{1}{4}}}$ 12 $\frac{3}{a^{\frac{2}{3}}}$ 13 $\frac{4a^3}{x^2}$ 14. $3a^2$.

15 $\frac{a^2}{4}$ 16 $\frac{x^{\frac{1}{2}}}{5}$ 17 $\frac{3c^4x^2}{5a^4y^3}$ 18 $\frac{x^a b^a}{y^b}$ 19 $\frac{6}{x^{\frac{1}{2}}}$

20 $\frac{a^{\frac{1}{2}}}{2}$ 21 y^2 22 $\frac{1}{3a^2x^3}$ 23 $\frac{1}{x^{\frac{3}{2}}}$ 24 $\frac{x^{\frac{3}{5}}}{4}$

25 $2y^{\frac{2}{3}}$ 26 $x^{\frac{5}{4}}$ 27 $\frac{a}{x^{\frac{1}{2}}}$ 28 $\frac{1}{a^{\frac{2}{3}}}$ 29 $\frac{1}{a^2}$

30 $\sqrt[5]{x^3}$. 31 $\frac{1}{\sqrt{a}}$ 32 $\frac{5}{\sqrt{x}}$ 33 $\frac{2}{\sqrt[3]{a}}$ 34 $\frac{1}{2\sqrt[3]{a}}$

35 $2\sqrt[4]{b^3}$ 36 $\frac{1}{2\sqrt[3]{c}}$ 37 $\sqrt[3]{x}$ 38 $\frac{2}{\sqrt[6]{a^5}}$ 39 $\frac{\sqrt{a}}{2\sqrt[4]{x^3}}$.

40 $\frac{21}{\sqrt{a^3}}$ 41. $\frac{2}{\sqrt{a}}$ 42 $\frac{1}{3\sqrt{a^3}}$ 43 $\frac{4}{\sqrt[3]{x^2}}$ 44. $\sqrt[6]{a^{13}}$

XXX b Page 327. 1 $a^{\frac{4}{3}}-2a^{\frac{2}{3}}-3$ 2 $x^{-4}-16$

3 $9c-6c^{\frac{1}{2}}+1$ 4. $x+5x^{\frac{1}{2}}-14$ 5 $x^{\frac{4}{3}}-4x^{\frac{2}{3}}+4$

6 $a^{\frac{2}{3}}-3a^{\frac{1}{3}}-10$ 7 $a^{\frac{2}{3}}-3$ 8 $2a^{\frac{1}{3}}-3$

9 $12x^{\frac{2}{3}}-20x^{\frac{1}{3}}+41-15x^{-\frac{1}{3}}+24x^{-\frac{2}{3}}$ 10 $7a^{\frac{2}{3}}-2a^{\frac{1}{3}}+1$

11. $15m-3m^{\frac{1}{3}}-2m^{-\frac{1}{3}}+8m^{-1}$ 12 $2c^{2x}-9c^{x}-34+31c^{-x}-6c^{-2x}$.

13 $5p^{2n}-8p^{n}-8p^{-n}-3p^{-3n}$ 14. $x+x^{\frac{1}{2}}y^{\frac{1}{3}}-2y^{\frac{2}{3}}$

15 $2x^{\frac{1}{4}}-1+x^{-\frac{1}{4}}$ 16 $8a^{-2}+7a^{-1}+6$

17 $1-x^{\frac{1}{6}}-2x^{\frac{1}{3}}+2x^{\frac{2}{3}}$ 18 $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}$ 19 $a^{2x}-3a^{x}-2$

XXX c. Page 330. 1. (i) 7, 5, 49, (ii) 23, 17, 529 2 $\frac{1}{a}$

3 $\frac{1}{x^{\frac{4}{3}}}$ 4. a 5 $x^{\frac{5}{6}}$. 6 x 7 1 8 $\frac{1}{a^{\frac{5}{6}}}$

9 $a^{\frac{1}{2}}b^{\frac{1}{2}}$ 10 $\frac{x}{y^{24}}$ 11 c^2 12 $b^{\frac{3}{8}}$. 13 a^6b^9 14. $\frac{x^{\frac{4}{3}}}{y}$.

15 $\frac{1}{y^{2a+3b}}$ 16 $\frac{1}{2x^{\frac{1}{2}}y^{\frac{1}{2}}}$ 17 $\frac{4}{9a^2x^3}$ 18 $16ac^4$ 19 $\frac{x^{\frac{1}{5}}}{y^{\frac{1}{4}}}$

20 $x^{\frac{1}{3}}$ 21 $\frac{3ax}{2}$. 22 x^{n-1} 23 $\frac{1}{x^{\frac{1}{a}}}$ 24. $\frac{1}{a^{\frac{2}{3}}b^{\frac{1}{2}}}$

25 x^{b+1} 26 $\frac{1}{x^{\frac{1}{3}}}$ 27 $ac^{\frac{1}{6}}$ 28 8 29 $a-b$

30 $\frac{1}{(x^2-y^2)^{3n}}$ 31. $\frac{1}{a^5}$ 32 $b^{\frac{3}{2}}$ 33 $x^{\frac{1}{6}}$ 34 $\frac{a+b}{(a-b)^{\frac{1}{2}}}$.

35 $c^{\frac{7}{3}}$ 36 $\frac{x^2}{a^3}$ 37 $(a-b)^2$ 38 $ab(b^6-a^6)^{\frac{1}{6}}$.

39 2^{n^2} 40 $\frac{1}{4}$ 41 4 42 864 43. 1

XXX d. Page 331. 1 $(\sqrt{a}+3)(\sqrt{a}+4)$ 2 $(x^{\frac{1}{2}}-6)(x^{\frac{1}{2}}+2)$

3 $(x^{\frac{n}{2}}+7)(x^{\frac{n}{2}}-7)$ 4. $(3a^{\frac{1}{4}}+2)(a^{\frac{1}{4}}+1)$ 5 $(x^{-c}+5)(x^{-c}-4)$

6 $(x^m+3)(x^{2m}-3x^m+9)$ 7 $x-x^{\frac{1}{2}}-42$ 8 $4x^n-25$.

9 $2a+\sqrt{a}-6$ 10 $a^{2x}-4+4a^{-2x}$ 11 $a^{2x}+2a^{x+\frac{1}{x}}+a^{\frac{2}{x}}$

12 $x+3x^{\frac{1}{2}}+9$ 13 $2a+2(a^2-b^2)^{\frac{1}{2}}$ 14. $x^2+2x^{\frac{3}{2}}+x-16$.

15 $a^xb^{-y}-a^{-x}b^y$ 16 $20x^{2a}y^{2b}+13-15x^{-2a}y^{-2b}$

17 $1+2a^{-1}+4a^{-2}$. 18 $9\sqrt[3]{m^{2n}}-3\sqrt[3]{m^n}+1$

19 $a^{\frac{1}{3}}(a^{\frac{1}{3}}-2b^{\frac{1}{3}})$ 20 1 21 $\frac{x^{\frac{2}{3}}-2}{x^{\frac{1}{3}}+2}$ 22 $\frac{a^{\frac{1}{2}}}{b}$.

XXXI a. Page 333. 1 (ii), (iii), (viii)

3 $\sqrt{16}$, $\sqrt{9x^4}$, $\sqrt{a^2-4ax+4x^2}$

4. (i) $\sqrt[12]{64}$, (ii) $\sqrt[12]{81}$; (iii) $\sqrt[12]{36}$, (iv) $\sqrt[12]{a^9}$, (v) $\sqrt[12]{81c^8d^{12}}$

5 $\sqrt{3}$, $\sqrt[4]{10}$, $\sqrt[3]{6}$ 6 $\sqrt[18]{a^9}$, $\sqrt[18]{a^{10}}$ 7 $\sqrt[10]{a^6}$, $\sqrt[10]{a^5}$ 8 $\sqrt[12]{x^3}$, $\sqrt[12]{x^{10}}$.

9 $\sqrt[6]{125}$, $\sqrt[6]{121}$, $\sqrt[6]{13}$ 10 $\sqrt[8]{64}$, $\sqrt[8]{81}$, $\sqrt[8]{6}$ 11 $\sqrt[3]{2}$, $\sqrt[4]{2}$, $\sqrt[3]{2}$.

XXXI. b Page 335. 3 $7\sqrt{2}$ 4. $5\sqrt{5}$ 5 $8\sqrt{6}$

6 $12\sqrt{5}$ 7 $6\sqrt[3]{2}$ 8 $5\sqrt[3]{3}$ 9 $3\sqrt[4]{7}$ 10 $2\sqrt[3]{5}$

11 $10\sqrt{7}$ 12 $55\sqrt{6}$ 13 $6a\sqrt{2}$ 14. $6a\sqrt{2a}$ 15 $3a\sqrt[3]{2b}$

16 $-4xy^2\sqrt[3]{2}$ 17 $3mn\sqrt{3n}$ 18 $10n^3p\sqrt{2n}$ 19 $(x-y)\sqrt{2}$

20 $(a+b)\sqrt{a-b}$ 21 $(x+3)\sqrt{3a}$ 22 $\sqrt{605}$ 23 $\sqrt[3]{256}$.

24. $\sqrt[3]{-432}$ 25 $\sqrt{18a^4x^5}$ 26 $\sqrt{21}$ 27 $\sqrt{12}$ 28 $\sqrt{27}$.

29 $\sqrt{\frac{11}{30}}$ 30 $\sqrt{\frac{3m}{n}}$ 31 $\sqrt{\frac{48a^3}{25x}}$ 32 $\sqrt[3]{\frac{ax}{3}}$ 33 $\sqrt[4]{2a}$

34 $6\sqrt{2}$ 35 $7\sqrt{3}$ 36 $\sqrt{7}$ 37 $\sqrt[3]{2}$

38 $2\sqrt[3]{3}$ 39 $20\sqrt{3}-13\sqrt{2}$ 40 $6\sqrt{7}-15\sqrt{6}$ 41 $19\sqrt{3}$

42 $5a^2\sqrt{x}$ 43 $-4pq\sqrt{2pq}$ 44. 9 90 45 11 18

46 12 12 47 22 63 48 13 23 49 4·24

50 23 81 51. 35 91. 52. 53 70 53 26 20

XXXI. c. Page 337 1 $10\sqrt{6}$ 2 30 3 $21\sqrt{10}$.

4. $48\sqrt{2}$ 5 $66\sqrt{21}$ 6 $x^3y\sqrt{xy}$ 7 $\frac{\sqrt{6}}{4}$ 8 $\frac{\sqrt{42}}{14}$

9 $\frac{2\sqrt{5}}{5}$ 10 $\sqrt[3]{a^2-b^2}$ 11. $\frac{\sqrt{14}}{10}$ 12 $3\sqrt{3}$ 13 $10\sqrt[6]{500}$

14. $6\sqrt[4]{27}$ 15 $\frac{5\sqrt{6}}{4}$ 16 $\frac{x\sqrt{10a}}{30}$. 17 $\frac{c^7\sqrt{2c}}{x^3}$

18 5·196 19 3·536 20 2·981 21 0·817
22 1·134 23 11·314 24 0·630 25 0·141

XXXI. d Page 339 1 $6a-3\sqrt{a}$ 2 $x\sqrt{5}-5\sqrt{x}$
3 $8\sqrt{p}+6p$ 4 $23+7\sqrt{10}$ 5 $5\sqrt{35}+1$ 6 $37+20\sqrt{3}$
7 $56-6\sqrt{55}$ 8 $273-70\sqrt{14}$ 9 7 10 $58+7\sqrt{70}$
11 10 12 $a+1-\sqrt{a^2+a}$ 13 117 14 $84-21\sqrt{30}$
15 x 16 $2p-q$ 17 $2p+2\sqrt{p^2-q^2}$
18 $13a^2+5b^2-12\sqrt{a^4-b^4}$ 19 $3x+2+2\sqrt{2x^2+5x-3}$
20 $5x-4-2\sqrt{6x^2-7x-5}$ 21 $7+4\sqrt{3}$ 22 $8-3\sqrt{7}$
23 $\frac{1}{5}(3\sqrt{7}-2\sqrt{3})$ 24 $49+20\sqrt{6}$ 25 $\frac{1}{7}(38-27\sqrt{2})$
26 $\frac{\sqrt{ab}}{a}$ 27 $\sqrt{5}-\sqrt{3}$ 28 1 29 2 30 $\frac{1}{5}(2\sqrt{2}+\sqrt{3})$
31 $\frac{153+5\sqrt{30}}{91}$ 32 $\frac{4}{23}(9\sqrt{5}+5\sqrt{7})$ 33 $\frac{2(5a+3\sqrt{ax})}{25a-9x}$
34 $8-\sqrt{42}$ 35 $\frac{a+\sqrt{a^2-x^2}}{x}$ 36 $\frac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}$
37 $\frac{2\sqrt{3}+3-\sqrt{21}}{3}$ 38 4 39 $6(5\sqrt{3}-\sqrt{2})$ 40 $\frac{7\sqrt{5}}{11}$
41 $\frac{y^2}{x^3}$ 42 1 43 $\frac{59-30\sqrt{2}}{41}=0·4042$ 44 $\sqrt{5}+2=4·2361$
45 $59+34\sqrt{3}=117·8897$ 46 $\frac{114-41\sqrt{6}}{30}=0·4524$

XXXI e Page 342 1 $\sqrt{3}+1$ 2 $\sqrt{3}+\sqrt{2}$ 3 $\sqrt{6}+2$
4 $2-\sqrt{3}$ 5 $\sqrt{6}-\sqrt{3}$ 6 $\sqrt{11}+\sqrt{5}$ 7 $2\sqrt{2}+3$
8 $3\sqrt{3}-1$ 9 $4\sqrt{3}+\sqrt{2}$ 10 $2\sqrt{5}-3\sqrt{2}$ 11 $5-2\sqrt{6}$
12 $\sqrt[4]{2}(\sqrt{3}-1)$ 13 $8\sqrt{2}+2\sqrt{30}$ 14 $\sqrt[4]{7}(\sqrt{\frac{7}{2}}-\sqrt{\frac{3}{2}})$
15 $\sqrt{15}+\sqrt{11}$ 16 $2\sqrt{3a}-\sqrt{2b}$ 17 $3\sqrt{m}+2\sqrt{2n}$
18 $\sqrt{2x+1}+\sqrt{x-2}$ 19 $\sqrt{m+3n}+\sqrt{m-3n}$
20 $4\sqrt{2}$ 21 1 22 4

XXXI. f. Page 344 1 3 2 4 3 7 4 345
5 83 6 2 7 5 8 9 9 9
10 6 11 2 12 7 13 $\frac{17}{6}$ 14 20
15 5 16 $\frac{5a}{4}$ 17 $-\frac{ab}{a+b}$ 18 21 19 $-\frac{ab}{a+b}$
20 9 21 49 22 64 23 8 24 8 25 $\frac{4a}{3}$
26 1 27 $3\pm2\sqrt{2}$, 5·83, 0·17. 28 8, 0 29 1, -4.
30 $\frac{3\pm\sqrt{5}}{2}$, 2·62, 0·38 31 $-1, -1$; $-1\pm\sqrt{15}$, (or $-4·90, 2·90$)
32 $\frac{-5\pm\sqrt{17}}{2}$, $-4·56$, $-0·44$. 33 5, $-\frac{33}{5}$

XXXII a. Page 350. 9 $\log a + \log b - \log c$

10 $3\log a - 2\log b - \log c$ 11 $\log a + \frac{1}{3}\log c - \frac{1}{2}\log b$

12 $\frac{1}{3}\log a + \frac{1}{4}\log b - \frac{2}{3}\log c$

13 (i) $\log 2 + 2\log 3$, 1 255; (ii) $\log 3 + 4\log 2$, 1 681, (iii) $5\log 2 + 5\log 3$, 3 890, (iv) $\frac{3}{2}\log 2 + \frac{1}{4}\log 3$, 0 571

17 (i) 2 8, (ii) 3 7, (iii) 3 9, 6 5, 0 9, 6 7

XXXII. b. Page 353 1 2, 3, $\bar{1}$, 0, $\bar{1}$, $\bar{4}$, 1

2 1 6592, $\bar{3}$ 6592, 6 6592, $\bar{1}$ 6592

3 2 8786, 4 8786, $\bar{3}$ 8786, $\bar{2}$ 8786, 756100, 75 61, 0 007561, 0 7561, 7561×10^{15}

4 $\bar{2}$ 1043 5 $\overline{14}$ 0476 6 $\overline{37}$ 7984 7. $\bar{3}$ 8291 8 0 9842

9 2 8841 10 $\bar{1}$ 3141 11 $\bar{1}$ 6354 12 $\bar{2}$ 5979 13 38

XXXII. c. Page 359. 3 186 8 4 0 6735 5 8496

6 0 6116 7 840 9 8 8 337 9 44 22 10 0 6797.

11 7 446 12 0 07784 13 0 07612 14 9 346

15 0 1070 16 0 5113 17 0 00008855 18 4150

19 0 7142 20 1 936 21 1 973 22 6 47

23 1 21. 24 0 83 25 4 59, 26 0 74

27 −5 90 28 $1\ 772 \times 10^{6}$ 29 $3\ 711 \times 10^{5}$ 30 $4\ 130 \times 10^{6}$

31 $2\ 510 \times 10^{-1}$ 32 $6\ 449 \times 10$ 33 $3\ 631 \times 10^{4}$ 34 $5\ 910 \times 10^{4}$

35 $5\ 606 \times 10$ 36 8 116 37 $4\ 102 \times 10^{-2}$ 38 (i) 178, (ii) 16

XXXII. d Page 362 1 2 59, 3 91, 4 32 2 $\frac{1}{2}$, 0 7181, 0 3047

4 £731 5 £4875 6 £514 7 20 8 10 76

9 4 535 litres 10 508 11 500,700 12 270·2

13 0 028, 0 0078 14 (i) 85250, (ii) 7 444 15 29 52 cm.

16 3319 Kg 17 16 cwt 18 276 lbs 19 42

20 99 4 21 4649 cu ft per min 22 30 in

Miscellaneous Examples VII. Page 368.

1 (i) $a^{6}b^{-1}$, (ii) $\sqrt{3}$ 2 $\frac{1}{3}$ The 13th 3 (i) 1 813, (ii) 56260.

4 (i) $\dfrac{2c(a-b)}{ab}$, (ii) 0, −3 5 £580 at 1s; £920 at 9d

6 1 82, −0 82 7 9 8 (i) $\dfrac{39\sqrt{7}}{7}$, (ii) $25\sqrt{3}$

9 (i) 20, (ii) −1, −1, 2, −4 10 (i) 275, (ii) −1705 11. 109 yds

12 $4\frac{1}{2}$, $1\frac{1}{2}$. 13 (i) $(a+y)(a-y)(y+1)(y-1)$, (ii) $(4x+9a)(3x-2a)$

14. −1 15 (i) 1, −1, 2, 4; (ii) 3, $-3\frac{1}{2}$ 16 $\dfrac{\sqrt[3]{x^2}+3}{3\sqrt[3]{x^2}+5}$

17. $18\left\{1-\left(\frac{5}{6}\right)^{n}\right\}$ 20. $n(n+1)(n-5)(n-6)$, 1

21 $-\frac{1}{2}$, 2, $4\frac{1}{2}$, $14\frac{1}{2}$. 22 5, −8.

23 (i) $S=\frac{1-3^{2n}}{4}$, $l=-3^{2n-1}$, (ii) $S=-2n$, $l=1-4n$

24 54 minutes after B's start 26 22 27 x^2+x+1

28 (i) $\frac{7}{8}$, (ii) $(a+b)^2$, (iii) $(a+b)$ 30 21 miles

31 $a=-5$, $b=7$ 32 $£\frac{(p+q)x}{a+b}$ 33 $\frac{n(3n-7)}{2}$

34 (i) $\sqrt{ab}$, (ii) $x=-11$, $y=13$, $x=9$, $y=-12$

35 $x-2y$, $y-2x+2$, $y-x-3$ 37 5 38 $\frac{\sqrt{7}}{3}$, 0 882

39 (i) $\frac{a-b}{a+b}$, $-\frac{a+b}{a-b}$, (ii) $\frac{-a(a-b)}{a+b}$ 40 10, 12, 15

41 9 50 in, 2 50 in Let AB be the line; find a point X in BA produced so that BX=45 5 approx Then $AX^2=3AB\ BX$

42 $5\frac{1}{3}$ 43 (i) 1 961, (ii) 15 77 44 $2n+5$

45 £3 15*s* 9*d*. 46 (i) $\sqrt{a+2x}-\sqrt{a-3x}$, (ii) $\sqrt{a-b}-\frac{2}{\sqrt{a-b}}$

47 (i) $\frac{1}{2}$, (ii) $\frac{1}{4}$, $\frac{9}{4}$ 48 $111\frac{1}{9}$ yds

XXXIII a. Page 373

1 (i) $\frac{4}{7}$, (ii) $\frac{6}{13}$, (iii) $\frac{2xyz}{3}$, (iv) $\frac{5a}{3b}$

2 3 $2a^2$ 3 5 tons 5 cwt, 8 tons 5 cwt 4 5 21

5 (i) 2, (ii) $\frac{7}{11}$ 6 2 1 7 1 3 or 2 1 8 5 1 or 2 5

9 b a or $3b$ $2a$ 10 15 11 5 1 12 14, 21 13 52, 91

XXXIII. b. Page 378

6 52, 78, 91 yards

13 $\frac{x}{bn-cm}=\frac{y}{cl-an}=\frac{z}{am-bl}$ 14 $\frac{x}{5}=\frac{y}{3}=\frac{z}{2}$ 15 $\frac{x}{6}=\frac{y}{3}=\frac{z}{4}$

16 $x=3$, $y=4$. 17 $x=\frac{38}{13}$, $y=\frac{59}{13}$ 18 $x=\frac{q^2-pr}{q^2-p^2}$, $y=\frac{pq-r^2}{q^2-p^2}$

19 $x=2$, $y=1$, $z=1$, $x=-2$, $y=-1$, $z=-1$ 20 $x=3$, $y=4$, $z=1$

XXXIII c Page 382

1 bc^3 2 $6a^4$ 3 9 6 lbs

4 $6pq^3$ 5 $45a^2b^2c^3$ 6 $12\sqrt{2}$ 7 x^2 8 0 126

9 $\sqrt{3}$ 10 3 15 tons 11 280 mi 12 4 13 $x=9$, $y=12$

30 13 28 34 ± 2 35 0, 5 36 $\sqrt{6}$ 37 $3a$

42 40, 20 43 A, £6000, B, £4000 44 20 1

45 3, 6, 5, 10 46 4 lbs from A, 5 lbs from B 47 5 4

48 3 1 49 11 50 Copper 72 5%; tin 27 5%

XXXIV. a. Page 388

2 (i) $\frac{1}{4}$, (ii) ± 4 3 21 6, 5 76

4 16, $\pm\frac{9}{2}$ 5 $5\frac{1}{4}$, 14, $\frac{1}{2}$ 6 $86\frac{3}{8}$ sq ft

7 $s=16\ 1t^2$ 16 ft, 145 ft 8 0 4, 3 6, 6 4 10 3 125 cu. ft.

11 $y=k/x$, where $k=24$ 12 6 in 13 $8\frac{1}{2}$

14 $6y=5x+5\sqrt{x}$ 15 64 cm 16 10 17 $\frac{1}{n}$, ab

XXXIV. b Page 392 1 (i) $\frac{64}{27}$, (ii) 1944. 2 $A=\frac{3}{10}\frac{B}{C}$ 9.
3 280 cu cm 4. 20 mi per hr 5 256 6 2.
7 27 35 8 £11 14s 5d 9 8 5 10 15, 57, 87.
11 £136 12 19s 2d 13 $a=0\ 6$, $b=0\ 8$ 68 lbs
15 80 16 1 212 17 30 mi per hr 18 1503
19 £80, £45 20 897 1000 21 770

XXXV. a Page 399 1 and 2 Rational and unequal
3 Real, but irrational 4 Rational and equal
5 Imaginary 6 Equal, but opposite in sign
7 (i) Unreal; (ii) real, but irrational, (iii) rational; (iv) unreal
8. (i) $\frac{1}{4}$, (ii) 1, or $-\frac{1}{5}$ 10 ± 4 11 5, or -7. 12 7, or -9
13 $x^2+x-20=0$ 14 $x^2+20x+91=0$ 15 $x^2-ax-42a^2=0$
16 $x^2-2cx+c^2-d^2=0$ 17 $18x^2-27x+10=0$ 18 $x^3-19x+30=0$
19 $3abx^2-(a^2-9b^2)x-3ab=0$ 20 $x^2-8x+13=0$
21 $2x^2-6x+1=0$ 22 $x^4-3x^3-25x^2+75x=0$
23 $x^2+6x+7=0$ 24 $x^2-2mx+m^2-n=0$
25 (i) 1, $\frac{r-p}{p-q}$, (ii) 1, $-\frac{a+b+c}{a+b}$, (iii) c, $\frac{b-ac}{a}$; (iv) 1, $-\frac{194}{5}$
26 (i) $(b-c)^2+4a^2$, (ii) $(a-b)^2+4c^2$

XXXV. b Page 401. 1 (i) 33, -175, (ii) $\frac{5}{4}$, $\frac{9}{8}$, (iii) $-\frac{99}{25}$, $\frac{149}{125}$.
2 (i) $\frac{q^2-2pr}{p^2}$, (ii) $\frac{q(3pr-q^2)}{p^3}$, (iii) $\frac{q^2-4pr}{p^2}$, (iv) $-\frac{qr}{p^2}$
3 (i) $\frac{b^4-4ab^2c+2a^2c^2}{a^4}$, (ii) $\frac{c(b^2-2ac)}{a^3}$, (iii) $\frac{b(b^2-3ac)}{a^2c}$, (iv) $\frac{(a+c)^2}{ac}$
4. $a^2x^2+4ac-b^2=0$ 5 $2b^2+9ac=0$ 6 $l^2x^2+3lmx+2m^2+ln=0$.
7 (i) ac, (ii) c^2, (iii) $\frac{b^2-2ac}{a^2c^2}$, (iv) $\frac{b(b^2-3ac)}{a^3c^3}$
8 $x^2\mp(m+n)x+m^2-mn+n^2=0$ 9 -2, or $-\frac{46}{11}$.
13 $125x^2-28x+27=0$

XXXV c Page 406. 1 Between -4 and 3
2 Between $2\frac{1}{2}$ and 4 3 Between -8 and $-\frac{2}{3}$
4. Between -2 and 3 5 Between $-\frac{1}{4}$ and 3
6 Between -8 and $4\frac{1}{2}$ 7 (i) Positive, (ii) negative
9 (i) Positive except when x lies between $-2\frac{1}{2}$ and 3,
(ii) negative except when x lies between a and $-b$, (iii) positive
10 Max. value $=6\frac{1}{4}$ Min value $=14$ $4+3x-x^2$ is negative except when x lies between -1 and 4. $4x^2-4x+15$ is always positive
14. (i) Any value except between -4 and 10, (ii) between $\frac{1}{3}$ and 3,
(iii) between -1 and $\frac{5}{7}$
15 (i) Positive; (ii) positive, (iii) negative 18 11 and 3

XXXVI a Page 410 1 $x+c$ 2 $x-q$ 3 x^2-3x+2

4 $a^2+(b+c)a+(b^2-bc+c^2)$ 5 ax^2-bx+c

6 $x^2-(m+n)xy+m(m-n)y^2$ 7 $a(a-2)x^2+2(2a-1)x-(a^2-1)$

8 $(m^2-9)x^2+2(m^2+3)xy+(m^2-1)y^2$

9 $x^3-2ax^2+(a^2+ab-b^2)x-ab(a-b)$ 10 $x-2a$

11 $(m-3)x-(m+1)$ 12 $2x+3$

13 H C F $=x^2-1$ L C M $=(x^2-1)(x^2-px+q)(x^2-qx+p)$

14 H C F $=px-(p-1)$

L C M $=\{px-(p-1)\}\{(p+1)x+p\}\{(p+2)x+p+1\}$

15 $25x^4-115x^2y^2+81y^4$ 16 x^6-y^6 17 x^6-64

18 $16x^4(1-4x^2)$ 19 $64m^4(9m^2-1)$ 20 $7x+y+z$

21 $x+5$ 24 x^8-16 $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$

25 $(bx-a)(ax-b)$ 26 $(3x-b)(x-2a)$

27 $(m-n)(m+n+x)(m+n-x)$ 28 $(a+b)(c+a-b)(c-a+b)$

29 $(x^2z^2+y^2)(xy+z)(xy-z)$ 30 $(2x+3y)(a^2+xy)$

31 $\{ax+(a+2)\}\{(a-3)x-a\}$ 32 $\{(a+1)x-(b-1)\}(ax+b)$

33 (i) $(a^4-4a^2b^2-b^4)(a^2+b^2)(a+b)(a-b)$,

(ii) $(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$

34 $-(b-c)(c-a)(a-b)$ 35 $(b-c)(c-a)(a-b)$

36 $(b+c)(a+b)(c+a)$ 37 $-(b-c)(c-a)(a+b)$

38 $-(b-c)(c-a)(a-b)(a+b+c)$

40 $(a+b+2c)(a^2+b^2+4c^2-ab-2bc-2ca)$

41 $(a-3b+c)(a^2+9b^2+c^2+3ab+3bc-ca)$

42 $(1+3x-2y)(1+9x^2+4y^2+6xy+2y-3x)$

43 $(x-2y-3)(x^2+4y^2+9-6y+3x+2xy)$ 46 0 57 0

XXXVI b Page 413 1 (i) 0, (ii) 1, (iii) 1 2 1

3 $\dfrac{1}{abc}$ 4 $a+b+c$ 5 d 6 $\dfrac{1}{(x-a)(x-b)(x-c)}$

7 $\dfrac{x^2}{(x+a)(x+b)(x+c)}$ 8 $a+b+c$ 9 $bc+ca-ab$

XXXVI c Page 416 1 20 2 -7

3 $a=-10$, $b=-24$ 4 $a=3$, $b=6$

6 (i) $-(b-c)(c-a)(a-b)$, (ii) $-(b-c)(c-a)(a-b)(a+b+c)$,

(iii) $(b-c)(c-a)(a-b)(bc+ca+ab)$

14 (i) $-\dfrac{a+b+c}{3}$, (ii) $bc+ca+ab$, (iii) $\frac{5}{3}(x^2+y^2+z^2-yz-zx-xy)$

XXXVI d Page 420 1 $A=8$, $B=-6$, $C=-8$

2 $2-3(x+1)+4x(x+1)$ 3 $l=3$, $m=-4$, $n=5$

4 $A=C=1$, $B=2$ 5 $A=3$, $B=-2$ 6 $A=5$, $B=-3$

7 $A=3$, $B=-2$ 8 $A=-2$, $B=1$, $C=10$

9 $A=3$, $B=-2$, $C=4$ 10 $A=4$, $B=-3$ $C=-5$

11 $3(x+1)^3-9(x+1)^2+8(x+1)$ 12 $(x-2y+1)(x+4y+3)$

13 $(2x+y+2)(x-5y+10)$ 14 -6

15 (i) $2+3x-4x^2+3x^3$, (ii) $4x^4-2x^2+x-5$

16 $3x^2-x+6$ 17. $16a^4$ 18 $l=-4$, $m=4$ 20 13

22 (i) $\dfrac{q^2-2pr}{p^2}$; (ii) $\dfrac{3pqr-q^3}{p^3}$ 26 $-(b-c)(c-a)(a-b)$

XXXVII. a. Page 425. 1 $(2n-1)x^{n-1}$, $\dfrac{1-(2n-1)x^n}{1-x}+\dfrac{2x(1-x^{n-1})}{(1-x)^2}$

2 $n\ 3^{n-1}$, $\dfrac{(2n-1)3^n+1}{4}$ 3 $(3n-2)x^{n-1}$, $\dfrac{1-(3n-2)x^n}{1-x}+\dfrac{3x(1-x^{n-1})}{(1-x)^2}$

4. $(2n-1)2^{n-1}$, $(2n-3)2^n+3$ 5 $\dfrac{1}{(1-x)^2}$ 6 $\dfrac{1+3a}{(1-a)^2}$

7 $\frac{25}{36}$ 8 $\frac{3}{16}$ 9 $\dfrac{6r-r^2}{(1-r)^3}$ 10 $\dfrac{1+x}{(1-x)^3}$

12 $(2n-1)^2$, $\dfrac{n}{3}(4n^2-1)$ 13 n^2+n; $\frac{1}{3}n(n+1)(n+2)$

14 $n(2n+1)$, $\frac{1}{6}n(n+1)(4n+5)$ 15 $(3n-2)(3n+1)$; $n(3n^2+3n-2)$

16 $n(n+1)(n+2)$, $\frac{1}{4}n(n+1)(n+2)(n+3)$

17 $n(n+3)(n+6)$, $\frac{1}{4}n(n+1)(n+6)(n+7)$

18 $\frac{1}{3}n(n+1)(n+5)$ 19 $2n(n+1)^2$ 20 $n(n+1)(2n+1)-2(2^n-1)$.

21 $\frac{1}{3}n(n+1)(2n^2-1)$ 22 $\left\{\dfrac{n(n+1)}{2}\right\}^2-\frac{3}{2}(3^n-1)$

23 $n(n+1)^2(n+2)$

XXXVII b. Page 428. 1 $\dfrac{140+99\sqrt{2}}{8}$

2 $14(\sqrt{3}+1)$. 9 1 and 7

13 (i) $\dfrac{1}{x-y}\left\{\dfrac{x^2(1-x^n)}{1-x}-\dfrac{y^2(1-y^n)}{1-y}\right\}$, (ii) $\dfrac{10}{81}(10^n-1)-\dfrac{n}{9}$

14 461 yds 15 £1912 16*s* 19 18

20 *A* and *B* will be *together* on the 10th and 33rd days They meet on the 10th day, on the 11th *B* does not walk at all, but on the 12th he begins to walk *back* 2 mi ; on the 13th 4 mi *back*, and so on On the 33rd day he overtakes *A*

21 44 hours 22 (i) $\frac{1}{12}n(n+1)(3n^2+19n+26)$, (ii) $2n-2+\dfrac{1}{2^{n-1}}$

25 1st term$=(b+x)^2$, common diff $=-2bx$, nth term$=b^2+x^2-2(n-2)bx$.

27 $\dfrac{PQ(p-q)}{pQ-qP}$

XXXVIII. a. Page 436. 18 $x=3$, $y=2$, $x=-3$, $y=-2$

20 $-1, 1, 2$ 21 $-2, 1, 4$ 23 (i) 1; (ii) 1, 2, -3.

25 1 37. 26 (i) 15, (ii) 10; (iii) $2a$; (iv) 0.

27 The gradient of the tangent is zero

XXXVIII. b. Page 440.

1 $n=3$, $c=27\times10^5$
2 $n=1.5$, $c=79500$ 3 $n=\frac{17}{16}$ 4 $n=0.86$, $c=207.$
5 $a=1.78$, $b=31.5$ 6 $T=36.31h^{1.5}$; $V=1271h^{1.5}$

Miscellaneous Examples VIII. Page 441.

1 $x+2$ 3 x^2+2x+3
5 (i) 0, 0, $\frac{5}{14}$, (ii) $x=\frac{c}{a}(a+b)$, $y=\frac{c}{b}(a+b)$, ∞, ∞, $2\frac{4}{5}$
7 (i) 1625, (ii) 9.093 8 66 mi 9 1
10 $\frac{1}{a}+\frac{1}{b}$ 11 (i) $9n(7-n)$, (ii) $162\{1-(\frac{2}{3})^n\}$
13 28, 27 14. $(x+1)(x-1)(y+1)(y^2-y+1)$
15 £120 at 4%, £80 at $3\frac{1}{2}$% 16 259, 621
17 (i) $(a-b)(a+b+x)(a+b-x)$, (ii) $2(3xy+4)(4xy-3)$
18 -7 19 $\frac{pr}{(p+q)(r+s)}$ ounces 20 (i) 0, (ii) $\frac{\sqrt{2}}{4}$
22 4.616 23 (i) $\frac{1}{7}$, (ii) $x=4$, or $\frac{3}{2}$
24 $v=6$, $p=45$ 25 $3x^2y^2-x^{\frac{5}{2}}y^{\frac{3}{2}}+2x^{\frac{3}{2}}y^{\frac{5}{2}}-4xy^3-4y^4$
26 (i) $\frac{3}{5}$, (ii) $x=4$, $y=6$, $x=6$, $y=4$ 28 500 29 96
30 Just before 11.30 a.m., rather more than $16\frac{1}{2}$ miles from A
31 $a^2+b^2=c^2$ 32 £700
33 (i) $x=3$, $y=1$, $x=1\frac{1}{6}$, $y=1\frac{11}{12}$, (ii) -1, $-\frac{1}{2}$
35 $(a+b-2c)(a-b-2c)$, $p=-6$ 36 £12 15s
37 1 38 5s 6d 39 3.1415 40 2
41 (i) $\frac{x-y}{x+y}$, (ii) $\frac{x}{(x-n)(x+n+1)}$
42 (i) $x=\frac{1}{3}$, $y=\frac{1}{2}$, (ii) $x=y=\frac{1}{2}ab$ 44. $\frac{2ab}{a+b}$
45 $(3c-2d)(2c-3d)(c+5d)$ 46 $x^2-nx+ml=0.$
47 $2-18x^2+27x^3-18x^4$ 48 $-2, 0, 2, 4, 6$
50 13 51 $1-2a^{\frac{1}{4}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}b^{\frac{2}{3}}$ 52 -0.9916
53 (i) $52\frac{1}{2}$, (ii) 20 54. $\frac{1}{4q-p^2}$ 56 Max height $=64$ ft; 4 secs.
58 (i) $3b$; (ii) 4 60 $82\frac{1}{2}$ m, 88 m 62 11
64 -0.382, -2.618 65 $\sqrt{2x+3}+\sqrt{x-2}$
66 $x=5$, $y=3$, $x=-3$, $y=-5$ 67 $48\sqrt[3]{4}$ 68 13 m
71 42 73 1 74 2, 5, 8, 11 75 (i) 3, 4, -2, -3, (ii) 1, 2, 3
77 $\frac{13a(3+\sqrt{3})}{9}$, $9-3\sqrt{3}$ 79 $5\frac{1}{16}$ 81 $\frac{x+3}{x-3}$ 82 $\sqrt{3}+1$
83 $b=-3$, $c=-2$ 84. $(3a^2+b^2)(a^2+3b^2)$
85 $2x^2+4ax-3a^2$ 86 $nx^2-2mx+m=0$

87. $x=2,\ y=4,\ z=6$; $x=-2,\ y=-4,\ z=-6$

89 $\left(\dfrac{a}{x}\right)^{a+x}$. 90 A=33, B=24, C=5

95 2 in 96. 2, 0·6, −2·6 97 $a=2\frac{1}{2},\ b=5$ 98. 65654.

99 0·02995. 100 29. 102 $\dfrac{n(n+1)(4n+11)}{6}$ 103. 1·5

104. 2, 2, −1. 106. 2 109. 19·78 in

110 46 miles from the starting point

111 $52n(n+12)$ and $26n(2n+25)$ shillings

112. 3 inches from the point of suspension 113. $a+b$

114. $2a-b$, $-(2a-3b)$

119. 22 lbs, $16\frac{1}{2}$ lbs, $14\frac{3}{4}$ lbs, $13\frac{1}{4}$ lbs, 11 lbs, $9\frac{1}{2}$ lbs, 8 lbs, $7\frac{1}{4}$ lbs. The curve is a rectangular hyperbola whose equation is $xy=22\times 12$.

120. E=0·84W+10·2, R=2·32W−30·5; P=1·48W−40·7 (i) 40; (ii) 28.

PART III

XXXIX. a. Page 453 1. 12 2 18 3. 120
4. (i) 144, (ii) 132 5 720 6 840 7. 24, 18
8 120, 24, 96, 6 9 72

XXXIX. b Page 455. 1. 5040, 120, 2520, 40320, 3024, 60480
2 (i) 720, (ii) 720 3 6 4. 120, 24 5 60
6 (i) 120, (ii) 600 7 144. 8 696 9 144 10 480.
12 6720 13 10368000 14. (i) 144, (ii) 72

XXXIX c. Page 458
1 (i) 1260, (ii) 3360, (iii) 166320, (iv) 64864800, 180, 83160
2 12, 18 3 420, 360 4. 302400 5 (i) 40, (ii) 8.
6 420 7 120 8 2520 9 48 10 243, 48
11 1296 12 81 13 624 14. 30 15 64

XXXIX d Page 464. 1 20, 84, 55, 1225, 105 2 1140
3 12250 4. (i) 210; (ii) 63, (iii) 35 5 (i) 126, (ii) 63, (iii) 203
6 8 7 13 9 55200 10 327600 11 24000
12 1485 13 1000 14. 35, 201600 15 255 16 246
17 1023 18 511 19 $\dfrac{\lfloor 30}{\lfloor 10\,\lfloor 12\,\lfloor 8}$ 20 (i) $\dfrac{\lfloor 52}{(\lfloor 13)^4\,\lfloor 4}$; (ii) $\dfrac{\lfloor 52}{(\lfloor 13)^4}$.
21 480 22 80 23 2903040 24. 2880 25 $\dfrac{\lfloor mn}{(\lfloor m)^n}$
26 315 27 (i) 40, (ii) 116 28 5760 29 $(n-2)\lfloor n-1$
30 (i) 60, (ii) 15120 31 (i) 378, (ii) 120 32 $(n+1)^m-1$
33 (i) 15, (ii) 208 34. 70, 35 35 $(n-2)(n-3)\lfloor n-2$.
36 18480 37 (i) 113, (ii) 2190 38 1343

XLI a. Page 476 1 (i) x^3+5x^2+2x-8,
(ii) $x^4+6x^3-21x^2-74x+168$, (iii) x^4-41x^2+400,
(iv) $a^3+6a^2b-37ab^2-90b^3$
2 $x^4+8x^3+24x^2+32x+16$ 3 $x^5+15x^4+90x^3+270x^2+405x+243$.
4. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$
5 $a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$.
6 $1+8b+28b^2+56b^3+70b^4+56b^5+28b^6+8b^7+b^8$
7 $1-10y+40y^2-80y^3+80y^4-32y^5$
8 $1+3x+\frac{15}{4}x^2+\frac{5}{2}x^3+\frac{15}{16}x^4+\frac{3}{16}x^5+\frac{1}{64}x$.

9 $16x^4+16x^3y+6x^2y^2+xy^3+\frac{y^4}{16}$

10 $64-96x+60x^2-20x^3+\frac{15x^4}{4}-\frac{3x^5}{8}+\frac{x^6}{64}$

11 $a^7-\frac{21a^6}{b}+\frac{189a^5}{b^2}-\frac{945a^4}{b^3}+\frac{2835a^3}{b^4}-\frac{5103a^2}{b^5}+\frac{5103a}{b^6}-\frac{2187}{b^7}$

12 $x^5-\frac{5x^3}{2}+\frac{5x}{2}-\frac{5}{4x}+\frac{5}{16x^3}-\frac{1}{32x^5}$

13 $a^9x^9+9a^7x^8y+36a^5x^7y^2+84a^3x^6y^3+126ax^5y^4$
$+\frac{126x^4y^5}{a}+\frac{84x^3y^6}{a^3}+\frac{36x^2y^7}{a^5}+\frac{9xy^8}{a^7}+\frac{y^9}{a^9}$

14 $2(5a^4b+10a^2b^3+b^5)$ 15 $2(729+4860x^2+2160x^4+64x^6)$.

16 $2(x^4+18x^2+9)$ 17 $140\sqrt{2}$

19 $x^6-3x^5-3x^4+11x^3+6x^2-12x-8$

20 $1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$.

21 $1-6a+21a^2-44a^3+63a^4-54a^5+27a^6$

22 (i) $a^8+4a^7x+4a^6x^2-4a^5x^3-10a^4x^4-4a^3x^5+4a^2x^6+4ax^7+x^8$;
(ii) $1+x-8x^2-2x^3+25x^4-11x^5-26x^6+28x^7-8x^8$

23 $280x^3$ 24 $-448y^5$ 25 $43750a^4b^4$ 26 $5440x^3$.

27 $\frac{210}{x^6}$ 28 $\frac{5103x^4a^5}{16}$ 29 (i) $20x^3$, (ii) $\frac{2835}{8}$

30 $126x$, $-\frac{126}{x}$ 31 -252 32 $\frac{2300b^{22}}{x^{16}}$ 33 3360

34 $-\frac{1001a^9}{256}$ 35 70, -56 36 7920

XLI. b. Page 480

1 The 20th 2 The 5th.

3 The 12th and 13th 4 The 6th. 5 The 5th.

6 The 4th and 5th, $=\frac{224}{9}$ 7 The 4th $=6048$

9 (i) $\frac{2816}{27}$, (ii) 4725000 10 5 11 22

12 4096 13 3125 22 -285

23 $729-2916x+4860x^2-4320x^3+2160x^4-576x^5+64x^6$ 24 $-167960x^7$.

25 $1+1+\frac{x-1}{2x}+\frac{(x-1)(x-2)}{6x^2}+\dots$, $(r+1)^{\text{th}}$ term $=\frac{\underline{|x}}{\underline{|r}\;\underline{|x-r}}\frac{1}{x^r}$, $2\cdot495$

26 $\frac{\underline{|n}}{\underline{|r-1}\;\underline{|n-r+1}}a^{n-r+1}(2x)^{r-1}$, $\frac{\underline{|n}}{\underline{|r-1}\;\underline{|n-r+1}}a^{r-1}(2x)^{n-r+1}$.

27 -105 29 $1\cdot061$ 30 49 31 56

32 $840a^4$ 35 $\frac{\underline{|n}}{\underline{|\frac{1}{5}(3n+r)}\;\underline{|\frac{1}{5}(2n-r)}}$

XLI. c. Page 486.

1 $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$

2 $1+\frac{3}{4}x-\frac{3}{32}x^2+\frac{5}{128}x^3$ 3 $1-\frac{2}{5}x-\frac{3}{25}x^2-\frac{8}{125}x^3$.

4 $1-6x+27x^2-108x^3$ 5 $1-3x^2+6x^4-10x^6$

6 $1-12x+90x^2-540x^3$ 7 $\frac{1}{8}-\frac{3}{16}x+\frac{3}{16}x^2-\frac{5}{32}x^3$

8 $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$ 9 $a^{-\frac{3}{2}}\left(1+\frac{3x}{a}+\frac{15x^2}{2a^2}+\frac{35x^3}{2a^3}\right)$

10 $1-\frac{5}{3}x+\frac{15}{8}x^2-\frac{5}{16}x^3$ 11 $3\left(1+\frac{1}{9}x-\frac{1}{162}x^2+\frac{1}{1458}x^3\right)$

12 $4\left(1+a-\frac{1}{4}a^2+\frac{1}{8}a^3\right)$ 13 $\frac{315}{128}x^4,\ -\frac{230945}{65536}x^9$

14 $\frac{99}{2}x^2,\ -\frac{77}{256}x^{10}$ 15 $-4x^3,\ (-1)^r(r+1)x^r$

16 $-\frac{21}{1024}x^6,\ -\frac{1\ 3\ 5\quad(2r-3)}{2r\lfloor\underline{r}}x^r$

17 $\frac{b^r}{a^{r+1}}x^r,\ -\frac{(n-1)(2n-1)\quad\{(r-1)n-1\}}{\lfloor\underline{r}}x^r$

18 $(-1)^{r-3}\frac{5\ 3\ 1\ 1\ 3\quad(2r-7)}{2^r\lfloor\underline{r}}x^r,\ (-1)^r\frac{3\ 5\ 7\quad(2r+1)}{\lfloor\underline{r}}x^r$

19 $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor\underline{4}}x^r$ 20 $(-1)^r\frac{(r+1)(r+2)}{2}x^{2r}$

21 $(-1)^r\frac{1\ 3\ 5\ 7\quad(2r-1)}{2^r\lfloor\underline{r}}x^r$ 22 $(-1)^r\frac{2\ 5\ 8\quad(3r-1)}{3^r\lfloor\underline{r}}x^r$

23 $\frac{(r+1)(r+2)\quad(r+9)}{\lfloor\underline{9}}\frac{x^r}{2^{r+10}}$

24 $(-1)^r\frac{p(p+q)(p+2q)\quad(p+\overline{r-1}\,q)}{\lfloor\underline{r}}\left(\frac{x}{q}\right)^r$

25 $(-1)^r\frac{1\ 3\ 5\quad(2r-1)}{\lfloor\underline{r}}x^r$ 26 $\frac{2\ 5\ 8\quad(3r-1)}{\lfloor\underline{r}}x^r$

27. $-\frac{2\ 1\ 4\quad(3r-5)}{3^r\lfloor\underline{r}}x^r$ 28 (i) $-\frac{5}{8}x^4$; (ii) $-\frac{4}{81}x^3$, (iii) $-\frac{22}{3}x^5$

XLI d Page 487 1 The 7th 2 The 2nd 3 The 38th and 39th

4 The 1st and 2nd 5 The 3rd 6 The 4th 7 (i) $\frac{5}{4}$, (ii) $\frac{77}{2}\ 3^{\frac{7}{4}}$

8 (i) 4, (ii) 201, (iii) -5050

9 (i) $3r$, (ii) $\frac{1}{2}(r^2+r+2)$, (iii) $(-1)^{r-1}(2r-1)$

11 $(-1)^r\frac{(r+1)(r+2)(5r+6)}{6}$ 12 $\frac{1}{2}\sqrt{3}$ 13 $\sqrt[3]{2}$

18 $\frac{3}{2}$, the series is the expansion of $\left(1-\frac{1}{3}\right)^{-5}\times\left(\frac{2}{3}\right)^4$

22 (i) $\frac{1\ 3\ 5\quad(2r-1)}{2^r\lfloor\underline{r}}$, (ii) $\frac{(r+1)(r+2)(r+3)}{\lfloor\underline{3}}$, (iii) $(-1)^r$

(iv) $\frac{\lfloor\underline{2n}}{\lfloor\underline{r}\,\lfloor\underline{2n-r}}$

XLI e Page 492 1 1·08243 2 0·85873 3 7·07106

4 10·0100 5 6·0092 6 1·9743 7 4·9900

8 9·9933 9 0·1459 10 0·1995 11 125·1500

12 1·0013 13 1·414214 14. $1+\frac{13}{6}x$ 15 $4-\frac{29}{48}x$

16 $1-\frac{5}{6}x$ 17 $1-\frac{7}{90}x$ 18 $1-\frac{9}{4}x$ 19 $1+2x$

20 $16\left(1-\frac{41}{40}x\right)$ 21 $1-\frac{35}{192}x$ 22 $1-3x+\frac{35}{4}x^2$

23 1 0032 24 0 9974 25 1 0054. 26 1 00017 27 0 99284.

28 1 00048 29 1 0076 30 0 676 31 1 0002 32 1 017.

33 1 06 34. 1 0015 35 0 9988 36 4 5 % 37. 0 5 sq cm.

38 1003 sq ft 40. 0 0000009. 41 360 42 1 3 in excess

43 2 6 % in defect 44. 7 8 % in excess 45 $\frac{5\ 11\ 13x^9}{2^{16}}$.

46 $\frac{1}{6}n(n-1)(n-38)$ 47 $\frac{1}{\sqrt{2}}\left(\frac{1}{4}+\frac{11x}{16}+\frac{179}{128}x^2\right)$

48 (i) -73, (ii) $-(8n^2+16n+9)$ 49 $\frac{1\ 3.5\ \ (2n-1)}{\lfloor n}2^n=\frac{\lfloor 2n}{\lfloor n\lfloor n}$

51 $\frac{n}{2a^3}\{2a-(n+1)b\}$ 52 $\sqrt{10}=3\ 162$, $\sqrt{26}=5\ 099$, $\sqrt{123}=11\ 091$.

XLII. Page 498. 1 $\frac{3}{x+5}+\frac{2}{x-3}$ 2 $\frac{15}{x-1}-\frac{11}{x-2}$

3 $\frac{8}{x+4}+\frac{6}{x-3}$ 4 $\frac{3}{1-2x}+\frac{5}{1+3x}$ 5 $\frac{3}{x-1}-\frac{3}{x-2}+\frac{1}{x+3}$.

6 $\frac{3}{2-x}+\frac{2x-1}{x^2+x+1}$ 7 $\frac{1}{x}+\frac{2}{x+1}-\frac{3}{(x+1)^2}$

8 $\frac{2}{x+2}-\frac{1}{x-1}+\frac{3}{(x-1)^2}$ 9 $x+2+\frac{2}{x-3}+\frac{3}{x+1}$

10 $\frac{3}{x+2}-\frac{2}{x+3}-\frac{4}{(x+3)^2}$ 11 $-\frac{1}{3(1+2x)}+\frac{5}{3(x+2)}-\frac{4}{(x+2)^2}$.

12 $\frac{15x+2}{x^2+1}-\frac{12}{x+6}$ 13 $\frac{1}{1-x}+\frac{2}{1-2x}$; $(1+2^{r+1})x^r$.

14. $\frac{1}{x-5}-\frac{1}{x-3}$, $\left(\frac{1}{3^{r+1}}-\frac{1}{5^{r+1}}\right)x^r$

15 $\frac{1}{2+x}+\frac{3}{1+3x}$, $(-1)^r\left\{\frac{1}{2^{r+1}}+3^{r+1}\right\}x^r$

16 $\frac{3}{1+2x}-\frac{5}{1-x}$, $\{(-1)^r 3\ 2^r-5\}x^r$

17 $\frac{1}{1-x}-\frac{1}{1+x}-\frac{4}{1-2x}$, $\{1+(-1)^{r-1}-2^{r+2}\}x^r$.

18 $\frac{1}{2(1-x)}+\frac{1}{6(1+x)}-\frac{4}{3(2-x)}$, $\left\{\frac{1}{2}-\frac{1}{3}\ \frac{1}{2^{r-1}}+\frac{(-1)^r}{6}\right\}x^r$.

19 $\frac{1}{4(1+4x)}+\frac{11}{4(1+4x)^2}$; $(-1)^r(12+11r)4^{r-1}x^r$

20 $\frac{2}{1+x}+\frac{3}{(1+x)^2}-\frac{6}{2+3x}$, $(-1)^r\left(3r+5-\frac{3^{r+1}}{2^r}\right)x^r$

21 $\frac{4}{2-x}-\frac{9}{4(3-x)}-\frac{7}{4(1-x)}+\frac{1}{2(1-x)^2}$, $\left(\frac{2r-5}{4}+\frac{1}{2^{r-1}}-\frac{1}{4}\ \frac{1}{3^{r-1}}\right)x^r$.

22 $\frac{1}{x-4}+\frac{2}{(x-4)^2}-\frac{3}{(x-4)^3}$; $\frac{1}{x+2}-\frac{3}{(x+2)^2}-\frac{2}{(x+2)^3}+\frac{4}{(x+2)^4}$.

23 $\frac{1}{3}\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$, $\frac{n}{3n+1}$

Miscellaneous Examples IX. Page 499. 1 30

2 $\frac{x-16}{19x+18}$, $\frac{18y+16}{1-19y}$ 3 174, $15x-4$, $8x+7$

4 167, $\frac{14}{83}$ 5 -5103 6 $x=1$, $y=2$, $x=\frac{4}{3}$, $y=\frac{5}{8}$

7 $y=\frac{4}{7}$, when $x=\frac{2}{3}$ $x=-\frac{1}{3}$, $y=\frac{1}{3}$

8 (i) $(mt+n)(m-nt)$, (ii) $(x^2+3xy+y^2)(x^2-3xy+y^2)$

9 (i) £$\frac{c-a+b}{2}$ in first, £$\frac{c+a-b}{2}$ in second;

(ii) £$\frac{2c-a+2b}{3}$,, £$\frac{c+a-2b}{3}$,,

10 (i) $\frac{1}{2}$, (ii) $a=1·25$, $c=1$ 11 7·8125 gallons 13 9.

14 $m+1$ 16 $\frac{2880q-23lp}{480(12q-p)}$ shillings 17 1·24 secs

18 $2x^2-9x-12=0$ 19 $9n+5$ 21 A=2, B=3, C=−4.

22 £$\{(a-b)x+a\}$ 23 (i) 0, $-\frac{17}{28}$, (ii) $\frac{q^2}{p}$, $-\frac{p^2}{q}$

24 $64\{(-\frac{1}{2})^n-1\}$; 16 27 1, 1·414, 2, 2·828, 4, 5·657, 8

29 (i) $(\frac{1}{2})^{2n-2}$, (ii) $\sqrt{6}-\sqrt{2}$ 33 0·002 too small (i) 1·809, (ii) 0·691

34 1 mile; *A*, 4′ 57″; *B*, 5′ 30″ 36 $\frac{a^2+ab+b^2}{a+b}$ 37 0, $\frac{20}{3}$

38 $7+4\sqrt{3}$; $2+\sqrt{3}$ 40 $n^2(b-a)$ 41 7 or 14.

42 −0·97 43 $x=\frac{b+c}{2a}$, $y=\frac{c+a}{2b}$, $z=\frac{a+b}{2c}$ 44 1, 2, 4, 8, 16,

45 $\frac{ac-b^2}{a+c-2b}$ 47 42 48 About 17·4 yrs 49 $\frac{1}{2}\sqrt{6}$

XLIII Page 510. 1 (i) $\frac{(-1)^r}{\lfloor r}(1+r)$; (ii) $\frac{(-1)^{r-1}}{\lfloor r}(ar-b)$ 2 $\frac{e^a b^n}{\lfloor n}$

4 $\frac{(-1)^r}{\lfloor r}(1+2r-r^2)$ 8 1 9 $e-1$ 10 $\frac{3}{2}e$. 11 $e^{a^2}-e^{b^2}$.

12 $\frac{1}{2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)$ 14 $-y+\frac{y^2}{2}-\frac{y^3}{3}+\ldots$ 15 0·69315.

16 $x-\frac{5x^2}{2}+\frac{7x^3}{3}-\frac{17x^4}{4}$, $\frac{(-1)^{r-1}2^r-1}{r}x^r$

17 $\frac{2^{2r}(-1)^{r-1}-2^r}{r}x^r$, $2x-10x^2+\frac{56x^3}{3}-68x^4$

18 $\frac{(-1)^{r-1}+3^r}{r}x^r$. 20 $3x-\frac{5x^2}{2}+3x^3-\frac{17x^4}{4}+\ldots$

21 0 0020000007 22 $\dfrac{x}{1-x}+\log_e(1-x)$.

23 Each series may be shewn to be equal to $\frac{1}{2}(1-3\log 2)$.

24 $\log 13=1$ 11394, $\log 17=1$ 23045 [For log 13, put $n=12$ in the series of Art 552 For log 17, put $n=50$ in the series (1) of Art 554 This gives log 51, or $\log 3+\log 17$]

25. 5569

XLIV Page 516. 1 £985 2 £737 3 £1793 4. 86 yrs.

5 £2184 6. 9 6 nearly 7 £9868 9 £3755 12s 6d

10 £11708 8s 11 £588 12 £1548 9s 13 £2231 12s 4d

14. $4\frac{1}{2}$% 15 16 16 £1020 17. £1340 1s 11d 18 £4200

XLV. Page 524. 1 22220 2 1380755 3 28001 4 251021.

5 (i) 244332343, (ii) 14320241 6 564, 26 7 345, 3024

8 32552, 3552, 215312024 9 1342, 34705 10 30523

11 123456 12 12^3+12^2+12 13 27 14 (i) 0203, (ii) 106314

15 132 12 16 26 17 200 211 18 (i) $\frac{21}{80}$, (ii) $\frac{111}{309}$

19 1892, 292 20 Five, Nine 21 Nine 22 Twelve

23 7 e 24 1 lb, 2 lbs, 32 lbs, 64 lbs, 128 lbs 25 $\frac{58}{105}$

XLVI Page 527. 9 All except such as lie between -2 and $+1$.

13 Unless $a=b=c$, or $a+b+c=0$

XLVII. a. Page 531. 1 $\dfrac{ab}{b-a}$ 2 $\dfrac{cd(a+b)-ab(c+d)}{ab-cd}$.

3 $\dfrac{a+b+3}{2}$ 4. $ab-bc-ca$ 5 $2ab-bc-2ca$ 6 $\dfrac{ab+bc+ca}{abc}$

7 $\dfrac{pq}{p-q}$ 8 $\dfrac{ab+c^2}{a-b-2c}$ 9 $-2(a+b+c)$ 10 $0,\ \dfrac{a^2+b^2+a^3+b^3}{a+b+a^2+b^2}$

11 2, -1 12 0, 3 13 9, 1 14 $\frac{4}{9}, \frac{1}{4}$ 15 0, -3

16 ± 1 17 1, 2, $\frac{1}{2}$, 4 18 3, -1, $\dfrac{3\pm\sqrt{21}}{2}$ 19 16, $-\frac{4}{11}$.

20 $\dfrac{a(b^2-c^2)}{b^2+c^2}$ 21 25, -3 22 20, 1 23 2, $\frac{1}{2}$ 24 $-2a$.

25 3, $-\frac{1}{2}$ 26 5, $\frac{1}{2}$ 27 2, $-\frac{14}{3}$. 28 3, $\frac{5}{87}$ 29 $\frac{21}{4}$.

30 7, $-\frac{31}{8}$ 31 2, $-\frac{1}{3}$ 32 2, $\frac{1}{2}$, 3, $\frac{1}{3}$ 33 2, $\frac{1}{2}$, $2\pm\sqrt{3}$

34 2, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{2}{3}$ 35 -2, $\frac{1}{2}$, -3, $\frac{1}{3}$ 36 2, -4, $-1\pm\sqrt{71}$

37 $-a$, $2a$, $-5a$, $6a$ 38 0, $\pm\frac{3}{10}$ 39 $\frac{9}{2}$, 10, -1

40 $\pm 4\sqrt{2}$ 41 0, $\dfrac{63a}{65}$. 42 1, $\dfrac{(\sqrt{a}-\sqrt{b})^2+4}{(\sqrt{a}+\sqrt{b})^2-4}$

XLVII. b. Page 536

1 $x=3,\ 4,\ \frac{1}{2}(7\pm\sqrt{-295})$; $y=4,\ 3,\ \frac{1}{2}(7\mp\sqrt{-295})$

2 $x=1,\ -2,\ 1\pm\sqrt{-15}$; $y=2,\ -4,\ -1\pm\sqrt{-15}$

3 $x=4,\ -1,\ \frac{1}{2}(3\pm\sqrt{-43})$; $y=-1,\ 4,\ \frac{1}{2}(3\mp\sqrt{-43})$

4 $x=0,\ \frac{10}{3}$; $y=\frac{19}{7},\ \frac{1}{3}$

5 $x=5,\ -5,\ \pm\frac{2}{5}\sqrt{145}$; $y=-2,\ 2,\ \mp\frac{1}{5}\sqrt{145}$

6 $x=2,\ y=-1,\ x=-\frac{6}{5},\ y=\frac{3}{5}$

7 $x=2,\ y=1,\ x=-\frac{17}{5},\ y=-\frac{1}{5}$

8 $x=7,\ y=3,\ x=-\frac{35}{4},\ y=-\frac{15}{4}$

9 $x=y=\frac{3}{4},\ x=0,\ y=1,\ x=1,\ y=0$

10 (i) $x=3,\ 2,\ -3,\ -2,\ y=-2,\ -3,\ 2,\ 3$ (ii) $x=3,\ y=1,\ x=-1,\ y=-3$

11 $x=5,\ y=1,\ x=\frac{21}{5},\ y=\frac{7}{5}$

12 $x=2,\ \frac{1}{2},\ x=-2,\ -\frac{1}{2},\ y=5,\ y=-5$

13 $x=8,\ y=2,\ x=3,\ y=7$

14 $x=\pm1,\ y=\pm6,\ z=\pm3$

15 $x=\pm3,\ y=\pm4,\ z=\pm6$

16 $x=\pm12,\ y=\pm6,\ z=\pm8$

17 $x=10,\ y=6,\ z=4,\ x=-\frac{10}{21},\ y=-\frac{2}{7},\ z=-\frac{4}{21}$

18 $x=3,\ y=2,\ z=1,\ x=-1,\ y=-12,\ z=-17$

19 $x=4,\ y=5,\ z=3,\ x=-6,\ y=-7,\ z=-5$

20 $x=\pm2,\ y=\pm1,\ z=\pm3$

21 $x=\pm\sqrt{2},\ y=\pm\sqrt{3},\ z=\pm\sqrt{6}$

22 $x=12,\ y=6,\ z=3,\ x=3,\ y=6,\ z=12$

23 $x=6,\ y=9,\ z=4,\ x=6,\ y=4,\ z=9$

24 $x=-5,\ y=3,\ z=1,\ x=-5,\ y=1,\ z=3$

25 $x=\pm\dfrac{a(b^2+c^2)}{2bc},\ y=\pm\dfrac{b(c^2+a^2)}{2ca},\ z=\pm\dfrac{c(a^2+b^2)}{2ab}$

26 $x=\pm4,\ y=\pm8,\ z=\mp6$

27 $x=3,\ y=2,\ z=1$; $x=1,\ y=2,\ z=3$; $x=\dfrac{-1\pm\sqrt{29}}{2},\ y=-3,\ z=\dfrac{-1\mp\sqrt{29}}{2}$

XLVII c Page 540

1 $x=7,\ 4,\ 1,\ y=2,\ 7,\ 12$

2 $x=11,\ 7,\ 3$; $y=2,\ 9,\ 16$

3 $x=3,\ 8,\ 13,\ y=6,\ 4,\ 2$

4 $x=2,\ y=4$

5 $x=10,\ y=8$

6 $x=26,\ 13,\ y=8,\ 19$

7 $x=3,\ y=2$

8 $x=17,\ 12,\ 7,\ 2,\ y=1,\ 4,\ 7,\ 10$

9 $x=12,\ 1,\ y=2,\ 6$

10 $x=13p-3,\ 10,\ y=6p-2,\ 4$

11 $x=5p-2,\ 3,\ y=7p+3,\ 10$

12 $x=21p+10,\ 10$; $y=8p+2,\ 2$

13 $x=7p+3,\ 10,\ y=8p-1,\ 7$

14 $x=11p-3,\ 8$; $y=7p-5,\ 2$

15 $x=13p+15,\ 15$; $y=10p+8,\ 8$

16 Of the first 28, 16, or 4, of the second 2, 9, or 16

17 13

18 140, 12, 56, 96

19 $\frac{5}{7},\ \frac{8}{11}$

20 8 florins, 1 half-crown An infinite number

21 30 tables, 9 sofas; or 5 tables, 32 sofas.

22 89 23, 65, 47, 41, 71, 17, 95 **23.** $x=3,\ y=2,\ z=5$.

24. $x=4, 7, 10, \ldots$
$y=2, 7, 12, \ldots$
$z=7, 13, 19, \ldots$

25 26 rams, 4 pigs, 6 oxen; or 11 rams, 17 pigs, 8 oxen

Miscellaneous Examples X. Page 541. **1.** $(2a-3b)(a+b)$

2 $(2a+3b)(2a-3b)(x-2a)(x^2+2ax+4a^2)$. **4.** 30240

5 (i) $2,\ 3,\ \dfrac{5\pm\sqrt{37}}{2}$, (ii) $20\frac{3}{4},\ 16\frac{1}{4}$. **7.** $\log 2$ 8 2449440

9 (i) $(x^2+2x+3)(x^2-2x+3)$, (ii) $(a+b)\{3(a+b)+2\}\{3(a+b)-2\}$.

10 y^4+3y. 11 (i) $-\dfrac{a+b}{4}$, (ii) $x=\dfrac{c}{c+1},\ y=-\dfrac{c}{c+1}$ 12 0·02 nearly

14 360 15 2080 A, 97, B, 163 16 $4r+1$.

17 $A=\frac{1}{2}(1+\sqrt{5}),\ B=\frac{1}{2}(1-\sqrt{5})$ **19.** 1, 2 20 $-1,\ -\frac{1}{2}$

21 3·742 22 $x=bc/(a-b)(a-c),\ y=ca/(b-c)(b-a),\ z=ab/(c-a)(c-b)$

24. £468 11s 2d, £1000 26 $x=-b,\ y=a$ 27 (i) 8, (ii) 81.

28 Cost price 8s, Sale price 9s 29 ab sq ft 30 14.

32 $\bar{3}\cdot698970$, 0·799340, $\bar{1}\cdot785248$ 33 $p-q-\dfrac{pq}{100}$ 35 $\frac{7}{4}$.

37. $3\frac{1}{4}$ mi per hr; 2 hrs, 36 min 38 14, 22 39 180.

40 $\{2\cdot 3^r+(-1)^{r+1}\}x^r$. 41 $y=\dfrac{(b+p)(b+q)}{b-a}$

44. (i) $\dfrac{\sqrt{2}}{2}(7-\sqrt{5})$; (ii) $\sqrt{\dfrac{a^2+ax+x^2}{2}}+\sqrt{\dfrac{a^2-ax+x^2}{2}}$

45 2·83, 4·95, 12·16, 5·5 46 A, 7 mi per hr, B, 12 mi per hr

47 $a=251,\ b=1$, $a=127,\ b=2$, $a=55,\ b=5$, $a=35,\ b=10$

48 1·105 49 $(a+b)(a-b)(a^2+b^2)(2a-b)(2a+b)$ 50 $1-x$. 51 100

52 (i) $-\dfrac{9}{4}$; (ii) -2 53 $\dfrac{2ax}{a^2-b^2}$ This $>\dfrac{2ax}{a^2}$, or $\dfrac{2x}{a}$ 55 14 miles

58 0 60 $1+2x+3x^2+\ldots$, $(n+1)x^n-nx^{n+1}$, $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

61 $\frac{1}{2}$. 62 $-\frac{1}{6}(n^3-15n^2+2n)$ 63 £439 19s.

66 $(x^2+7x+15)(x^2+7x+7)$ 69 (i) $x=3, 2, 5, 1$, $y=2, 3, 1, 5$ (ii) $x=\pm 2\sqrt{3},\ \pm 3$, $y=\pm 3,\ \pm 2\sqrt{3}$.

70 $\dfrac{231x^{12}}{4}$ 71 $\dfrac{2}{1-x}+\dfrac{2x+3}{1+x^2}$ **72.** $\dfrac{e^x-e^{-x}}{2}$.

73 $\dfrac{1}{m+1}+\dfrac{1}{n+1}+\dfrac{1}{p+1}=1$, or $mnp=m+n+p+2$ 74 $a+b$

75 69·7 in, 30·3 in. 76 $a^2x^2+(2ac-4b^2)x+c^2=0$

78. $3+\dfrac{12}{x-2}+\dfrac{1}{(x-2)^3}$ **79.** 642·2. **81.** $(2x-3y)(x+2y)(3x-y)$

82 $\dfrac{p+q}{p-q}$ 83 -2 84. $x=y=\frac{3}{4}$, $x=0$, $y=1$, $x=1$, $y=0$

85 8 40 a m at C 86 31% 88 $x=\pm3$, $y=\mp2$, $z=\pm1$,
$x=\pm\dfrac{11}{\sqrt{19}}$, $y=\pm\dfrac{1}{\sqrt{19}}$, $z=\pm\dfrac{7}{\sqrt{19}}$

91 None 92 (i) 4, (ii) $3xyz$ 93 $\frac{1}{2}$ 94 26 95 19

96 $a=0\ 32$, $b=8\ 8$ 99 $2c-a-b$ 100 $a^3-3ab^2+2c^3=0$

101 4 12 p m 103 59 104. 2 8% too much 105 $2+4x-9x^2+3x^3$.

108 324 110 1680 111 $8x=\sqrt{yz}+2\sqrt[3]{yz}$.

113 $a=2$, $b=3$, $p=10$, $q=12$; $a=2$, $b=-3$, $p=-2$, $q=-12$.

115 $2^{n+1}-2+\dfrac{n(n^2-1)}{3}$ 117 (i) $-\dfrac{a+2b}{2}$, (ii) $a+b+c$.

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